Proof. The proof goes along the same lines as the one of Corollary 1, with Theorem 2 (resp. Theorem 4) in place of Theorem 1.

Analogously we get

COROLLARY 3. *Let* (M_p) *satisfy* (M.1) *and* (M.3′), *and let* $\mathring{u} \in L'_{(\mathring{\nu},\mathring{\omega})}(\mathbb{R})$ (*resp.* $L^{(M_p)'}_{(\mathring{\nu},\mathring{\omega})}(\mathbb{R})$) *with some* $\mathring{\nu} \in \mathbb{R}\cup\{-\infty\}$ *and* $\mathring{\omega} \in \mathbb{R}\cup\{\infty\}$. *Then there exists at most one* $F^{\pm} \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ *of exponential type in* $\{\pm \operatorname{Im} z \ge \varepsilon\}$ *for all* $\varepsilon > 0$ *such that* $b(F^{\pm}) \in L'_{(\nu^{\pm},\omega^{\pm})}(\mathbb{R})$ (*resp.* $L^{(M_p)'}_{(\nu^{\pm},\omega^{\pm})}(\mathbb{R})$) *with some* $\nu^{\pm} \in \mathbb{R}\cup\{-\infty\}$ *and* $\omega^{\pm} \in \mathbb{R}\cup\{\infty\}$, *and* $b(F^{\pm}) - \mathring{u} \in L'_{(\nu,\omega)}(\mathbb{R})$ (*resp.* $L^{(M_p)'}_{(\nu,\omega)}(\mathbb{R})$) *with some* $\nu < \omega$. *Furthermore, if* $\mathring{\nu} < \mathring{\omega}$ *then* $F^{\pm} \equiv 0$.

We remark that in the case $\mathring{\nu} \ge \mathring{\omega}$, in general, the problem of existence of such an $F^{\pm}$ remains open.

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Compact homomorphisms between algebras of analytic functions

by

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Abstract. We prove that every weakly compact multiplicative linear continuous map from $H^\infty(D)$ into $H^\infty(D)$ is compact. We also give an example which shows that this is not generally true for uniform algebras. Finally, we characterize the spectra of compact composition operators acting on the uniform algebra $H^\infty(B_E)$, where B_E is the open unit ball of an infinite-dimensional Banach space E.

Let E denote a complex Banach space with open unit ball B_E and let $\phi : B_E \to B_E$ be an analytic map. We will consider the composition operator C_ϕ defined by $C_\phi(f) = f \circ \phi$, acting on the uniform algebra $H^\infty(B_E)$ of all bounded analytic functions on B_E. This operator may also be regarded as acting on the smaller uniform algebra $A_{\mathrm{u}}(B_E)$ of all analytic functions on B_E which are uniformly continuous, in which case we assume that $f\circ\phi \in A_{\mathrm{u}}(B_E)$ whenever f is in $A_{\mathrm{u}}(B_E)$. These algebras, which are natural generalizations of the classical algebras $H^\infty(D)$ and $A(D)$ of analytic functions on the complex open disc D, have been studied in [ACG].

Several results automatically yielding compactness of composition operators from weak compactness have appeared recently. For instance, D. Sarason in [Sa] proved that every weakly compact composition operator on $H^1(D)$ is compact, and K. Madigan and A. Matheson [MM] obtained the analogue for the little Bloch space $\mathcal{B}_0$. In the first section we study compactness of C_ϕ and prove that every weakly compact homomorphism from $H^\infty(D)$ into $H^\infty(D)$ is automatically compact. This result has also inde-

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pendently been obtained by A. Ülger in [Ü], where he proves that every weakly compact homomorphism from a logmodular algebra into any uniform algebra is compact. In fact, this also follows by combining the Wermer embedding theorem (see, e.g., [Ga], VI.7.2) with results of J. Galé, T. Ransford and M. White [GRW] and K. Hoffman [H]. Earlier S. Ohno and J. Wada [OW] had found sufficient conditions on function algebras A and B to ensure that weakly compact homomorphisms between them are actually compact.

In [Ü] A. Ülger also asks whether every weakly compact homomorphism between uniform algebras is compact without the logmodularity condition. We conclude the first section with an example which shows that this is not generally true. Our example is the uniform algebra $H^\infty(B_E)$ when E is the Tsirelson space.

In Section 2 we determine the spectrum of compact composition operators of the form C_ϕ. In the one-dimensional case H. Kamowitz [K] used the Denjoy–Wolff theorem to determine the spectra of compact composition operators on $A(D)$, and in the setting of the open unit ball B_N in $\mathbb{C}^N$, B. MacCluer [Ma] characterized the spectra by proving the existence of a unique fixed point of $\phi : B_N \to B_N$ when C_ϕ is compact. For E infinite-dimensional the main tool we use to determine the spectra of compact composition operators C_ϕ is the remarkable fixed point theorem of C. Earle and R. Hamilton [EH].

