Standard exact projective resolutions
relative to a countable class of Fréchet spaces

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Abstract. We will show that for each sequence of quasinormable Fréchet spaces
\((E_n)_{n \in \mathbb{N}}\) there is a Köthe space \(\lambda(A)\) such that
\[
\text{Ext}^1(\lambda(A), \lambda(A)) = \text{Ext}^1(\lambda(A), E_n) = 0
\]
and there are exact sequences of the form
\[
\ldots \to \lambda(A) \to \lambda(A) \to \lambda(A) \to E_n \to 0.
\]
If, for a fixed \(n \in \mathbb{N}\), \(E_n\) is nuclear or a Köthe sequence space, the resolution above may be reduced to a short exact sequence of the form
\[
0 \to \lambda(A) \to \lambda(A) \to E_n \to 0.
\]
The result has some applications in the theory of the functor \(\text{Ext}^1\) in various categories of Fréchet spaces by providing a substitute for non-existing projective resolutions.

Let us recall that \(\text{Ext}^1(E, F) = 0\) for Fréchet spaces \(E, F\) means that every short exact sequence
\[
0 \to F \xrightarrow{\beta} G \xrightarrow{\gamma} E \to 0
\]
of Fréchet spaces splits (i.e., \(\gamma\) has a continuous linear right inverse). We will prove the following main result:

**MAIN THEOREM.** Let \((E_n)_{n \in \mathbb{N}}\) and \((F_n)_{n \in \mathbb{N}}\) be two sequences of quasinormable Fréchet spaces. There exists a Köthe space \(\lambda(A)\) such that:

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(0) \( \lambda(A) \oplus \lambda(A) \simeq \lambda(A, \lambda(A)) \simeq \lambda(A) \);

(1) \( \text{Ext}^1(\lambda(A), \lambda(A)) = 0 \);

(2) \( \text{Ext}^1(\lambda(A), E_n) = 0 \);

(3) For every \( n \) there is an exact sequence
\[
\cdots \to \lambda(A) \to \lambda(A) \to \lambda(A) \to F_n \to 0.
\]

(4) For every \( F_n \) which is a reduced projective limit of Banach spaces \( l_1 \) we have a short exact sequence
\[
0 \to \lambda(A) \to \lambda(A) \to F_n \to 0.
\]

Moreover, the space \( \lambda(A) \) may be chosen Schwartz, whenever all \( F_n \) are Schwartz spaces.

The result is a far reaching refinement of an unpublished result of the second named author [Kr, 2.2.4]. In view of Th. 2.1 below the obtained theorem is optimal (recall that a quotient of a quasi-normable space is quasi-normable). Although we believe that the result is interesting in itself we explain some external motivations for it.

There are two main sources of motivation for the result. First of all, it provides a substitute for non-existing projective resolutions for the class of quasi-normable Fréchet spaces. Let us recall that a locally convex space \( (X, \mathcal{L}) \) is called \textit{projective} if it is a counterexample for the class \( \mathcal{L} \) of LCS if for any \( Y \in \mathcal{L} \) and any topological quotient map \( q: Y \to X \), the operator \( q \) has a linear continuous right inverse. Moreover, the following topologically exact diagram is called a \textit{projective resolution} of \( X \) in \( \mathcal{L} \):
\[
\cdots \to P_3 \to P_2 \to P_1 \to X \to 0
\]

if \( P_i \) are projective in \( \mathcal{L} \). Geiger proved [G1] (see also [G2]) that, contrary to the Banach case and the LB-space case (see [K1], comp. [D1]), in many classes of Fréchet spaces (like the classes of nuclear, Schwartz or Montel spaces) there are no infinite dimensional projective spaces. In particular, there are no projective resolutions. The latter fact produces an annoying asymmetry in the theory of the functor \( \text{Ext}^4 \) for Fréchet spaces (as developed in [P1], [V6] or [V5]): we may use injective resolutions but we cannot apply homological constructions based on projective resolutions.

Our result gives a resolution (2) for any countable class \( \mathcal{L} := \{ F_n : n \in \mathbb{N} \} \) of quasi-normable Fréchet spaces, and (if we take \( E_n = F_n \)) (2) is "relatively projective" for \( \mathcal{L} \) in the sense of condition (1) and (2), which suffices for applications.

The authors are mainly interested in applications to the theory of the functor \( \text{Ext}^1 \) in the category of "graded" Fréchet spaces [DV]. Using our Main Theorem we can obtain an essential ingredient of that theory: the fact that if \( \text{Ext}^1(E, F) = 0 \), then \( \text{Ext}^1(E, G) = 0 \) for any "graded" quotient \( G \) of \( F \). The theory of \( \text{Ext}^1 \) for "graded" Fréchet spaces allows to give a proper splitting theory of short exact sequences containing spaces \( C^\infty(\Omega) \) and a splitting theory of differential complexes. It is also a base of a structural theory of the space \( C^\infty(\Omega) \). For more details see [DV].

The Main Theorem could also be applied to the classical theory of \( \text{Ext}^1 \) in the category of Fréchet spaces. For example, we can get an analogue of [V6, 1.1] (and the asymmetry mentioned above disappears). The fact that our resolution is "short" in case of locally \( l_1 \) spaces amounts to the known equality \( \text{Ext}^1(E, F) = 0 \) for \( E \) nuclear or locally projective [V5, 1.4] or [V6, 1.2, 1.6], which in turn implies some permanence properties for \( \text{Ext}^1 \) (see [V6, 1.5 and 1.7] or [V6, 1.6 and 1.7]). The theory of the functor \( \text{Ext}^1 \) for Fréchet spaces has proved its importance in the structure theory of Fréchet spaces (see [A2], [V2], [V3], [V5], [VW1], [VW2] etc.).

The second source of motivation comes from the problem whether a Fréchet space of a certain class is a quotient of a "nice" Kôthe sequence space. This problem serves to reduce some questions on general Fréchet spaces to questions on Kôthe spaces, where we have methods of calculations with matrices at our disposal. There are some known positive results ([A1], [VW6], [W]); in particular, it is known that each nuclear, Schwartz or quasi-normable Fréchet space is a quotient of a nuclear, Schwartz, or quasi-normable Kôthe sequence space, respectively, see [W], [VW3] and [MV1].

