A Phragmén–Lindelöf type quasi-analyticity principle

by

GRZEGORZ ŁYSIK (Warszawa)

Abstract. Quasi-analyticity theorems of Phragmén–Lindelöf type for holomorphic functions of exponential type on a half plane are stated and proved. Spaces of Laplace distributions (ultradistributions) on $\mathbb{R}$ are studied and their boundary value representation is given. A generalization of the Painlevé theorem is proved.

1. Introduction and statement of the main results. The well-known Phragmén–Lindelöf theorem consists of two parts. The first one ([H]), called the maximum principle, says that a function holomorphic and of exponential type on a sector $S$ of opening less than $\pi$ is bounded if it is bounded on the boundary of $S$. The second one ([T]), called the quasi-analyticity principle, says that a holomorphic function $F$ on a sector $S$ vanishes if the opening of $S$ is greater than $\pi$ and $F$ is exponentially decreasing in $S$.

In the present paper we study the quasi-analyticity principle in the critical case of a half plane $\Pi$. To ensure vanishing of $F$ in that case we assume that $F$ is of exponential type in $\Pi$ and decreases exponentially along the boundary of $\Pi$. More precisely, we have

**Theorem 1** (Quasi-analyticity principle, continuous version). Let $F \in \mathcal{O}(\{\text{Re } z > 0\}) \cap C^\infty(\{\text{Re } z \geq 0\})$ be of exponential type, i.e.

(1) $|F(z)| \leq Ce^{\|z\|}$ for $\text{Re } z \geq 0$ with some $C < \infty$ and $c < \infty$.

If

(2) $|F(\pm ir)| \leq Ce^{\pm r}$ for $r \geq 0$

with some $c^\pm \in \mathbb{R}$ such that $c^+ + c^- < 0$ then $F \equiv 0$.

The elementary proof of Theorem 1 is based on the Laplace integral representation of holomorphic functions of exponential type.

1991 Mathematics Subject Classification: 30D15, 44A15, 46F12, 46F20.

*Key words and phrases:* quasi-analyticity, Laplace distributions, Laplace ultradistributions, boundary values.

Partially supported by KBN grant No 2 PO3A 006 08.
Next we give the distributional version of Theorem 1. In that case it is more convenient to assume that $F$ is holomorphic in the upper half plane $\{\Im z > 0\}$ and of exponential type. The condition (2) is replaced by the assumption that the boundary value $b(F)$ of $F$ is a Laplace distribution on $\mathbb{R}$ (see Section 3). More precisely, we have

Theorem 2 (Quasi-analyticity principle, distributional version). Let $F \in \mathcal{O}(\{\Im z > 0\})$ be of exponential type in $\{\Im z \geq \varepsilon\}$ for all $\varepsilon > 0$. If $b(F) \in L^{(0,\omega)}(\mathbb{R})$ with some $\nu, \omega \in \mathbb{R}$ then for every $a > \nu$ and $b < \omega$ there exist $0 < R' < R$ and $k \in \mathbb{N}$ such that

$$|H(z)| \leq \begin{cases} Ce^{-\alpha Rz}/(\Im z)^{\frac{k}{2}} & \text{for } Re z \leq 0, 0 < \Im z \leq R', \\ Ce^{-\beta Rz}/(\Im z)^{\frac{k}{2}} & \text{for } Re z \geq 0, 0 < \Im z \leq R'. \end{cases}$$

In the proof of Theorem 3 we follow an idea presented by Z. Szmyd and B. Ziemian in [SZ].

In Section 4 we give the ultradistributional versions of Theorems 2 and 3; namely

Theorem 4 (Quasianalyticity principle, ultradistributional version). Let $(M_p)$ be a sequence of positive numbers satisfying the conditions (M.1) and (M.3) (see Section 4). Let $F \in \mathcal{O}(\{\Im z > 0\})$ be of exponential type in $\{\Im z \geq \varepsilon\}$ for all $\varepsilon > 0$. If $b(F) \in L^{(M_p)}(\mathbb{R})$ with some $\nu < \omega$ then $F \equiv 0$.