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Preliminaries. The reader is referred to [D] and [M2] for background information on analytic functions on an infinite-dimensional Banach space. The algebra $H^\infty(B_E)$ is a Banach algebra with the natural norm $\|f\| = \sup_{x\in B_E} |f(x)|$. $A_{\mathrm{u}}(B_E)$ is a uniformly closed subalgebra of $H^\infty(B_E)$. A *homomorphism* between Banach algebras is a continuous linear multiplicative map. By an *operator* we mean a continuous linear map from a Banach space into another Banach space. The space of all operators from E into F is denoted by $L(E,F)$. We denote the adjoint operator of $T \in L(E,F)$ by $T^{\mathrm{t}} : F' \to E'$. We say that $T \in L(E,F)$ is *(weakly) compact* if T maps bounded sets in E into relatively (weakly) compact sets in F. Equivalently, $T \in L(E,F)$ is weakly compact if and only if $T^{\mathrm{tt}}(E'') \subset F$. If $T \in L(E,E)$ the spectrum of T is denoted by $\sigma(T)$. Recall that if $T \in L(E,E)$ is compact and $0 \neq \lambda \in \sigma(T)$, then λ is an eigenvalue of both T and T^{t} [R, p. 109]. We also recall that a Banach space E is called a *Grothendieck space* if every weak* null sequence in E' is weakly null [Di, p. 121].

1. Compactness of homomorphisms. Suppose that $T : H^\infty(D) \to A(D)$ is an operator. For any bounded sequence (ϕ_j) in $A(D)'$, one can find a weak* convergent subsequence, since $A(D)$ is separable. The image of this subsequence is weak* convergent in $H^\infty(D)'$. Using J. Bourgain's result that $H^\infty(D)$ is a Grothendieck space ([B1]), it follows that T^{t} is weakly compact and therefore so is T. Further, J. Bourgain also showed that $H^\infty(D)$ has the Dunford–Pettis property [B2], and so the weakly compact operator T is in fact always completely continuous; that is, T maps weakly convergent sequences into norm convergent sequences. Is T always compact? The answer is no. Consider the natural inclusion map of l_1 into c_0. This map factors through $H^\infty(D)$ (see, e.g., [Di, p. 223]). Since c_0 has a complemented copy in $A(D)$, we obtain a non-compact operator from $H^\infty(D)$ into $A(D)$. On the other hand, it is an easy exercise using the Dunford–Pettis property of $H^\infty(D)$ that for any two operators $S, T : H^\infty(D) \to A(D) \subset H^\infty(D)$, the composition $S \circ T$ is compact.

Now, one can ask when a completely continuous operator T from a Banach space E into a Banach space F is compact. It is well known that this is the case if E does not contain a copy of l_1. For completeness we give a proof: Suppose that T is not compact. Then there is a bounded sequence (x_n) in E and $\varepsilon > 0$ such that $\|T(x_n) - T(x_m)\| \geq \varepsilon$ for all $m, n \in \mathbb{N}$ with $n \neq m$. By Rosenthal's l_1 theorem [Di, p. 201] there is a subsequence (x_{n_k}) of (x_n) which is weakly Cauchy. The sequence $(x_{n_{2k}} - x_{n_{2k-1}})$ is weakly null, so the complete continuity of T gives a contradiction. Recently, J. Cima and A. Matheson [CM, Prop. 2, Theorem 2] have studied completely continuous composition operators and obtained a somewhat weaker result. Notice also that $T^{\mathrm{t}} : A(D)' \to H^\infty(D)'$ is completely continuous since the dual of $A(D)$ has the Dunford–Pettis property [B2, p. 3].

After this introduction we are ready to formulate our problem:

Is every weakly compact homomorphism from $H^\infty(D)$ into $H^\infty(D)$ compact?

To study this problem we shall use results due to K. Hoffman [H], [G] concerning the analytic structure of $M(H^\infty)$, the set of all complex-valued homomorphisms of $H^\infty(D)$. When endowed with the weak* topology, $M(H^\infty)$ is a compact Hausdorff space. We may regard D as a subset of $M(H^\infty)$ by identifying $\lambda \in D$ with the evaluation homomorphism δ_λ. For $m, n \in M(H^\infty)$, the pseudo-hyperbolic distance is given by

$$\varrho(m,n) = \sup\{|\widehat{f}(n)| : f \in H^\infty(D),\ \|f\|_\infty \leq 1,\ \widehat{f}(m) = 0\},$$

where $\widehat{f}$ is the Gelfand transform of f. The norm topology induced by the dual space $H^\infty(D)'$ is the topology of the metric space $(M(H^\infty), \varrho)$. If $\lambda, \mu \in D$, then $\varrho(\lambda,\mu) = |(\lambda-\mu)/(1-\overline{\mu}\lambda)|$. For $m \in M(H^\infty)$, $P(m) =$