We make another step in this direction: in our result \( F \) is a quotient of \( \lambda(A) \), where also the corresponding kernel is "nice", in the locally projective case even isomorphic to \( \lambda(A) \) and the spaces involved have nice splitting properties. Moreover, for any countable class \( F \) we find a universal \( \lambda(A) \). Our proof in the Schwartz locally projective case is modelled after a proof of a result of that type due to the second named author [Kr, 2.2.4].

Now, we summarize the content of the paper. In Section 0 we introduce some (known) definitions and in Section 1 we describe a construction of Kôthe sequence spaces \( \lambda_\Phi(\alpha) \). In Section 2 we prove a splitting theorem for \( \lambda_\Phi(\alpha) \) and in Section 3 we construct some auxiliary short exact sequences containing \( \lambda_\Phi(\alpha) \). Section 4 is devoted to the proof of the main result (based on results of Section 2 and 3) in the case of \( (E_n) \) being Schwartz spaces and projective limits of \( l_1 \) Banach spaces. Section 5 shows how the general case follows from the special one.

We should point out that our method of proof uses certain elements of the proof of [Kr, 2.2.4] and some other results from that unpublished paper of the second named author. Nevertheless, we give here a self-contained proof of our main result.
0. Preliminaries. By an operator we always mean a linear continuous
map. A Köthe space will be defined as follows:
\[
\lambda(A) := \{ x = (x_i)_{i \in I} \in \mathbb{K}^I : \| x \|_k := \sum_{i \in I} |x_i| a_{i,k} < \infty \},
\]
where \( I \) is an arbitrary set and \( A = (a_{i,k}) \) is a matrix of positive numbers
such that \( a_{i,k} \leq a_{i,k+1} \) for \( i \in I, k \in \mathbb{N} \). In our definition \( \lambda(A) \) always is a
Fréchet space with a continuous norm. A Köthe sequence space is separable
if \( I \) is countable and then we assume \( I = \mathbb{N} \). If \( (E, (\| \cdot \|_k)_{k \in \mathbb{N}}) \) is a Fréchet
space, then
\[
\lambda(A, E) := \{ x = (x_k) \in E^I : \| x \|_k^{\lambda(A,E)} := \sum_{i \in I} |x_i| \| E_{a_{i,k}} < \infty \},
\]
We call \( \lambda(A) \) shift stable, stable or tensor stable if
\[
\lambda(A) \simeq \lambda(A) \times \mathbb{K}, \quad \lambda(A) \simeq \lambda(A) \times \lambda(A), \quad \lambda(A) \simeq \lambda(A, \lambda(A)),
\]
respectively. If \( \lambda(A) \) is tensor stable, then the Pelczyński decomposition method implies (see [V4, Lemma 1.1] applied to \( A(E) := \lambda(A, E) \)):

**Proposition 0.1.** Let \( \lambda(A) \) be tensor stable. If \( E \) is a Fréchet space
isomorphic to a complemented subspace of \( \lambda(A) \), and if \( E \) contains a com-
plemented subspace isomorphic to \( \lambda(A) \), then \( E \simeq \lambda(A) \).

We will be interested in Köthe spaces of the form \( \lambda(B) = l_1(J) \overline{\otimes}_\pi \lambda(A) \),
where \( \lambda(A) \) is a separable Köthe sequence space and
\[
l_1(J) = \{ x = (x_i) \in \mathbb{K}^J : \| x \| := \sum_{i \in J} |x_i| < \infty \}.
\]
It is easily seen that for \( I = J \times \mathbb{N} \) we have
\[
B = (b_{j,n,k})_{(j,n) \in J, k \in \mathbb{N}}, \quad b_{j,n,k} := a_{n,k}.
\]
A matrix \( A \) is called regular if \( I = \mathbb{N} \) and \( a_{i,k+1}/a_{i,k} \) is decreasing as \( i \rightarrow \infty \). A projective limit \( \text{proj}(E_n, \iota_k) \) is called reduced if
\[
\forall k \exists l \forall m > 1 : \quad \iota_k(E) \supseteq \iota_k(E_m),
\]
where \( \iota_k : E \to E_k \) is the canonical map. Finally, we call a sequence of
Fréchet spaces and operators
\[
\ldots \overset{T_3}{\rightarrow} G_3 \overset{T_2}{\rightarrow} G_2 \overset{T_1}{\rightarrow} G_1 \overset{T_0}{\rightarrow} E \overset{\lambda}{\rightarrow} \ldots
\]
exact if \( \text{im } T_l = \ker T_{l-1} \) for each \( l \in \mathbb{N} \). For example,
\[
0 \rightarrow F \overset{f}{\rightarrow} G \overset{q}{\rightarrow} E \rightarrow 0
\]
is said to be short (topologically) exact if \( j \) is a topological embedding, \( q \) is a
topological quotient map and \( \ker q = \text{im } j \).

One can find more about the functors \( \text{Ext}^k \) for Fréchet spaces in [V6],
[V5], [MV] or [P1].

For other unexplained functional analytic notions see [J] and [K].

1. \( G_\infty \)-spaces with increasing transition functions. From now on
we denote by \( \alpha = (\alpha_i) \) an increasing unbounded sequence of positive numbers
and by \( \Phi = (\phi_1, \phi_2, \ldots) \) an increasing sequence of functions \( \phi_i : \mathbb{R}_+ \to \mathbb{R}_+ \)
such that
\[
r_\infty^2 \leq \phi_1(r) \leq \phi_2(r) \leq \ldots \quad \text{for any } r \in \mathbb{R}_+.
\]
Let us define \( \phi_0 := \phi_\infty \circ \phi_{\infty-1} \circ \ldots \circ \phi_1 \). We call a Köthe sequence space \( \lambda(A) \), where
\[
a_{i,k} := \alpha_i, \quad a_{i,k+1} := \phi_{i,k+1}(a_{i,k}),
\]
a \( G_\infty \)-space with increasing transition functions and we denote it by \( \lambda(A, \alpha) \).