Theorem 5. Let $(M_p)$ be a sequence of positive numbers satisfying the conditions (M.1), (M.2) and (M.3) (see Section 4). Let $F \in \mathcal{O}(\{0 < \Im z < R\})$ with some $R > 0$. If $b(F) \in L^{(M_p)}(\mathbb{R})$ with some $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$ then for every $a > \nu$ and $b < \omega$ there exist $0 < R' < R$ and $L < \infty$ such that

$$|H(z)| \leq \begin{cases} C\exp\{-a Rz + M^*(L/\Im z)\} & \text{for } Re z \leq 0, 0 < \Im z \leq R', \\ C\exp\{-b Rz + M^*(L/\Im z)\} & \text{for } Re z \geq 0, 0 < \Im z \leq R'. \end{cases}$$

In the above $M^*$ is the growth function of the sequence $(M_p)$ defined by

$$M^*(\varphi) = \sup_{p \in N_\omega} \frac{M_p \varphi^p}{M_p} \quad \text{for } \varphi > 0.$$

One can try to prove Theorem 5 by the method used in the proof of Theorem 3, replacing smooth functions by ultradifferentiable ones. In this way we get a slightly worse estimate than (4) (see Theorem 8), but still good enough to deduce Theorem 4 and only under the assumptions (M.1) and (M.3') on the sequence $(M_p)$. To get the precise estimate (4) we represent the space of Laplace ultradistributions on $\mathbb{R}$ as a suitable quotient space of holomorphic functions on a tubular neighbourhood of $\mathbb{R}$ with some growth restrictions (see Theorem 9).

Theorems 1, 2 and 4 allow us to prove, in the final section, some generalizations of the Painlevé theorem.

2. Proof of the continuous version. Let $F \in \mathcal{O}(\{Re z > 0\}) \cap \mathcal{C}^0(\{Re z \geq 0\})$ satisfy (1) and (2). Let

$$\Psi_0(\zeta) = \int_0^\infty F(x)e^{\zeta x} \, dx, \quad Re \zeta < -c,$$

be the Laplace transform of $F|_{\mathbb{R}_+}$. By the uniqueness theorem for the Laplace transformation ([W], Theorem 6.3) it is sufficient to show that $\Psi_0 \equiv 0$. To this end define for $|\varphi| \leq \pi/2$,

$$\Psi_\varphi(\zeta) = \int_0^\infty F(x)e^{\zeta x} \, dx$$

for $\zeta \in \Omega_\varphi$,

where $l_\varphi = \{z \in \mathbb{C} : z = r e^{i\varphi}, 0 \leq r < \infty\}$ and $\Omega_\varphi$ is the set of all $\zeta \in \mathbb{C}$ such that

$$\{c + \Re \zeta \cos \varphi - \Im \zeta \sin \varphi < 0 \quad \text{if } |\varphi| < \pi/2, \quad \pm \Im \zeta > c \pm \epsilon \quad \text{if } \varphi = \pm \pi/2.$$
and
\[
\kappa_{a,b}(x) = \begin{cases} 
  e^{-ax} & \text{for } x \leq 0, \\
  e^{-bx} & \text{for } x \geq 0.
\end{cases}
\]

The spaces \( L_{(\nu, \delta)}(\mathbb{R}_-) \) and \( L_{(\omega, \theta)}(\mathbb{R}_+) \) of Laplace distributions on \( \mathbb{R}_- \) and \( \mathbb{R}_+ \) are defined in an analogous way replacing \( L_{a,b,m}(\mathbb{R}) \) respectively by
\[
L_{a,b,m}(\mathbb{R}_-) = \{ \varphi \in C^m(\mathbb{R}_-) : \sup_{\alpha \leq m} \sup_{x \in \mathbb{R}_-} |D^\alpha \varphi(x)| e^{-ax} < \infty \}
\]
and
\[
L_{\theta,b,m}(\mathbb{R}_+) = \{ \varphi \in C^m(\mathbb{R}_+) : \sup_{\alpha \leq m} \sup_{x \in \mathbb{R}_+} |D^\alpha \varphi(x)| e^{-bx} < \infty \}.
\]

We have the topological inclusions
\[
L_{(\nu, \delta)}(\mathbb{R}_-) \hookrightarrow L_{(\nu, \omega)}(\mathbb{R}) \quad \text{for any } \omega \in \mathbb{R} \cup \{\infty\},
\]
\[
L_{(\theta, \omega)}(\mathbb{R}_+) \hookrightarrow L_{(\theta, \omega)}(\mathbb{R}) \quad \text{for any } \nu \in \mathbb{R} \cup \{-\infty\}.
\]

Since \( C_0^\infty(\mathbb{R}) \) is dense in \( L_{(\nu, \omega)}(\mathbb{R}) \) we also have the topological inclusion
\[
L_{(\nu, \omega)}(\mathbb{R}) \hookrightarrow D'(\mathbb{R}).
\]
The following theorem characterizes the image of \( L_{(\nu, \omega)}(\mathbb{R}) \) under the above inclusion.