$\{n \in M(H^\infty) : \varrho(m,n) < 1\}$ is called the *Gleason part containing* m. If $P(m) \neq \{m\}$, the Gleason part is called *non-trivial*. In [H] K. Hoffman shows that for each $m \in M(H^\infty)$ there is a continuous map L_m from D onto $P(m)$ such that $\widehat{f} \circ L_m \in H^\infty(D)$ for every $f \in H^\infty(D)$ and $L_m(0) = m$. Moreover, he proves [H, pp. 103–105] that if $P(m)$ is non-trivial, then the map L_m preserves pseudo-hyperbolic distances: For $\lambda, \mu \in D$, $\varrho(\lambda,\mu) = \varrho(L_m(\lambda), L_m(\mu))$. Thus L_m is a homeomorphism into the metric topology. Further, by [H, pp. 90–91] there exists a sequence of Blaschke products $A_n \in H^\infty(D)$ such that for every $\lambda \in D$ we have $\widehat{A}_n(L_m(\lambda)) \to \lambda$ as $n \to \infty$. Since the sequence (A_n) is bounded, this shows that L_m is also a homeomorphism into the weak topology of $H^\infty(D)'$. This fact is very crucial for us.

Let $T : H^\infty(D) \to H^\infty(D)$ be a homomorphism. We will now show that if T is weakly compact or if T factors through $A(D)$, then T is compact. As we mentioned in the Introduction, this result has also been obtained by A. Ülger. However, there are differences in the proofs, although both essentially use the result that, unless the Gleason part P of the maximal ideal space of a logmodular algebra reduces to a singleton, there is a homeomorphism of D onto P, when P is given the metric topology or the weak topology. In the proof of Ülger it is shown that the weakly compact and the compact subsets of the maximal ideal space of a logmodular algebra are the same. We do not need such a result since in our specific case we apply Carleson's corona theorem which implies that the dual map of the given weakly compact homomorphism takes the maximal ideal space of $H^\infty(D)$ strictly inside one single Gleason part. Therefore the proof below contains some information that cannot directly be found in Ülger's proof.

THEOREM 1 (cf. [Ü]). *Let* $T : H^\infty(D) \to H^\infty(D)$ *be a homomorphism. Suppose that one of the following conditions holds*:

(i) T *is weakly compact*;
(ii) *The range of* T *is contained in* $A(D)$.

Then T *is compact.*

Proof. We have already observed that every homomorphism $T : H^\infty(D) \to A(D) \subset H^\infty(D)$ is weakly compact. Thus, it suffices to prove the result under assumption (i).

Now, the induced map $\phi_T : (M(H^\infty), w^*) \to (M(H^\infty), w^*)$ defined by $n \mapsto n \circ T$ is continuous and analytic on D. Hence, by Lemma 1.1 in [G, p. 402], there is an $m_0 \in \phi_T(D) \subset M(H^\infty)$ such that $\phi_T(D) \subset P(m_0)$. We only need to consider the case when $P(m_0)$ is non-trivial, so from now on we assume that this is the case.

CLAIM 1. $\phi_T(M(H^\infty)) \subset P(m_0)$.

Indeed, let $n_0 \in M(H^\infty) \setminus D$. We shall use Theorem VI.2.1 in [Ga], which states that two homomorphisms θ and ϕ are in the same Gleason part of $M(H^\infty)$ if and only if whenever (f_j) is a sequence in H^∞ such that $\|f_j\| \le 1$ and $|\widehat{f}_j(\theta)| \to 1$, then $|\widehat{f}_j(\phi)| \to 1$. Consider (f_j) in $H^\infty(D)$ with $\|f_j\| \le 1$ and $|\widehat{f}_j(\phi_T(\delta_\lambda))| \to 1$ for all $|\lambda| < 1$. If $|\widehat{f}_j(\phi_T(n_0))| \not\to 1$, then there exists a subsequence (f_{j_k}) of (f_j) such that $|\widehat{f}_{j_k}(\phi_T(n_0))| \le \beta < 1$ for some β and all k. Since T is weakly compact, there is another subsequence, which we still denote by (f_{j_k}), with $(T(f_{j_k}))$ converging weakly to some $f \in H^\infty(D)$. In particular, $n(T(f_{j_k})) \to n(f)$ for all $n \in M(H^\infty)$, so that $|\widehat{f}_{j_k}(\phi_T(n))| \to |n(f)|$ for all $n \in M(H^\infty)$. The corona theorem asserts that D is dense in $(M(H^\infty), w^*)$, and hence there exists a net $(\delta_{\lambda_\alpha})$, with $|\lambda_\alpha| < 1$, converging in the w^*-topology to n_0 as $\alpha \to \infty$. For all α, $|\delta_{\lambda_\alpha}(f)| = \lim_k |\widehat{f}_{j_k}(\phi_T(\delta_{\lambda_\alpha}))| = 1$ and, further, $|n_0(f)| = \lim_k |\widehat{f}_{j_k}(\phi_T(n_0))| \le \beta < 1$. Thus, $1 = \lim_\alpha |\delta_{\lambda_\alpha}(f)| = |n_0(f)| \le \beta < 1$, and we have obtained a contradiction.