**Remark.** We should point out that without changing the space \( \lambda(A, \alpha) \) we may assume that: (i) \( (1.1) \) holds only for large \( r \) (i.e., for \( r > R \)); (ii) \( (\phi_i) \)
are continuous; (iii) \( (\phi_i) \) are strictly increasing; (iv) \( \alpha_0 = 1 \); (v) \( \phi_1(1) = 1 \).

There are two typical examples of \( G_\infty \)-spaces with increasing transition functions:

(a) any power series space of infinite type \( \Lambda_\infty(\alpha) \simeq \lambda(A, \alpha) \), where \( \alpha_i := 2^{\alpha_i}, \phi_i(r) := r^{\alpha_i} \);

(b) any Dragelev space of infinite type \( L_r(\alpha, \infty) \simeq \lambda(A, \alpha) \), where
\[
\alpha_i := \exp(f(i\alpha_i)), \quad \phi_i(r) := \exp(f(2^{i-1}(\log r)))
\]

The following easy proposition summarizes elementary properties of \( \lambda(A, \alpha) \).

**Proposition 1.1.** Let \( \alpha \) and \( \Phi \) be as above.

(1) \( \lambda(A, \alpha) \) is a Schwartz space.

(2) \( \lambda(A, \alpha) \) is regular whenever \( \phi_i(r)/r \) is increasing for each \( i \in \mathbb{N} \).

(3) If \( \sup_{n \in \mathbb{N}} \alpha_{n+1}/\alpha_n < \infty \), then \( \lambda(A, \alpha) \) and \( l_1(J) \overline{\otimes}_\pi \lambda(A, \alpha) \) are shift
stable.

(4) If \( \sup_{n \in \mathbb{N}} \alpha_{n}/\alpha_n < \infty \), then \( \lambda(A, \alpha) \) and \( l_1(J) \overline{\otimes}_\pi \lambda(A, \alpha) \) are stable.

(5) If \( \sup_{n \in \mathbb{N}} \alpha_{n+1}/\alpha_n < \infty \), then \( \lambda(A, \alpha) \) and \( l_1(J) \overline{\otimes}_\pi \lambda(A, \alpha) \) are tensor stable.

**Remark.** We call sequences \( \alpha \) satisfying conditions from (3), (4)
and (5) shift stable, stable and tensor stable, respectively.

2. A splitting theory for \( \lambda(A, \alpha) \). We characterize those Fréchet spaces
\( E \) for which \( \text{Ext}^k(\lambda(A, \alpha), E) = 0 \). The characterization will be given in terms
of the so-called \( \Omega \)-type conditions introduced in [VW1, Def. 3.2] (cf. [MV1]).
From now on \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing function and \((U_n)_{n \in \mathbb{N}}\) is a decreasing \(0\)-neighbourhood basis in \(E\).

The space \(E\) satisfies the condition \((\Omega_\psi)\) if
\[
(2.1) \quad \forall u \exists k \forall K \exists C > 0 \forall r > 0 : \ U_k \subseteq C \psi(r) U_K + \frac{1}{r} U_u.
\]
The space \(E\) satisfies the condition \((\Omega_\phi)\) if
\[
(2.2) \quad \forall u \exists k \forall K \exists n, C > 0 \forall r > 0 : \ U_k \subseteq C \phi^n(r) U_K + \frac{C}{r} U_u.
\]
It is clear that \((\Omega_\phi)\) implies \((\Omega_\psi)\) whenever \(\phi \geq \psi\) and \((\Omega_\phi)\) implies \((\Omega_\psi)\) whenever
\[
(2.3) \quad \forall u \exists R \forall r > R : \psi(r) \geq \phi^n(r).
\]
Thus we get by [MV1, Th. 7] and [V5, 5.11]:

**Theorem 2.1.** For any Fréchet space \(E\) the following conditions are equivalent:

1. \(E\) is quasinormable.
2. There is \(\Phi\) such that \(E\) satisfies \((\Omega_\Phi)\).
3. There is \(\Psi\) such that \(E\) satisfies \((\Omega_\Psi)\).
4. There is a non-Banach Fréchet space \(F\) such that \(\text{Ext}^1(F, E) = 0\).

The following theorem is the main result of the present section:

**Theorem 2.2** [Kr, 2.2.2]. Let \(E\) be a Fréchet space and let \(\alpha\) be shift stable. The following conditions are equivalent:

1. The space \(E\) satisfies \((\Omega_\Phi)\).
2. \(\text{Ext}^1(\lambda(\alpha), E) = 0\).
3. \(\text{Ext}^1(l_1(J) \otimes_\alpha \lambda(\alpha), E) = 0\).

For the sake of completeness we give the proof based on the following splitting result of Vogt [V6, 3.1, 3.4, 2.5] (comp. also [KrV]):

**Theorem 2.3.** Let \(\lambda(A)\) be a Köthe sequence space and let \(E\) be an arbitrary Fréchet space.

(a) \(\text{Ext}^1(\lambda(A), E) = 0\), whenever \((\lambda(A), E)\) satisfies the following condition \((S_1)\):
\[
(2.4) \quad \exists p \forall u \exists k \forall n, K, R > 0 \exists n, S \forall i \in \mathbb{N} : \ a_{i,n} U_k \subseteq S a_{i,n} U_K + \frac{a_{i,n} E}{R} U_u.
\]
(b) If \(\text{Ext}^1(\lambda(A), E) = 0\), then the pair \((\lambda(A), E)\) satisfies the following condition \((S_2)\):
\[
(2.5) \quad \forall u \exists k \forall n, K, R > 0 \exists n, S \forall i \in \mathbb{N} : \ a_{i,n} U_k \subseteq S a_{i,n} U_K + S a_{i,p} U_u.
\]