**Theorem 6 (Structure theorem).** In order that a distribution \( S \in D'(\mathbb{R}) \) belong to \( L_{(\nu, \omega)}(\mathbb{R}) \), it is necessary and sufficient that for any \( a > \nu \) and \( b < \omega \) there are differential operators \( P_a(D) \) and \( P_b(D) \) and functions \( S_a, S_b \in C^0(\mathbb{R}) \) such that \( \text{supp } S_a \subseteq \mathbb{R}_-, \quad |S_a(x)| \leq Ce^{-ax} \), \( \text{supp } S_b \subseteq \mathbb{R}_+, \quad |S_b(x)| \leq Ce^{-bx} \) for \( x \geq 0 \) and
\[
S = P_a(D)S_a + P_b(D)S_b \quad \text{in } L_{(a,b)}(\mathbb{R}).
\]

The proof of Theorem 6 follows easily from the structure theorem for the Laplace distributions on \( \mathbb{R}_- \) and \( \mathbb{R}_+ \) (cf. [L1, Theorem 2]), and

**Lemma 1.** Let \( S \in L_{(\nu, \omega)}(\mathbb{R}) \). Then one can find \( S^- \in L_{(\nu, \delta)}(\mathbb{R}_-) \) and \( S^+ \in L_{(\theta, \omega)}(\mathbb{R}_+) \) such that \( S = S^- + S^+ \in L_{(\nu, \omega)}(\mathbb{R}) \). The decomposition is unique modulo a distribution with support at zero.

**Proof.** Take a cut-off function \( \chi \in D(\mathbb{R}) \) equal to one in a neighbourhood of zero. Then \( \chi S \in E'(\mathbb{R}) \) and so one can find a differential operator \( J(D) \) and a function \( g \in C^0(\mathbb{R}) \) such that \( \chi S = J(D)g \). Put \( g^+ = g \cdot H \), \( g^- = g \cdot (1-H) \), \( H \) the Heaviside function, \( g^+ = \nu g^+ \) and \( S^+ = J(D)g^+ + (1-\chi)H \cdot S \), \( S^- = J(D)g^- + (1-\chi)H \cdot S \). Then \( S^+ \) and \( S^- \) satisfy the conclusion of Lemma 1.

Let \( \delta, \sigma \in \mathbb{R} \). Define
\[
\chi_{\delta,\sigma}(z) = \frac{1}{\pi}(e^{-\delta z} + e^{\sigma z}) \quad \text{for } z \in \mathbb{C}.
\]

Note that the function \( \chi_{\delta,\sigma}(z) \) does not vanish on a tubular neighbourhood of \( \mathbb{R} \). If \( \delta + \sigma \geq 0 \) then multiplication by \( \chi_{\delta,\sigma}(z) \) gives an isomorphism of \( L_{a,b}(\mathbb{R}) \) onto \( L_{a,b-\delta,\sigma}(\mathbb{R}) \) and consequently an isomorphism of \( L_{(\nu, \delta, \omega, \sigma)}(\mathbb{R}) \), of \( L_{(\omega, \theta)}(\mathbb{R}) \) onto \( L_{(1+b, \nu, \delta, \omega, \sigma)}(\mathbb{R}) \) with the inverse being multiplication by \( 1/\chi_{\delta,\sigma} \).

The next two lemmas will be useful in the proof of Theorem 3.

**Lemma 2.** Let \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp } \varphi \subseteq [-1,1] \) and let \( a, b \in \mathbb{R} \) and \( l \in \mathbb{N}_0 \). Then for any \( \frac{\nu}{r} \in \mathbb{R} \) and \( 0 < r < 1 \),
\[
\left\| \varphi \left( \frac{\nu - \frac{\nu}{r}}{r} \right) \right\|_{a,b,l} \leq C_{a,b} \left( \frac{l}{r} \right)^{\kappa_{a,b} \left( \frac{\nu}{r} \right)} \| \varphi \|_{a,b,l},
\]
where \( C_{a,b} = \max\{e^{a|l|}, e^{b|l|}, e^{a-b|l|}\} \).

**Proof.** Assume that \( \frac{\nu}{r} \leq -1 \). Since \( \text{supp } \varphi \subseteq [-1,1] \) we derive
\[
\left\| \varphi \left( \frac{\nu - \frac{\nu}{r}}{r} \right) \right\|_{a,b,l} = \sup_{0 \leq a \leq l} \sup_{x \in \mathbb{R}_-} |D^a \varphi \left( \frac{\xi - \frac{\nu}{r}}{r} \right) e^{-ax} |
\leq \frac{1}{r} \sup_{0 \leq a \leq l} \sup_{x \in \mathbb{R}_-} |D^a \varphi (x - \frac{\nu}{r})| e^{-a(r(x - \frac{\nu}{r}) + \frac{\nu}{r})} = \frac{1}{r} \max \left( \sup_{0 \leq a \leq l} \sup_{x \in \mathbb{R}_-} |D^a \varphi (x - \frac{\nu}{r})| e^{-a(r(x - \frac{\nu}{r}) + \frac{\nu}{r})}, \sup_{0 \leq a \leq l} \sup_{x \in \mathbb{R}_-} |D^a \varphi (x - \frac{\nu}{r})| e^{-b(x - \frac{\nu}{r})} \right).
\]

The other cases \(-1 \leq \frac{\nu}{r} \leq 0, 0 \leq \frac{\nu}{r} \leq 1 \) and \( \frac{\nu}{r} \geq 1 \) can be treated in the same way.