CLAIM 2. $L_{m_0}(D_r)$ *is open in* $(P(m_0), w)$ *for all* $0 < r < 1$, *where* $D_r = \{\lambda \in D : |\lambda| < r\}$.

Indeed, equip $P(m_0)$ with the weak topology of the dual $H^\infty(D)'$, that is, $(P(m_0), w)$. The map $L_{m_0} : D \to (P(m_0), w)$ is a homeomorphism, so every $L_{m_0}(D_r)$ is open in the weak topology.

Now we are ready to finish the proof. We have $\phi_T(M(H^\infty)) \subset L_{m_0}(D)$ and by weak compactness of T we deduce that $\phi_T(M(H^\infty))$ is compact in the weak topology. Thus

$$\phi_T(M(H^\infty)) \subset \bigcup_{n=2}^{\infty} L_{m_0}(D_{1-1/n}),$$

from which it follows by Claim 2 that there is an integer N_0 such that $\phi_T(M(H^\infty)) \subset L_{m_0}(D_{1-1/N_0})$. This means that $\phi_T(M(H^\infty))$ is compact in the norm topology, that is, in $(P(m_0), \|\cdot\|)$. Therefore $\phi_T : (M(H^\infty), w^*) \to (P(m_0), \|\cdot\|)$ is continuous. In order to show that $T : H^\infty(D) \to H^\infty(D)$ is compact, we take a sequence (f_j) in $H^\infty(D)$ with $\|f_j\| \le 1$. According to the Arzelà–Ascoli theorem we must show that $\{\widehat{T(f_j)} : j \in \mathbb{N}\}$ is equicontinuous in $C(M(H^\infty))$ endowed with the uniform topology. By the continuity of ϕ_T we infer that, for every $\varepsilon > 0$ and for every $m_0 \in M(H^\infty)$, there is a $\delta > 0$ such that

$$|\widehat{T(f_j)}(m) - \widehat{T(f_j)}(m_0)| = |\phi_T(m)f_j - \phi_T(m_0)f_j| < \varepsilon$$

for all $m \in M(H^\infty)$ with $|m(g_i) - m_0(g_i)| < \delta$ for $g_i \in H^\infty(D)$, $i = 1, \ldots, k$, and all $j \in \mathbb{N}$. This completes the proof.

Our next aim is to show that Theorem 1 is not generally true for uniform algebras. To do so, we describe the relation between compactness and weak compactness of composition operators C_ϕ. This is done in a way that is also convenient for the characterization of the spectra of compact C_ϕ in the next section. But we start with some general facts about the uniform algebras $H^\infty(B_E)$ and $A_u(B_E)$.

Using a theorem of K. Ng [N], it follows that there is a Banach space $G^\infty(B_E)$ whose dual is isometrically isomorphic to $H^\infty(B_E)$. This fact has been pointed out by S. Dineen in his book [D] and developed by J. Mujica in [M1].

PROPOSITION 2. *$H^\infty(B_E)$ and $A_u(B_E)$ contain complemented copies of $H^\infty(D)$ and $A(D)$, respectively. Consequently, $A_u(B_E)$ contains a complemented copy of c_0, and $H^\infty(B_E)$ contains a complemented copy of l_∞ but not a complemented copy of c_0. Furthermore, $G^\infty(B_E)$ contains a complemented copy of l_1.*

Proof. Let $x_0 \in E$ with $\|x_0\| = 1$. Choose $l \in E'$ with $l(x_0) = 1 = \|l\|$, so that $|l(x)| < 1$ for all $x \in B_E$. Define $J : H^\infty(D) \to H^\infty(B_E)$ by $g \mapsto g \circ l$, and define $P : H^\infty(B_E) \to H^\infty(D)$ by $P(f)(\lambda) = f(\lambda x_0)$. Both J and P are well-defined continuous linear maps. For every $g \in H^\infty(D)$ and every $\lambda \in D$, $P(J(g))(\lambda) = g(\lambda l(x_0)) = g(\lambda)$, so $J(H^\infty(D))$ is a complemented subspace of $H^\infty(B_E)$. In a similar way one can show that there is a complemented copy of $A(D)$ in $A_u(B_E)$.

Recall next that F. Delbaen [De] has shown that $A(D)$ contains a copy of c_0. The rest of the assertions in Proposition 2 follow from results obtained by C. Bessaga and A. Pełczyński (cf. p. 48 in [Di]), by using the facts that c_0 has a complemented copy in $A_u(B_E)$, and $H^\infty(B_E)$ is the dual space of $G^\infty(B_E)$.

Remarks. (i) $H^\infty(B_E)$ contains a copy of l_1 and $A_u(B_E)$ is never a Grothendieck space.