**Proof of 2.2.** We may assume that \(\alpha \geq 1\), \(\phi(1) = 1\) and apply 2.3.

(2) \(\Rightarrow\) (1). We apply \((S_2)\) for \(\lambda(A) = \lambda(\alpha)\) and \(m = p + 1\). Dividing (2.5) by \(a_{i,m}\) we obtain
\[
U_k \subseteq S a_{i,n} U_K + S a_{i,p} U_u.
\]
Since \(\phi_{p+1}(r) \geq r^2\) and \(\alpha_1 \geq 1\), we get \(a_{i,n}/a_{i,p+1} \leq 1/a_i \leq 1/a_i\) and \(a_{i,p+1} \leq 1\). Moreover, shift stability implies that for large \(i\) we get
\[
a_{i,n} = \phi^n(a_i) \leq \phi^n(a_i^2) \leq \phi^{n+1}(a_i - 1) = a_i - 1, n+1.
\]
Hence increasing \(S\), we obtain, for all \(i\),
\[
U_k \subseteq S a_{i-1,n+1} U_K + S a_{i-1} U_u.
\]
Let \(a_{i-1} \leq r < a_i\). Then \(a_{i-1,n+1} \leq \phi^{n+1}(r)\) and we get
\[
(2.6) \quad \forall u \exists k \forall n, S > 0 \exists n, S \forall r \geq a_0 : \ U_k \subseteq S a_{i,n} U_K + S a_{i,n} U_u.
\]
In order to get the same inclusion for all \(r > 0\), it suffices to increase \(S\) appropriately.

(1) \(\Rightarrow\) (3). We take \(p = 1\) and we choose \(k\) for any \(u\) according to \((\Omega_\phi)\). Then for \(K\) we find \(n, C\) again as in \((\Omega_\psi)\). We put \(R := c R a_{i,m}\) in \((\Omega_\psi)\):
\[
U_k \subseteq S \phi^n(C R a_{i,m}) U_K + S \phi^n R a_{i,m} U_u.
\]
Since \(a_{i,m}/a_{i,m-1} \to 0\) as \(i \to \infty\) (see Prop. 1.1.1), for large \(i\) we obtain
\[
a_{i,m} \phi^n(C R a_{i,m}) \leq a_{i,m} \phi^n(a_{i,m-1}) \leq \phi^n(a_{i,m-1})^2 \leq \phi^{n+1}(a_{i,m-1}) \leq a_{i,m-1} + 1.
\]
Hence putting \(S\) suitably large we obtain, for all \(i\),
\[
a_{i,n} U_k \subseteq S a_{i,m} U_K + S a_{i,p} U_u.
\]
Now taking \(B = (b_{j,i,k})_{i,j,k} \in \mathbb{N}^3, i = j \times \mathbb{N}, b_{j,i,k} = a_{i,k}\), we obtain \((S_1)\) for \((\lambda B), E\), where \(\lambda B = l_1(J) \otimes_\alpha \lambda(\alpha)\).

(3) \(\Rightarrow\) (2). This is obvious, since \(\lambda(\alpha)\) is a complemented subspace of \(l_1(J) \otimes_\alpha \lambda(\alpha)\).

**Corollary 2.4** [Kr, 1.2.5 and 1.1.2]. For arbitrary \(\alpha\) and \(\Phi\) as in Section 1 we have
\[
\text{Ext}^1(\lambda(\alpha), \lambda(\alpha)) = 0 \quad \text{and} \quad \text{Ext}^1(l_1(J) \otimes_\alpha \lambda(\alpha), l_1(J) \otimes_\alpha \lambda(\alpha)) = 0.
\]

**Proof.** We assume that \(\alpha_0 = 1\) and \(\phi(1) = 1\). It suffices to show that both \(\lambda(\alpha)\) and \(l_1(J) \otimes_\alpha \lambda(\alpha)\) satisfy \((\Omega_\phi)\). Since the two cases are nearly identical we consider only \(\lambda(\alpha)\).
We take \( k = u + 1, n = K, C = 1 \) and \( U_l := \{ \xi = (\xi_i) : \|\xi\| = \sum_{i=0}^{\infty} \phi^K(\alpha_i)|\xi_i| \leq 1 \} \). Let \( \alpha_{i0} \leq r < \alpha_{i0+1}, \xi \in U_k \). We take \( \xi = \eta + \zeta \) where \( \eta := (\xi_1, \xi_2, \ldots, \xi_{i0}, 0, \ldots), \quad \zeta := (0, 0, \ldots, 0, \xi_{i0+1}, \xi_{i0+2}, \ldots) \).

Obviously

\[
\|\eta\|_K = \sum_{i=0}^{i_0} \phi^K(\alpha_i)|\xi_i| \leq \phi^K(\alpha_{i0}) \sum_{i=0}^{i_0} |\xi_i| \leq \phi^K(r).
\]

On the other hand, since \( r^{\alpha_i}(\alpha_i) \leq \alpha_i r^{\alpha_i}(\xi_i) \leq (\phi(\alpha_i))^2 \leq \phi^{u+1}(\alpha_i) \) for \( i \geq i_0 + 1 \), we get

\[
\|\xi\|_u = \sum_{i=0}^{\infty} \phi(\alpha_i)|\xi_i| = \sum_{i=i_0+1}^{\infty} \phi^{u+1}(\alpha_i)|\xi_i| \leq \frac{1}{r}.
\]

3. Auxiliary short exact sequences. First we show that each locally projective Schwartz Fréchet space is a reduced projective limit of spaces \( \lambda_{\phi}(\alpha) \) for arbitrary \( \phi \) and suitably chosen \( \alpha \).

**Proposition 3.1.** Let \( (F_n) \) be a sequence of Schwartz Fréchet spaces which are reduced projective limits of \( l_1 \). Then there exists a tensor stable sequence \( \alpha \) such that for each \( n \in \mathbb{N} \), \( F_n \) is a reduced projective limit of \( \lambda_{\phi}(\alpha) \). In particular, there is a short exact sequence of the form

\[
0 \to F_n \to \prod_{i \in \mathbb{N}} \lambda_{\phi}(\alpha) \to \prod_{i \in \mathbb{N}} \lambda_{\phi}(\alpha) \to 0.
\]

**Proof.** The last statement follows from the previous one by a Fréchet analogue of [V5, 1.3] or [V6, 1.1] (see also [P1, Cor. 5.1]).