**Lemma 3.** Let \( g \in C_0^\infty(\mathbb{C}) \) be a radial function such that \( \text{supp } g \subseteq \{ z \in \mathbb{C} : |z| < 1 \} \) and \( g(x,y) \overline{dx dy} = (2\pi)^{-1} \). Then for any \( F \in \mathcal{O}(\{ z \in \mathbb{C} : |z| < 1 \}) \),
\[
\int g(x+i\gamma) F(x+i\gamma) dx dy = F(0).
\]

**Definition.** Let \( H \in \mathcal{O}(\{ \theta < \text{Im } z < R \}) \) for some \( R > 0 \) and let \( \nu \in \mathbb{R} \cup \{-\infty\} \) and \( \omega \in \mathbb{R} \cup \{\infty\} \). Assume that for any \( a > \nu \) and \( b < \omega \) there exists \( R' > 0 \) such that \( H \in L_{(a,b)}(\mathbb{R}) \) for \( 0 < y < R' \), where
\[ H_y[\varphi] := \int_\mathbb{R} H(x + iy) \varphi(x) \, dx \quad \text{for} \ \varphi \in L_{a,b}(\mathbb{R}). \]

If for every \( \varphi \in L_{(\nu,\infty)}(\mathbb{R}) \) the limit \( \lim_{y \to 0} H_y[\varphi] =: u[\varphi] \) exists then, by the Banach–Steinhaus theorem, \( u \in L'_{(\nu,\infty)}(\mathbb{R}) \) and we say that \( H \) has boundary value \( u = b(H) \) in \( L'_{(\nu,\infty)}(\mathbb{R}) \).

**Proof of Theorem 3.** Suppose \( H \in \mathcal{O}([0 < \Im z < R]) \) has a boundary value \( u = b(H) \in L'_{(\nu,\infty)}(\mathbb{R}) \). Fix \( a > \nu \) and \( b < \infty \). By definition \( H_y \in L_{a,b}(\mathbb{R}) \) for \( 0 < y \leq R' \) with some \( R' > 0 \) and the limit \( \lim_{y \to 0} H_y = u \in L_{0,\nu}(\mathbb{R}) \) exists. So one can find \( \tilde{C} < \infty \) and \( l \in \mathbb{N}_0 \) such that

\[ |H_y[\varphi]|, |u[\varphi]| \leq \tilde{C} \|\varphi\|_{a,b,l} \quad \text{for} \ \varphi \in L_{a,b,l}(\mathbb{R}) \text{ and } 0 < y \leq R'. \]

Take \( 0 < R' < \min(1, R'') \) and \( 0 < c < \min(1, R'/R' - 1) \). Fix \( \tilde{z} \in \mathbb{R} \) and \( \tilde{y} \in (0, R') \) and set \( \tilde{\zeta} = \tilde{z} + i \tilde{y} \). Then \( H \in \mathcal{O}(\mathbb{R} + i[y : |y - \tilde{y}| \leq c \tilde{y}]) \). Let \( \psi \in C_c^\infty(\{x : |x - \tilde{z}| \leq c \tilde{y}\}) \) and define \( \mu(x) = \psi(x + \tilde{z}) \) and \( g(x) = H(x + \tilde{z}) \). Note that \( \mu \in C_c^\infty(\{|x| \leq c \tilde{y}\}) \), \( g \in \mathcal{O}(\mathbb{R} + i[|y| \leq c \tilde{y}]) \) and \( H(\tilde{z}) = g(0) \). So if we take \( \psi \) with \( \|\psi\|_{a,b,l} \leq 1/\tilde{C} \) then by (7),

\[ \left| \int g(x + iy) \mu(x) \, dx \right| = \left| \int H(x + i(y + \tilde{y})) \psi(x + \tilde{z}) \, dx \right| = \left| H_{y + \tilde{y}}[\varphi] \right| \leq 1 \quad \text{for } \|\psi\|_{a,b,l} \leq 1/\tilde{C}. \]