(ii) If E and F are Banach spaces, then the Banach space $A_u(B_E, F)$ of all analytic maps from B_E into F which are uniformly continuous on B_E, endowed with the supremum norm, contains a complemented copy of c_0. This follows from Proposition 2 and the fact there is a complemented copy of $A_u(B_E)$ contained in $A_u(B_E, F)$. To see this elementary fact, take $y_0 \in F$ and $l \in F'$ with $\|l\| = l(y_0) = \|y_0\| = 1$. Define the continuous linear maps $J : A_u(B_E) \to A_u(B_E, F)$ and $P : A_u(B_E, F) \to A_u(B_E)$ by $J : f \mapsto (x \mapsto f(x)y_0)$ and $P : f \mapsto l \circ f$. Finally, for each $f \in A_u(B_E)$ and each $x \in B_E$, $P(J(f))x = l(f(x)y_0) = f(x)$.

Clearly, C_ϕ will act on $A_u(B_E)$ if $\phi : B_E \to B_E$ is analytic and *uniformly* continuous. Example 1 shows that when dealing with composition operators on $A_u(B_E)$, we have to require that ϕ be uniformly continuous in general, while in Example 2 we will see that in the special case of $E = c_0$, the assumption of uniform continuity is unnecessary.

EXAMPLE 1. Define an analytic map $\phi : B_{l_2} \to B_{l_2}$ by $x \mapsto (x_n^n)_n$. If $r = 1 - 1/n$ and $s = 1 - 2/n$, then $\|\phi(re_n) - \phi(se_n)\| = |r^n - s^n| \to 1/e$ as $n \to \infty$. From this we conclude that ϕ is not uniformly continuous. In addition, if $f \in A_u(B_{l_2})$ is defined by $f(x) = \sum x_n^2$, then $f \circ \phi \notin A_u(B_{l_2})$.

EXAMPLE 2. It is trivial that the analytic mapping $\phi : B_{c_0} \to B_{c_0}$ defined as in Example 1 is not uniformly continuous. On the other hand, $f \circ \phi \in A_u(B_{c_0})$ for every $f \in A_u(B_{c_0})$. To see this, observe that every such f is uniformly approximable on B_{c_0} by polynomials. By the Littlewood–Bogdanowicz–Pełczyński theorem (see, e.g., [ACG, p. 58]), every polynomial on c_0 is weakly uniformly continuous on B_{c_0}, and so for any $\varepsilon > 0$ there exist $\delta > 0$ and $u_1, \ldots, u_k \in l_1$ such that if $x, y \in B_{c_0}$ with $|u_i(x) - u_i(y)| < \delta$, $i = 1, \ldots, k$, then $|f(x) - f(y)| < \varepsilon$. Now, we can find a positive integer N and a constant $\mu > 0$ such that $|u_i(x - y)| < \delta$, $i = 1, \ldots, k$, provided x and y are in B_{c_0}, $|x_j - y_j| < \mu$, $j = 1, \ldots, N$. For $|x_j - y_j|$ so small that $|x_j^j - y_j^j| < \mu$ $(j = 1, \ldots, N)$, we get $|f(\phi(x)) - f(\phi(y))| < \varepsilon$. Thus $f \circ \phi \in A_u(B_{c_0})$ for all $f \in A_u(B_{c_0})$.

An easy argument using Cauchy's inequality and the Hahn–Banach theorem shows that if the analytic map $\phi : B_E \to B_E$ maps B_E into a relatively (weakly) compact set in E, then the differential operator $d\phi(x) : E \to E$ is (weakly) compact for every $x \in B_E$. Indeed, there is an absolutely convex zero-neighbourhood U in E such that $x + U \subset B_E$. Since $d\phi(x)(U)$ is a subset of the absolutely closed convex hull of $\phi(x + U)$, we are done. Following [EH], we say that the subset $\phi(B_E)$ of B_E *lies strictly inside* B_E if there exists $0 < \varepsilon < 1$ with $\|\phi(x)\| \le 1 - \varepsilon$ for all $x \in B_E$.

The following result was obtained independently by Manuel Maestre in the case of $A_u(E)$.

PROPOSITION 3. *Consider C_ϕ as a composition operator on $H^\infty(B_E)$ or $A_u(B_E)$. The following statements are equivalent:*

(1) *C_ϕ is compact;*
(2) *C_ϕ is weakly compact and $\phi(B_E)$ is relatively compact in E;*
(3) *$\phi(B_E)$ lies strictly inside B_E and $\phi(B_E)$ is relatively compact in E.*

Proof. We only give the proof for $H^\infty(B_E)$ since the proof for $A_u(B_E)$ is identical.

(1)⇒(2). If $\phi(B_E) \subset E$ is not relatively compact, then there exist a sequence $(x_n) \subset B_E$ and $\varepsilon > 0$ so that $\|\phi(x_n) - \phi(x_m)\| \ge \varepsilon$ for all $m \neq n$.

For each pair (m,n) with $m\neq n$, choose a unit vector $l_{mn}\in E'$ such that

$$|l_{mn}(\phi(x_n))-l_{mn}(\phi(x_m))|\geq\varepsilon.$$

By compactness of $C_\phi^{\mathrm t}$, the set $\{\delta_{\phi(x)}:x\in B_E\}=\{C_\phi^{\mathrm t}(\delta_x):x\in B_E\}$ is relatively compact in $H^\infty(B_E)'$, where δ_y is the evaluation (at y) map. But the above inequality yields $\|\delta_{\phi(x_n)}-\delta_{\phi(x_m)}\|\geq\varepsilon$, for $m\neq n$. Thus, we obtain a contradiction.