Without loss of generality we may assume that each \( F_n \) is a reduced projective limit of spaces \( l_1 \) with compact linking maps.

Let \( T : l_1 \to l_1 \) be an arbitrary compact map. The image of the unit ball \( T(B_{l_1}) \) is relatively compact, thus contained in a closed absolutely convex hull of a null sequence \( (x_n) \). As easily seen, there is a real (monotonic) null sequence \( (\alpha_n x_n) \) is still null. We take maps \( R : l_1 \to l_1 \) and \( \sigma : l_1 \to l_1 \) defined by

\[
R(x_n) = \alpha_n^{-1} x_n \quad \text{and} \quad \sigma(x_n) = \alpha_n x_n.
\]

Since \( R \circ \sigma(B_{l_1}) \supseteq T(B_{l_1}) \), there is a map \( S : l_1 \to l_1 \) such that \( T = R \circ \sigma \circ S \).

As is easily seen \( \sigma \) and hence also \( T \) factorize through a finite type power series space. Now, it suffices to show our proposition for a sequence \( (F_n) \) of Schwartz spaces \( F_n = \lambda(A^n) \), where \( A^n \) are regular matrices.

We construct a tensor stable sequence \( \alpha \) satisfying

\[
\forall n, k, l \exists C(n, k, l) \forall i \in \mathbb{N} : \phi(i)(\alpha_i) \leq C(n, k, l) \frac{a_{n,k+1}^i}{a_{n,k}^i}.
\]

First, we obtain easily a sequence \( \beta = (\beta_i) \in c_0 \) decreasing to zero and satisfying

\[
\forall n, k \exists C(n, k) \forall i(n, k) : \beta_i \geq \frac{a_{n,k}^i}{a_{n,k+1}^i}
\]

and an increasing function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying

\[
\phi(1) = 1, \quad \forall k \exists C(k) : \forall r \geq 1 : \phi(k) \leq C(k)\phi(r).
\]

Now, we take simply \( \alpha_i = \min(\phi^{-1}(1/\beta_i), 2a_{2n+2}) \) for \( 2n < i \leq 2n+1 \).

A reduced projective spectrum representing \( F_n \) is given by \( (\lambda_{\phi}(\alpha))_{k \in \mathbb{N}} \) and linking maps \( \sigma_{k+1} : \lambda_{\phi}(\alpha) \to \lambda_{\phi}(\alpha) \) are defined by

\[
\sigma_{k+1}(\xi) := (a_{n,k}^i \xi_i)_{i \in \mathbb{N}}.
\]

The following map is an isomorphism:

\[
W : \lambda(A^n) \to \prod_{k \in \mathbb{N}} \lambda_{\phi}(\alpha), \quad W(\xi) := (a_{n,k}^i \xi_i)_{i,k}.
\]

Indeed, \( W \) is continuous since

\[
\|a_{n,k}^i \xi_i\|_{\lambda_{\phi}(\alpha)} = \sum_{i=0}^{\infty} \phi(\alpha_i)|\xi_i| \leq C(n, k, p) \sum_{i=0}^{\infty} |\xi_i| a_{n,k}^i = C(n, k, p)\|\xi\|_{\lambda_{\phi}(\alpha)}.
\]

Moreover, \( W \) is open since

\[
\|\xi\|_{\lambda_{\phi}(\alpha)} = \sum_{i=0}^{\infty} \alpha_i \phi(\alpha_i)|\xi_i| \leq \sum_{i=0}^{\infty} \alpha_i \phi(\alpha_i)|\xi_i| = \|\sigma(\xi)\|_{\lambda_{\phi}(\alpha)}.
\]

This completes the proof because the image of \( W \) contains the space of all finitely non-zero sequences which is dense in \( \prod_{k \in \mathbb{N}} \lambda_{\phi}(\alpha) \).

**Proposition 3.2.** Let \( \alpha \) be tensor stable and let \( \phi_i(r)/r \) increase monotonically for any \( i \in \mathbb{N} \). Then there exists a short exact sequence

\[
0 \to \lambda_{\phi}(\alpha) \to \lambda_{\phi}(\alpha) \to \lambda_{\phi}(\alpha)^N \to 0.
\]

The first result of the type above was proved in [V1, proof of Lemma 1.6] for the space \( s \). Now, there is a whole family of similar theorems very useful in the structural theory of Fréchet spaces (see [VW2, Th. 2.3 and 2.4], [V3], [V7, Th. 3.2]). Our result follows from the following theorem due to Apiola [A1, Prop. 3.3]:

**Theorem 3.3.** Let \( \lambda(A) \) be a Schwartz Köthe sequence space with a regular matrix such that \( (a_{n,k})_{k \in \mathbb{N}} \) increases for each \( k \in \mathbb{N} \). Assume that there exists a bijection \( \beta : \mathbb{N} \to \mathbb{N} \) such that

\[
\forall n, k, l \exists C(n, k, l) \forall i \in \mathbb{N} : \phi(i)(\alpha_i) \leq C(n, k, l) \frac{a_{n,k+1}^i}{a_{n,k}^i}.
\]
(i) \( \beta \) increases in each variable and \( j \leq \beta(0,j,0) \).
(ii) \( \forall k \exists p(k), C > 0 \forall n \leq k: a_{\beta(n,j,i+1),k} \leq C a_{\beta(n,j,i),p(k)} \).
(iii) \( \forall k \exists p(k): \sum_{i=0}^{\infty} a_{\beta(0,0,i),k} / a_{\beta(0,0,i),p(k)} < \infty \).
(iv) \( \forall k \exists p(k), C > 0 \forall n \leq k: a_{\beta(n,j,0),k} \leq C a_{\beta(j,0,p(k))} \).