Define \( g_{\tilde{y}}(x + iy) = g(c \tilde{y} x + ic \tilde{y} y) \). Then \( g_{\tilde{y}} \in \mathcal{O}(\mathbb{R} + i[|y| \leq 1]) \) and by Lemma 3 we get

\[ H(\tilde{z}) = g_{\tilde{y}}(0) = \int g(x + iy) g(c \tilde{y} x + ic \tilde{y} y) \, dx \, dy \]

\[ = \frac{1}{(c \tilde{y})^2} \int g \left( \frac{\xi}{c \tilde{y}} + i \frac{\eta}{c \tilde{y}} \right) g(\xi + i \eta) \, d\xi \, d\eta, \]

where \( g \) is given in Lemma 3. Now fix \( \eta \) such that \( |\eta| \leq c \tilde{y} \). We shall apply (8) to the function \( \xi \to \sigma(\xi, \eta) \), where

\[ \sigma(\xi, \eta) = \frac{(c \tilde{y})^2}{M_1 C_{a,b}} \left( \frac{\xi}{c \tilde{y}} + i \frac{\eta}{c \tilde{y}} \right) \kappa_{-a,-b}(\tilde{z}) \]

with \( M_1 = \sup_{|\xi| \leq 1} \|g(\cdot + iy)\|_{a,b,l} \) and \( C_{a,b} \) given in Lemma 2. Then \( \|\sigma(\cdot, \eta)\|_{a,b,l} \leq 1/\tilde{C} \). So by (9) and (8) with \( \tilde{C} < \infty \) we derive

\[ |H(\tilde{z})| = \frac{1}{(c \tilde{y})^2} \left| \int g \left( \frac{\xi}{c \tilde{y}} + i \frac{\eta}{c \tilde{y}} \right) g(\xi + i \eta) \, d\xi \, d\eta \right| \]

\[ \leq \frac{M_1 C_{a,b}}{(c \tilde{y})^{2+l}} \kappa_{a,b}(\tilde{z}) \left| \int d\eta \int g(\xi + i \eta) \sigma(\xi, \eta) \, d\xi \right| \]

\[ \leq \frac{M_1 C_{a,b}}{(c \tilde{y})^{2+l}} \kappa_{a,b}(\tilde{z}) \int \frac{d\eta}{|\eta|_{\leq c \tilde{y}}} \leq C \frac{\kappa_{a,b}(\tilde{z})}{c \tilde{y}^l} \]

with \( k = l + 1 \).

**Proof of Theorem 2.** Let \( F \in \mathcal{O}((\Im z > 0)) \) be of exponential type in \( (\Im z > 0) \) and have a boundary value \( b(F) \in L'_{(\nu,\infty)}(\mathbb{R}) \) with some \( \nu < \infty \). Fix \( \nu < a < b < \infty \). By Theorem 3 we can find \( R > 0 \) and \( k \in \mathbb{N} \) such that

\[ |P(z)| \leq \begin{cases} Ce^{-a Re z} \left/ (\Im z)^k \right. & \text{if } \Re z \leq 0, 0 < \Im z \leq R, \\ Ce^{-b Re z} \left/ (\Im z)^k \right. & \text{if } \Re z \geq 0, 0 < \Im z \leq R, \\ Ce^{|z|} & \text{if } \Im z \geq R. \end{cases} \]

Thus, the function \( H(z) = F(i(z + R)), \Re z \geq 0 \), satisfies the assumptions of Theorem 1 with \( c^+ = a \) and \( c^- = b \). Since \( c^+ + c^- = a - b < 0 \) we get \( H \equiv 0 \) and consequently \( F \equiv 0 \).

4. **Laplace ultradistributions on \( \mathbb{R} \).** Let \( (M_p)_{p \in \mathbb{N}_0} \) be a sequence of positive numbers. We consider the conditions

\[ \begin{align*} 
&M_0^p \leq M_{p-1} M_{p+1} \quad \text{for } p \in \mathbb{N}; \\
&M_{2p} \leq H^p M_{p-1} \quad \text{for } p \in \mathbb{N} \text{ with some } H < \infty; \\
&M_p \leq H^p M_q M_{p-q} \quad \text{for } p, q \in \mathbb{N}, 0 < q \leq l \text{ with some } H < \infty; \\
&M_p \to 0 \quad \text{as } p \to \infty; \\
&M_p \to Aq \quad \text{as } q \to \infty. 
\end{align*} \]

We refer to [K] or [M] for the significance of these conditions. We always assume (M.1), (M.3') and \( M_0 = 1 \).