(2)⇒(3). Assume that there is a sequence $(x_j)\in B_E$ such that $\|\phi(x_j)\|>1-1/j$ for all j. Since $\phi(B_E)$ is relatively compact in E, we may assume that $(\phi(x_j))$ converges in norm to some y_0 with $\|y_0\|=1$. Choose $l\in E'$ with $l(y_0)=1=\|l\|$. Since C_ϕ is weakly compact and the set $\{l^n{}_{|B_E}:n\in\mathbb N\}$ is bounded in $H^\infty(B_E)$, we may assume without loss of generality that $(l^n\circ\phi)$ converges weakly to some $f\in H^\infty(B_E)$. For every j, the norm of the evaluation map $\delta_{x_j}\in H^\infty(B_E)'$ is 1, so by Alaoglu's theorem (δ_{x_j}) has a weak*-cluster point $u\in H^\infty(B_E)'$. Therefore, for every n we have $1=l(y_0)^n=\lim_j l(\phi(x_j))^n=\lim_j\delta_{x_j}(l^n\circ\phi)=u(l^n\circ\phi)$. Now, since $\|\phi(x_j)\|<1$ for all j, $0=\lim_n l(\phi(x_j))^n=f(x_j)=\delta_{x_j}(f)$. Hence, $0=\lim_j|\delta_{x_j}(f)|=|u(f)|=\lim_n|u(l^n\circ\phi)|=1$, and we have a contradiction. Actually, the function f must be identically 0.

(3)⇒(1). Suppose that $\phi(B_E)\subset E$ is relatively compact and $\phi(B_E)\subset rB_E$ for some $0<r<1$. If C_ϕ is not compact, then there exist $\varepsilon>0$ and a sequence $(f_n)\subset H^\infty(B_E)$ such that $\|f_n\|\leq1$ and $\|C_\phi(f_n)-C_\phi(f_m)\|\geq\varepsilon$ for all $n\neq m$. By Montel's theorem [C, p. 274] the set $\{f_n:n\in\mathbb N\}$ is relatively compact in $H(B_E)$ with respect to the compact open topology. Therefore (f_n) has a subnet, say (f_α), which converges uniformly on compact sets in B_E. For every α, choose $\beta>\alpha$ such that $f_\alpha\neq f_\beta$. Then

$$\sup_{y\in\phi(B_E)}|f_\alpha(y)-f_\beta(y)|=\sup_{\|x\|<1}|f_\alpha(\phi(x))-f_\beta(\phi(x))|=\|C_\phi(f_\alpha)-C_\phi(f_\beta)\|\geq\varepsilon.$$

Since $\phi(B_E)$ is relatively compact in B_E, we have a contradiction. Thus C_ϕ is compact, and we are done.

Examples of mappings ϕ which satisfy condition (3) in the above proposition abound. For example, any polynomial mapping $\phi:c_0\to l_p$, $1\leq p<\infty$, with norm strictly less than 1, works.

EXAMPLE 3. We now show that there is a weakly compact, non-compact composition operator on $H^\infty(B_E)$ when E is the Tsirelson space, thus answering negatively a question raised by A. Ülger in [Ü, §6]. We note in passing that there is no completely trivial counterexample to this question, i.e. there is no infinite-dimensional uniform (Banach) algebra which is reflexive [Go, p. 285].

Recall that the original Tsirelson space $E=T'$ is a reflexive space with an unconditional basis such that every continuous polynomial on E is automatically weakly uniformly continuous on weakly compact subsets of E (cf. [AAD]). Let $\phi(x)=x/2$. Since $\phi(B_E)$ is not relatively compact in E it follows from Proposition 3 that the composition operator $C_\phi:H^\infty(B_E)\to H^\infty(B_E)$ is non-compact. Notice that $C_\phi(f)\in A_{\rm wu}(B_E)$, the Banach algebra of analytic functions on B_E which are weakly uniformly continuous equipped with the natural norm. Indeed, we have

$$f(x/2)=\sum_{m=0}^\infty\frac{\widehat{d}^mf(0)}{m!}(x)/2^m$$

and this series is uniformly convergent in B_E since $\|\widehat{d}^mf(0)/m!\|\leq\|f\|$. Since every continuous polynomial on E is weakly uniformly continuous on B_E, the assertion follows.

Recall that any function in $A_{\rm wu}(B_E)$ has a continuous extension to the closed unit ball $\overline{B}_E$ endowed with the weak topology (also, $\overline{B}_E$ is a weakly compact set). Thus we may also consider $A_{\rm wu}(B_E)$ as a closed subalgebra of $C(\overline{B}_E)$, and hence the weak topology of $A_{\rm wu}(B_E)$ is the one induced by the weak topology of $C(\overline{B}_E)$.