Then there exists a short exact sequence
\[ 0 \rightarrow \lambda(A) \rightarrow \lambda(A) \rightarrow \lambda(A)^N \rightarrow 0. \]

Proof of 3.2. It suffices to check the assumptions of Th. 3.3 for \( a_{i,k} = \phi^k(a_i), a_0 = 1 \).

First we have to define \( \alpha \). Assume that \( \alpha_{2i} \leq C \alpha_i \) for each \( i \in \mathbb{N} \). For any \( i \) there is \( \lambda \geq 4^i \) such that \( 2C \alpha_{i+1} \leq \alpha_i \leq 2C^3 \alpha_i \). We can construct inductively an increasing sequence \( K_i = 4^i, K_0 = 0 \) such that
\[ 2a_{2K_i} \leq \alpha_{K_{i+1}} \leq S \alpha_{K_i} \quad \text{for} \quad i \in \mathbb{N}, S = 2C^3. \]

We order all natural numbers not of the form \((2j+1)K_i \) for \( i > 0 \) in an increasing sequence \( s(j) \). Obviously \( j \leq s(j) \leq 2j+1 \) for \( j > 0 \). We define a bijection \( \delta : \mathbb{N} \rightarrow \mathbb{N} \):
\[ \delta(i,j) = \begin{cases} s(j) & \text{for } i = 0, j \in \mathbb{N}, \\ (2j+1)K_i & \text{for } i \geq 1, j \in \mathbb{N}, \end{cases} \]
and
\[ \beta(n,j,i) := 2^n(2\delta(i,j) + 1) - 1 \]
\[ = \begin{cases} 2^{n+1} s(j) + 2^n - 1 & \text{for } i = 0, \\ 2^{n+2} K_i(2j+1) + 2^n - 1 & \text{for } i \geq 1, \end{cases} \]

The condition (i) of 3.3 is obvious. In order to show (ii) we first observe that
\[ \beta(n,j,i+1) = \max(2\delta(n,j,i+1), 2K_{i+1}). \]

Indeed, if \( 2^{n+1}(2j+1) \leq K_{i+1} \), then \( \beta(n,j,i+1) \leq 2K_{i+1} \). Otherwise, if \( 2^{n+1}(2j+1) > K_{i+1} \), then \( \beta(n,j,i+1) \leq 8(\beta(n,j,i)) \).

By the tensor stability of \( \alpha \) we find a constant \( M \) such that
\[ a_{\beta(\delta(n,j,0),0)}/M a_{\beta(n,j,0)} \quad \text{and} \quad a_{2K_{i+1}^2}/M a_{K_{i+1}}. \]

Since \( \alpha_{K_i} \leq a_{\beta(n,j,0)} \), we find by (3.2) and (3.4) a constant \( L > 0 \) such that
\[ a_{\beta(n,j,i+1)} = L a_{\beta(n,j,i)} \quad \text{for any } n, j, i. \]

Finally, for \( a_{\beta(n,j,i)} > L \) we get \( a_{\beta(n,j,i+1)} \leq a_{\beta(n,j,i),k} \). Taking a suitable constant \( C(k) \) we obtain (ii) for all \( \beta(n,j,i) \).

We now prove (iii). Since \( \phi_k(r)/r \) increases, \( \phi_k(2r) \geq 2\phi_k(r) \) and \( \phi_k(2r) \geq 2\phi_k(r) \). Moreover, \( \beta(0,0,i) = 2K_i \) and, by (3.2),
\[ \phi^k(a_{\beta(0,0,i+1)}) = \phi^k(a_{2K_{i+1}}) \geq \phi^k(2a_{2K_i}) \geq 2a_{\beta(0,0,i)} = 2a_{\beta(0,0,i)}. \]
Because \( \phi_{k+1}(r) \geq r^2 \), we obtain
\[ \sum_{i=0}^{\infty} \frac{1}{\phi^k(\alpha_{\beta(0,0,i)})} \leq \sum_{i=0}^{\infty} \frac{1}{\phi^k(\alpha_{\beta(0,0,i)})} \leq \sum_{i=0}^{\infty} \frac{1}{2^{i}a_{\beta(0,0,i)}} < \infty. \]

Finally, we prove (iv). We have \( \beta(n,j,0) = 2^{n+1} s(j) + 2^n - 1 \leq 2^{n+2} j + 2^{n+2} \). For \( j > 0, n \leq k \) and some \( M > 0 \) we have \( a_{\beta(n,j,0)} \leq a_{2^{n+2} j} \leq M a_j \).

Thus for \( \alpha_j \geq M \) we get
\[ \phi^k(\alpha_{\beta(n,j,0)}) \leq \phi^k(\alpha_j). \]

For \( j \) such that either \( \alpha_j < M \) or \( j = 0 \) we obtain (3.5) multiplying the right hand side by a suitable constant \( C(k) \).

This completes the proof of 3.2.

4. Proof of the main result for locally projective Schwartz spaces. We assume that all \( F_n \) are reduced projective limits of \( l_1 \) spaces with compact linking maps (i.e., Schwartz spaces).

By Th. 2.1, we find easily \( \Phi = \phi_i \), where \( \phi_i(r)/r \) increases as \( r \) increases, such that each \( E_n \) and \( F_n \) satisfies (\( \alpha_\Phi \)). We define \( \alpha \) according to Prop. 3.1. By 2.2 and 2.4,
\[ \text{Ext}^1(\lambda_\Phi(\alpha), \lambda_\Phi(\alpha)) = 0, \quad \text{Ext}^1(\lambda_\Phi(\alpha), E_n) = 0, \]
\[ \text{Ext}^1(\lambda_\Phi(\alpha), F_n) = 0. \]