Let \( \nu, \omega \in \mathbb{R} \cup \{-\infty\} \) and \( \omega \in \mathbb{R} \cup \{\infty\} \). The space \( L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) \) of Laplace ultradistributions on \( \mathbb{R} \) is defined as the dual space of

\[ L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) = \lim_{a\to\nu, b\to\omega} L^{(M_p)}_{a,b}(\mathbb{R}) \]
where for any \( a, b \in \mathbb{R} \),

\[
L^{(M_p)}_{a,b,h}(\mathbb{R}) = \lim_{h \to 0} L^{(M_p)}_{a,b,h}(\mathbb{R})
\]

with

\[
L^{(M_p)}_{a,b,h}(\mathbb{R}) = \left\{ \varphi \in C^\infty(\mathbb{R}) : \|\varphi\|_{a,b,h} = \sup_{x \in \mathbb{R}} \sup_{0 < \alpha < h} \frac{|D^\alpha \varphi(x)|}{\alpha^\alpha M_n} < \infty \right\}
\]

and \( \kappa_{a,b} \) given by (5).

Multiplication by the function \( \chi_{\delta,\sigma} \) with \( \delta + \sigma \geq 0 \) gives an isomorphism of \( L^{(M_p)}_{\alpha}(\mathbb{R}) \) onto \( L^{(M_p)}_{\alpha + \delta,\sigma}(\mathbb{R}) \) and consequently an isomorphism of \( L^{(M_p)}_{\nu}(\mathbb{R}) \) onto \( L^{(M_p)}_{\nu - \delta,\sigma}(\mathbb{R}) \), and of \( L^{(M_p)}_{\nu}(\mathbb{R}) \) onto \( L^{(M_p)}_{\nu + \delta,\sigma}(\mathbb{R}) \) with the inverse being multiplication by \( 1/\chi_{\delta,\sigma} \).

As in the case of Laplace distributions we can also define the spaces \( L^{(M_p)}_{\nu}(\mathbb{R}+) \) and \( L^{(M_p)}_{\nu}(\mathbb{R}-) \) of Laplace ultradistributions on \( \mathbb{R}+ \) and \( \mathbb{R}- \).

We have the topological inclusions

\[
L^{(M_p)}_{\nu}(\mathbb{R}-) \hookrightarrow L^{(M_p)}_{\nu}(\mathbb{R}) \quad \text{for any } \nu \in \mathbb{R},
\]

\[
L^{(M_p)}_{\nu}(\mathbb{R}+) \hookrightarrow L^{(M_p)}_{\nu}(\mathbb{R}) \quad \text{for any } \nu \in \mathbb{R}.
\]

Since \( D^{(M_p)}(\mathbb{R}) \) is dense in \( L^{(M_p)}_{\nu}(\mathbb{R}) \) the space \( L^{(M_p)}_{\nu}(\mathbb{R}) \) is imbedded into \( D^{(M_p)}(\mathbb{R}) \) and we have a counterpart of Theorem 6.

**Theorem 7 (Structure theorem).** Assume (M.1), (M.2) and (M.3). In order that an ultradistribution \( S \in D^{(M_p)}(\mathbb{R}) \) belong to \( L^{(M_p)}(\mathbb{R}) \) it is necessary and sufficient that for any \( a, b, \nu \) there are ultraspherical operators \( J_a(D) \) and \( J_b(D) \) of class \( M_p \) (cf. [K]) and functions \( S_a, S_b \in C^0(\mathbb{R}) \) such that supp \( S_a \subset \mathbb{R}+ \), \( |S_a(x)| \leq Ce^{-\alpha x} \) for \( x \leq 0 \), supp \( S_b \subset \mathbb{R}+ \), \( |S_b(x)| \leq Ce^{-\beta x} \) for \( x \geq 0 \) and

\[
S = J_a(D)S_a + J_b(D)S_b \quad \text{in } L^{(M_p)}_{\nu}(\mathbb{R}).
\]

The proof of Theorem 7 is based on the structure theorem for Laplace ultradistributions on \( \mathbb{R}+ \) and \( \mathbb{R}- \) (cf. [L2], Theorem 4), and on the following analogue of Lemma 1.

**Lemma 4.** Assume (M.1), (M.2) and (M.3). Let \( S \in L^{(M_p)}_{\nu}(\mathbb{R}) \). Then one can find \( S^- \in L^{(M_p)}_{\nu}(\mathbb{R}-) \) and \( S^+ \in L^{(M_p)}_{\nu}(\mathbb{R}+) \) such that \( S = S^- + S^+ \) in \( L^{(M_p)}_{\nu}(\mathbb{R}) \). The decomposition is unique modulo an ultradistribution of class \( M_p \) with support at zero.

**Proof.** We follow the proof of Lemma 1, this time taking \( \chi \in D^{(M_p)}(\mathbb{R}) \), an ultraspherical operator \( J(D) \) of class \( M_p \) and a function \( g \in C^0(\mathbb{R}) \) such that \( \chi S = J(D)g \).