We shall now prove that $C_\phi:H^\infty(B_E)\to A_{\rm wu}(B_E)$ is a weakly compact operator. First of all, observe that $C_\phi(f)(x)=f(x/2)$ even for $x\in\overline{B}_E$. Let $(g_n)\subset H^\infty(B_E)$ be a sequence with $\|g_n\|\leq1$, and let (x_m) be a norm dense subset of B_E. By a diagonal argument one can find a pointwise convergent subsequence (g_{n_k}) on (x_m). Since the family (g_{n_k}) is uniformly bounded, it is equicontinuous on B_E. Therefore, (g_{n_k}) converges pointwise on B_E. On the other hand, since $(g_{n_k})\subset H(B_E)$ is uniformly bounded, there is an analytic function g which is a cluster point of (g_{n_k}) for the compact-open topology of $H(B_E)$. Necessarily, $g\in H^\infty(B_E)$ and $g(x)=\lim_k g_{n_k}(x)$, $x\in B_E$.

Finally, $(C_\phi(g_{n_k}))$ converges weakly in $C(\overline{B}_E)$ (hence in $A_{\rm wu}(B_E)$) to $C_\phi(g)$ because it is uniformly bounded and $\lim_k C_\phi(g_{n_k})(x)=\lim_k g_{n_k}(x/2)=g(x/2)=C_\phi(g)(x)$ ([Di], Theorem 1, p. 66).

2. Spectra of compact composition operators. In order to be able to determine the spectrum of C_ϕ we need a fixed point theorem for analytic maps from B_E into B_E when E is an infinite-dimensional Banach space. T. Hayden and T. Suffridge [HS] pointed out that analyticity of ϕ is not sufficient, by considering the analytic map $\phi:B_{c_0}\to B_{c_0}$ given by $\phi(\xi_1,\xi_2,\ldots)=\left(\frac12,\xi_1,\xi_2,\ldots\right)$. The map ϕ has no fixed point, since the only possible fixed point of ϕ is $\left(\frac12,\frac12,\ldots\right)$, which is not in B_{c_0}. In [W] K. Włodarczyk has considered the problem of existence of a unique fixed point in B_E of ϕ when $\phi(B_E)$ does not lie strictly inside B_E. He showed that this problem is solved whenever (i) $\phi(B_E)$ is relatively compact in E; (ii) ϕ extends continuously to $\overline{B}_E$ and $\phi(\overline{B}_E) \subset \overline{B}_E$; (iii) ϕ has no fixed point on the boundary of B_E; and (iv) for each $x \in B_E$, $1 \notin \sigma(d\phi(x))$.

If $\phi(B_E)$ lies strictly inside B_E, C. Earle and R. Hamilton [EH] completely solved the existence and uniqueness of a fixed point of ϕ by proving the following remarkable result:

EARLE–HAMILTON FIXED POINT THEOREM. *If $\phi : B_E \to B_E$ is an analytic mapping and $\phi(B_E)$ lies strictly inside B_E, then ϕ has a unique fixed point z_0 in B_E.*

Now, when we know how to get a unique fixed point for ϕ, the characterization of the spectra of compact C_ϕ is not hard. The proof is based on the following two lemmata. In their proofs we only deal with $H^\infty(B_E)$ but everything works as well for $A_u(B_E)$.

LEMMA 4. *If C_ϕ is compact, then $\{0,1\} \subset \sigma(C_\phi)$ and $\{\prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(d\phi(z_0)),\ i = 1,\ldots,n \text{ and } n \in \mathbb{Z}^+\} \subset \sigma(C_\phi)$.*

Proof. Clearly $0 \in \sigma(C_\phi)$, as otherwise $H^\infty(B_E)$ would be finite-dimensional since C_ϕ is compact. The function $e(x) = 1$ belongs to $H^\infty(B_E)$. There is no $f \in H^\infty(B_E)$ with $f(x) - f(\phi(x)) = e(x)$, since for $x = z_0$ we get $0 = 1$. Thus $1 \in \sigma(C_\phi)$. Take $0 \neq \lambda \in \sigma(d\phi(z_0))$. Since $d\phi(z_0)$ is a compact operator, λ is an eigenvalue of $d\phi(z_0)$. Thus there is $0 \neq x_0 \in E$ with $d\phi(z_0)x_0 = \lambda x_0$. Choose $l \in E'$ such that $l(x_0) \neq 0$. Suppose that there exists $f \in H^\infty(B_E)$ with

$$\lambda f(x) - f(\phi(x)) = l(x).$$

Differentiation gives $\lambda df(x) - df(\phi(x)) \circ d\phi(x) = dl(x) = l$. For $x = z_0$ we get $\lambda df(z_0) - df(z_0) \circ d\phi(z_0) = l$ and consequently $0 = \lambda df(z_0)x_0 - df(z_0)(d\phi(z_0)x_0) = l(x_0) \neq 0$, which is a contradiction. Thus $\lambda \in \sigma(C_\phi)$. Suppose now that $\lambda_1,\ldots,\lambda_n$ are non-zero eigenvalues of $d\phi(z_0)$, and hence they are eigenvalues of C_ϕ. Let $f_1,\ldots,f_n$ be corresponding non-zero eigenfunctions in $H^\infty(B_E)$. Hence