Now, by 3.1 and 3.2, we obtain the first row and the last column of the following commutative diagram with exact rows and columns:

\[ \begin{array}{ccc}
0 & \rightarrow & 0 \\
0 & \rightarrow & F_n \\
\downarrow & & \downarrow \phi_1 \\
0 & \rightarrow & H \\
\downarrow & & \downarrow Q_1 \\
0 & \rightarrow & 0 \\
\end{array} \]

\[ \begin{array}{ccc}
\lambda_\Phi(\alpha) & \rightarrow & \lambda_\Phi(\alpha)^N \\
\downarrow id & & \downarrow q_1 \\
\lambda_\Phi(\alpha) & \rightarrow & \lambda_\Phi(\alpha)^N \\
\downarrow id & & \downarrow q_2 \\
\lambda_\Phi(\alpha) & \rightarrow & \lambda_\Phi(\alpha) \\
\end{array} \]

where \( H = \{(x,y) \in \lambda_\Phi(\alpha)^6 \times \lambda_\Phi(\alpha) : q_1 x = q_2 y \}, \)
\[ J_1(x) := (j_1 x, 0), \quad J_2(x) := (0, j_2 x), \quad Q_1(x,y) := y, \quad Q_2(x,y) := x. \]
By (4.1), $H \cong F_0 \oplus \lambda_\phi(\alpha)$. Similarly, we obtain another commutative diagram with exact rows and columns:

\[
\begin{array}{cccccccc}
0 & \rightarrow & \lambda_\phi(\alpha) & \rightarrow & \lambda_\phi(\alpha) \oplus F_0 & \rightarrow & \lambda_\phi(\alpha)^{ul} & \rightarrow & 0 \\
& \downarrow & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \lambda_\phi(\alpha) & \rightarrow & G & \rightarrow & \lambda_\phi(\alpha) & \rightarrow & 0 \\
& & & \downarrow & & & \downarrow & & \\
& & & \lambda_\phi(\alpha) & \rightarrow & \lambda_\phi(\alpha) & \rightarrow & 0 & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

By stability of $\alpha$ and (4.1), $G \cong \lambda_\phi(\alpha)$. Finally, we get a short exact sequence

\[0 \rightarrow \lambda_\phi(\alpha) \rightarrow \lambda_\phi(\alpha) \rightarrow \lambda_\phi(\alpha) \oplus F_0 \rightarrow 0.\]

Since $\text{Ext}^1(\lambda_\phi(\alpha), \lambda_\phi(\alpha)) = 0$, the space $K := \ker(q^*(F_0))$ is complemented in $\lambda_\phi(\alpha)$ and

\[0 \rightarrow \lambda_\phi(\alpha) \rightarrow K \rightarrow \lambda_\phi(\alpha) \oplus F_0 \rightarrow 0\]

is exact. Multiplying the above sequence by $\lambda_\phi(\alpha)$ we obtain the sequence we are looking for (by Prop. 0.1, $K \oplus \lambda_\phi(\alpha) \cong \lambda_\phi(\alpha)$). This completes the proof of the Main Theorem for locally projective Schwartz spaces.

We conclude this section with a simple consequence of the above case of the Main Theorem.

**Proposition 4.1.** If all $F_n$ are of the form $l_1(J) \otimes \lambda(D^n)$, where $\lambda(D^n)$ are Köthe Schwartz spaces, then the Main Theorem holds for $\lambda(A) = l_1(J) \otimes \lambda(A^0)$, $\lambda(A^0)$ is a Köthe Schwartz space and the resolution $(\ast)$ is short as in (4).

**Proof.** We apply the locally projective Schwartz case to the sequence of spaces $\lambda(D^n)$ instead of $F_n$. We find a Köthe Schwartz space $\lambda(A^0) = \lambda_\phi(\alpha)$. Thus for any $n \in \mathbb{N}$ there exists a short exact sequence

\[0 \rightarrow \lambda(A^0) \rightarrow \lambda(D^n) \rightarrow 0.\]

It is known ([J, 15.7.3]) that the tensored sequence

\[0 \rightarrow l_1(J) \otimes \lambda(A^0) \rightarrow l_1(J) \otimes \lambda(D^n) \rightarrow 0\]

is exact as well. Obviously (0) is satisfied for $\lambda(A) = l_1(J) \otimes \lambda(A^0)$, whenever it is satisfied for $\lambda(A^0)$. By Th. 2.2 and Cor. 2.4, we get (1) and (2).

5. **Proof in the general case.** We will use the following two results:

**Theorem 5.1.** (Vogt and Walldorf [VWd]). Every Schwartz Fréchet space is isomorphic to a quotient of a Schwartz Köthe space.

**Theorem 5.2.** (Meise and Vogt [MV1, Prop. 7]). Every quasinormable Fréchet space is isomorphic to a quotient of a space $l_1(J) \otimes \lambda(D)$, where $\lambda(D)$ is a nuclear Köthe space.

**Proof of the Main Theorem.** For every $n \in \mathbb{N}$ we define inductively, by use of 5.1 or 5.2, short exact sequences, setting $K_{n,0} = F_0$:

\[0 \rightarrow K_{n,k+1} \rightarrow \lambda_{n,k} \rightarrow K_{n,k} \rightarrow 0,\]

where $\lambda_{n,k} = l_1(J) \otimes \lambda(D^{n,k})$, $\lambda(D^{n,k})$ nuclear Köthe spaces, or (if all $F_n$ are Schwartz spaces) $\lambda_{n,k}$ are Schwartz Köthe spaces.

We then apply the special case of the Main Theorem to the spaces $F_n$, $n \in \mathbb{N}$, on one side and $\lambda_{n,k}$, $n, k \in \mathbb{N}$ on the other side. We obtain $\lambda(A)$ fulfilling (0), (1) and (2) which in case of Schwartz spaces $F_n$ is Schwartz as well. We proceed as follows. We set up the following diagram:

\[
\begin{array}{cccccccc}
0 & \rightarrow & K_{n,k+1} & \rightarrow & \lambda_{n,k} & \rightarrow & K_{n,k} & \rightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L_{n,k+1} & \rightarrow & \lambda(A) & \rightarrow & K_{n,k} & \rightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & \lambda(A) & \rightarrow & \lambda(A) & \rightarrow & 0 & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

Here the upper row is (5.1), and the middle column is the short exact sequence obtained from (4).