**Definition.** Let \( H \in C(\{0 < \text{Im} z < R\}) \) with some \( R > 0 \) and let \( \nu \in \mathbb{R} \cup \{-\infty\} \) and \( \omega \in \mathbb{R} \cup \{\infty\} \). Assume that for any \( a > \nu \) and \( b < \omega \) there exists \( R' > 0 \) such that \( H_y \in L^{(M_p)}_{\nu,b}(\mathbb{R}) \) for \( 0 < y \leq R' \). If for every \( \varphi \in L^{(M_p)}_{\nu,b}(\mathbb{R}) \) the limit \( \lim_{y \to 0} H_y[\varphi] = : u[\varphi] \) exists then, by the Banach–Steinhaus theorem, \( u \in L^{(M_p)}_{\nu,b}(\mathbb{R}) \) and we say that \( H \) has boundary value \( u = b(H) \) in \( L^{(M_p)}_{\nu,b}(\mathbb{R}) \).

One would like to prove Theorem 5 by the method used in the proof of Theorem 3. To this end we should state analogues of Lemmas 2 and 3. Since there is no problem with Lemma 3 (we only have to replace a smooth function \( \rho \) by that of class \( D^{(M_p)}(\{z \in \mathbb{C} : |z| < 1\}) \), we shall concentrate on Lemma 2.

So take \( \varphi \in D^{(M_p)}(\mathbb{R}) \) with supp \( \varphi \subset [-1, 1] \) and fix \( \xi \in \mathbb{R} \), \( 0 < r \leq 1 \) and \( a, b \in \mathbb{R} \). Following the proof of Lemma 2 we easily arrive at the estimate

\[
\left\| \varphi \left( \frac{\cdot - \xi}{r} \right) \right\|_{a,b,r,h}^{(M_p)} \leq C_{a,b,\kappa,\kappa}(\xi)^\alpha \|\varphi\|_{a,b,r,h}^{(M_p)},
\]

where \( \kappa_{a,b} \) is given by (5). Now, in general, we cannot estimate \( \|\varphi\|_{a,b,r,h}^{(M_p)} \) by \( C(\varphi)^{\alpha} \) with some \( c > 0 \) independent of \( 0 < r \leq 1 \).

But we have

**Lemma 5.** Let \( M_p \) satisfy (M.1) and (M.3'). Then there exists a sequence \( (Q_p) \) satisfying (M.1) and (M.3') such that \( Q_p \prec (M_p) \) (i.e. for any \( h > 0 \) there exists \( C < \infty \) such that \( Q_p \leq CH^pM_p \) for \( p \in N_0 \)). Moreover, given such sequences \( (M_p) \) and \( (Q_p) \) one can find a sequence \( (N_p) \rangle \) (pl) such that

\[
N_pQ_p \leq pM_p \quad \text{for } p \in N_0.
\]

Furthermore, for \( \varphi \in D^{(Q_p)}(\mathbb{R}) \) with supp \( \varphi \subset [-1, 1] \) and for \( \xi \in \mathbb{R} \), \( a, b \in \mathbb{R} \), \( h > 0 \) and \( 0 < r \leq 1 \) we have

\[
\left\| \varphi \left( \frac{\cdot - \xi}{r} \right) \right\|_{a,b,r,h}^{(M_p)} \leq C_{a,b,\kappa,\kappa}(\xi)^\alpha \exp(N^* \left( \frac{1}{r} \right))^\alpha \|\varphi\|_{a,b,r,h}^{(M_p)}.
\]

Note that the condition (pl) \( \prec (N_p) \) is equivalent to the finiteness of the growth function \( N^* \) for \( (N_p) \).
Proof. The existence of a sequence \((Q_p)\) satisfying (M.1), (M.3') and \((Q_p) \prec (M_p)\) is well known (cf. [R], pp. 66–67). Now if we put
\[ N_p = \frac{p!M_p}{Q_p} \quad \text{for } p \in \mathbb{N}_0 \]
then \((p!) \prec (N_p)\) and (11) holds. Thus, we only have to prove (12), and this follows by (10) and (11), since
\[ \|\varphi\|_{a,b,h}^{(M_p)} \leq \exp N^*(\frac{1}{p}) \|\varphi\|_{a,b,h}. \]

Now following the proof of Theorem 3 with Lemma 5 in place of Lemma 2 we obtain

**Theorem 8.** Let \((M_p)\) satisfy (M.1) and (M.3'). Let \(H \in L^{(\nu)}(\mathbb{R})\) with some \(\nu > 0\) and assume that \(H\) has a boundary value \(b(H) \in L^{(\nu)}(\mathbb{R})\) for some \(\nu \in (0, \infty)\). Then there exists a sequence \((N_p)\) such that \((M_p) \succ (N_p) \succ (p!)\) and for every \(a \prec \nu\) and \(b \prec 1\) one can find \(0 < R < R' < R \prec \infty\) such that
\[ |H(z)| \leq \begin{cases} C \exp(-a\Re z + N^*(L/\Im z)) & \text{for } \Re z \geq 0, 0 < \Im z \leq R', \\ C \exp(b\Re z + N^*(L/\Im z)) & \text{for } \Re z \leq 0, 0 < \Im z \leq R'. \end{cases} \]

Proof of Theorem 4. We follow the proof of Theorem 2, this time using (13) instead of (3).