$$\lambda_i f_i(x) = f_i(\phi(x)), \quad 1 \le i \le n.$$

Let $\mu := \prod_{i=1}^n \lambda_i$ and $g(x) := f_1(x)\ldots f_n(x)$. Then $g \in H^\infty(B_E)$ and, since B_E is connected, $g \not\equiv 0$. Now

$$\mu g(x) = f_1(\phi(x))\ldots f_n(\phi(x)) = g(\phi(x)).$$

Thus $\mu \in \sigma(C_\phi)$, and we are done.

For the proof of our next lemma we need the following result due to M. Schechter [S] (see also A. Brown and C. Pearcy [BP]) concerning the spectrum of tensor products of n operators on complex Banach spaces:

Let E_i, $i = 1,\ldots,n$, be complex Banach spaces and let $T_i \in L(E_i, E_i)$. Then $\sigma(T_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi T_n) = \prod_{i=1}^n \sigma(T_i)$.

Actually, we only need the fact that $\sigma(T_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi T_n) \subset \prod_{i=1}^n \sigma(T_i)$, which follows from a standard application of the following result (see [R, p. 293]): Let A be a Banach algebra with unit, $x, y \in A$ and $xy = yx$. Then $\sigma(xy) \subset \sigma(x)\sigma(y)$.

LEMMA 5. *Assume that $\phi : B_E \to B_E$ is analytic and maps B_E strictly inside B_E. If $\mu \neq 0$ is an eigenvalue of C_ϕ, then $\mu \in \{\prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(d\phi(z_0)),\ i = 1,\ldots,n \text{ and } n \in \mathbb{Z}^+\} \cup \{1\}$.*

Proof. Let $f \in H^\infty(B_E)$ be an eigenfunction corresponding to μ, so that

$$\mu f(x) = f(\phi(x)).$$

Suppose that $\mu \neq 1$ and is not a product of elements in the spectrum of $d\phi(z_0)$. Our aim is to show that $f \equiv 0$. In some neighbourhood of z_0 in B_E we have the uniformly convergent Taylor series of f around z_0:

$$f(x) = \sum_{m=0}^\infty \frac{d^m f(z_0)}{m!}(x - z_0)^m.$$

Thus we must show that $d^m f(z_0) \equiv 0$ for $m = 0,1,2,\ldots$ For $x = z_0$, $\mu f(z_0) = f(z_0)$ so $f(z_0) = 0$ as $\mu \neq 1$. Assume now that $d^m f(z_0) \equiv 0$ for $m < n$. Thus

$$f(x) = \frac{d^n f(z_0)}{n!}(x - z_0)^n + \sum_{m=n+1}^\infty \frac{d^m f(z_0)}{m!}(x - z_0)^m.$$

Since

$$\phi(x) = z_0 + d\phi(z_0)(x - z_0) + \sum_{m=2}^\infty \frac{d^m \phi(z_0)}{m!}(x - z_0)^m$$

converges uniformly in a neighbourhood of z_0, it follows from $\mu f(x) = f(\phi(x))$ by comparing the terms of $(x - z_0)^n$ that

$$\mu \overline{d^n f(z_0)} = \overline{d^n f(z_0)} \circ (d\phi(z_0) \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi d\phi(z_0)),$$

where we have used the isometric isomorphism between $L_s(^nE) \simeq (\widehat{\otimes}_{n,s,\pi} E)'$, which associates $A \in L_s(^nE)$ to $\overline{A} \in (\widehat{\otimes}_{n,s,\pi} E)'$. Thus we have

$$\mu \overline{d^n f(z_0)} = (d\phi(z_0) \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi d\phi(z_0))^{\mathrm{t}}\, \overline{d^n f(z_0)}.$$

As is well known $\sigma(d\phi(z_0) \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi d\phi(z_0)) = \sigma((d\phi(z_0) \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi d\phi(z_0))^{\mathrm{t}})$. If $d^n f(z_0) \not\equiv 0$, this means that $\mu \in \sigma(d\phi(z_0) \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi d\phi(z_0))$. In view of the above-mentioned result of M. Schechter this would imply that $\mu = \prod_{i=1}^n \lambda_i$, where all $\lambda_i \in \sigma(d\phi(z_0))$. But this is a contradiction, so that $d^n f(z_0) \equiv 0$ and hence $f \equiv 0$.

From Lemmata 4 and 5 we obtain

THEOREM 6. *If C_ϕ is compact and z_0 is the unique fixed point of ϕ, then*

$$\sigma(C_\phi) = \Big\{ \prod_{i=1}^{n} \lambda_i : \lambda_i \in \sigma(d\phi(z_0)), i = 1, \ldots, n \text{ and } n \in \mathbb{Z}^+ \Big\} \cup \{0,1\}.$$

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