For every $n, k \in \mathbb{N}$ we obtain exact sequences

\[0 \rightarrow \lambda(A) \rightarrow L_{n,k} \rightarrow K_{n,k} \rightarrow 0\]
and

\[ 0 \to L_{n,k+1} \to \lambda(A) \to K_{n,k} \to 0. \]

We use these to set up the following diagram (like the diagram (4.2)):

\[
\begin{array}{ccc}
0 & \to & \lambda(A) \\
& \uparrow & \uparrow \\
0 & \to & L_{n,k} \\
& \uparrow & \uparrow \\
0 & \to & H_{n,k} \\
& \uparrow & \uparrow \\
0 & \to & \lambda(A) \\
& \uparrow & \uparrow \\
L_{n,k+1} & \xrightarrow{id} & L_{n,k+1}
\end{array}
\]

Since the middle row splits on account of (1) and \( \lambda(A) \oplus \lambda(A) \cong \lambda(A) \) we obtain for all \( n, k \in \mathbb{N} \) an exact sequence

\[ 0 \to L_{n,k+1} \to \lambda(A) \to L_{n,k} \to 0. \]

Putting all these together we get a long exact sequence

\[ \cdots \to \lambda(A) \xrightarrow{\delta} \lambda(A) \oplus \lambda(A) \xrightarrow{\delta} F_n \to 0, \]

where \( \lambda(A) \) is either of the form \( \lambda_N(\alpha) \) or \( L_1(J) \otimes \alpha \).

Now, assume that \( F_n \) is a reduced projective limit of Banach spaces \( l_1 \) and consider a short exact sequence

\[ 0 \to \ker q_0 \xrightarrow{\delta} \lambda(A) \xrightarrow{\delta} F_n \to 0. \]

We then obtain an exact sequence of the form (see the condition (III) in [V6] or [P2, p. 49])

\[ 0 \to L(F_n, \ker q_1) \to L(\lambda(A), \ker q_1) \to L(\ker q_0, \ker q_1) \]

\[ \to \text{Ext}^1(F_n, \ker q_1) \to \text{Ext}^1(\lambda(A), \ker q_1) \to \text{Ext}^1(\ker q_0, \ker q_1) \]

\[ \to \text{Ext}^2(F_n, \ker q_1) \to \text{Ext}^2(\lambda(A), \ker q_1) \to \cdots \]

Since \( \lambda(A) \) has property (Q2) which is inherited by quotients, the space \( \ker q_1 = \ker q_0 \) has it as well. Thus, by [V6, Cor. 1.5] and Th. 2.2,

\[ \text{Ext}^2(F_n, \ker q_1) = 0 \quad \text{and} \quad \text{Ext}^2(\lambda(A), \ker q_1) = 0. \]

Hence \( \text{Ext}^1(\ker q_0, \ker q_1) = 0 \) and the sequence

\[ 0 \to \ker q_1 \to \lambda(A) \xrightarrow{\delta} \ker q_0 \to 0 \]

splits. This means that \( \ker q_0 \) is isomorphic to a complemented subspace of \( \lambda(A) \). Finally,

\[ 0 \to \ker q_0 \oplus \lambda(A) \xrightarrow{\delta \oplus \delta} \lambda(A) \oplus \lambda(A) \xrightarrow{\delta \oplus 0} F_n \to 0 \]

is the short exact sequence we are looking for because \( \ker q_0 \oplus \lambda(A) \cong \lambda(A) \), by Prop. 0.1.

References


The splitting spectrum differs from the Taylor spectrum

by

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Abstract. We construct a pair of commuting Banach space operators for which the splitting spectrum is different from the Taylor spectrum.

Let \( A_1, \ldots, A_n \) be mutually commuting operators in a Banach space \( X \). The Koszul complex of the \( n \)-tuple \((A_1, \ldots, A_n)\) is the complex

\[
0 \longrightarrow A^0(X,e) \xrightarrow{\delta_0} A^1(X,e) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} A^n(X,e) \longrightarrow 0
\]

where \( A^p(X,e) \) denotes the vector space of all forms of degree \( p \) in indeterminates \( e_1, \ldots, e_n \) with coefficients in \( X \) and the linear mappings \( \delta_p : A^p(X,e) \to A^{p+1}(X,e) \) are defined by

\[
\delta_p(xe_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^n A_j xe_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}.
\]

It is well known that \( \delta_{p+1} \circ \delta_p = 0 \) for every \( p \). The Taylor spectrum \( \sigma_T(A_1, \ldots, A_n) \) is the set of all \( n \)-tuples \((\lambda_1, \ldots, \lambda_n)\) of complex numbers for which the Koszul complex of \((A_1 - \lambda_1, \ldots, A_n - \lambda_n)\) is not exact [5].

Instead of the Taylor spectrum it is sometimes useful to use the following variation (see e.g. [1], [3], [4]). We say that the \( n \)-tuple \((A_1, \ldots, A_n)\) is splitting-regular if its Koszul complex is exact and the ranges of the operators \( \delta_p \) are complemented in \( A^{p+1}(X,e) \). Equivalently, there exist operators \( \varepsilon_p : A^{p+1}(X,e) \to A^p(X,e) \) \((p = 0, \ldots, n-1)\) such that \( \varepsilon_p \delta_p + \delta_{p-1} \varepsilon_{p-1} \) is the identity operator on \( A^p(X,e) \) for \( p = 0, \ldots, n \) (formally we set \( \delta_{-1} = \delta_n = 0 \)).

The splitting spectrum \( \sigma_S(A_1, \ldots, A_n) \) is the set of all \((\lambda_1, \ldots, \lambda_n)\) \( \in \mathbb{C}^n \) such that the \( n \)-tuple \((A_1 - \lambda_1, \ldots, A_n - \lambda_n)\) is not splitting-regular.

The splitting spectrum has similar properties as the Taylor spectrum. Clearly, \( \sigma_S(A_1, \ldots, A_n) \subseteq \sigma_T(A_1, \ldots, A_n) \). For Hilbert space operators these two spectra coincide and the same is true for \( n \)-tuples of operators in \( \ell^1 \).

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