5. Boundary value representation. Throughout this section \(W\) denotes a tubular neighbourhood of \(R\), i.e. a set of the type \(R + iU\), where \(U\) is a bounded neighbourhood of zero in \(R\).

Let \(a,b \in \mathbb{R}\) and \(h > 0\). We define the spaces
\[ L^{(M_p)}_{a,b,h}(W \setminus \mathbb{R}) = \{ F \in L^{(M_p)}(W \setminus \mathbb{R}) : q_{a,b,h}(F) < \infty \} \]
for any closed tubular subset \(W \subset W\subset W\); or
\[ L^{(M_p)}_{a,b}(W \setminus \mathbb{R}) = \lim_{h \to 0} L^{(M_p)}_{a,b,h}(W \setminus \mathbb{R}); \]
\[ L_{a,b}(W) = \{ G \in L^{(M_p)}(W \setminus \mathbb{R}) : q_{a,b,h}(G) < \infty \} \]
for any closed tubular subset \(W \subset W\).

It follows by the 3-line theorem that \(L_{a,b}(W)\) is a closed subspace of \(L^{(M_p)}_{a,b,h}(W \setminus \mathbb{R})\). Thus, we can define the quotient space
\[ L^{(M_p)}_{a,b}(W \setminus \mathbb{R}) = \frac{L^{(M_p)}_{a,b,h}(W \setminus \mathbb{R})}{L_{a,b}(W)} \]

Now, let \(\nu \in \mathbb{R} \cup \{\infty\}\) and \(\omega \in \mathbb{R} \cup \{\infty\}\). We define
\[ L^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R}) = \lim_{a \to \nu, b \to \omega} L^{(M_p)}_{a,b}(W \setminus \mathbb{R}); \]
and
\[ H^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R}) = \lim_{a \to \nu, b \to \omega} H^{(M_p)}_{a,b}(W \setminus \mathbb{R}). \]

By the Mittag–Leffler lemma (cf. [K]) we also have
\[ H^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R}) = L^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R}). \]

In Theorem 9 we shall prove that the space \(H^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R})\) is isomorphic to \(L^{(M_p)}_{(\nu,\omega)}(\mathbb{R})\) and consequently, it does not depend on the choice of a tubular neighbourhood \(W\) of \(R\). In the proof we shall use, as a Cauchy kernel, the function
\[ A(z) = \frac{\exp(-(z - x)^2)}{z - x}. \]

Lemma 6. Assume (M.1), (M.2') and (M.3'). Fix \(\xi \in (0, \infty)\). Then for any \(a,b \in \mathbb{R}\) the function \(R \ni x \mapsto A(z)\) belongs to \(L^{(M_p)}_{a,b}(R)\). In particular, for any \(h > 0\) one can find \(C = C(a,b)\) such that
\[ |A(z)| \leq \begin{cases} \frac{C_{a,b}(R \xi)}{\xi} \exp\left\{M^*(L/\Im z) \right\} & \text{for } 0 < |\Im z| \leq 1, \\ \frac{C_{a,b}(R \xi)}{\xi} \exp\left\{(|\Im z| + 1)^2 \right\} & \text{for } |\Im z| > 1. \end{cases} \]

Proof. Applying the Leibniz formula we get by (M.1))
\[ |A(z)| \leq \sup_{a \in R} \frac{C_{a,b}(z)}{h^a M^a} \]
\[ \leq \sup_{a \in R} \frac{h^a M^a}{\left(\sum_{\beta \leq a} \frac{2^\beta h^a M^a}{\beta! \Gamma(1/2)}\right)} \]

By (M.2') the second factor under the sum sign can be estimated by \(C \exp\{M^*(L/\Im z)\} \) with some \(L < \infty\). To estimate the first factor define
\[ F_\zeta(x) = \exp\{-(\zeta - z)^2\} \quad \text{for } z \in C, \ z \in C. \]

Using the formula
\[ D^\gamma F_\zeta(x) = \frac{\gamma!}{2\pi} \int_0^{2\pi} F_\zeta(x + e^{i\varphi}) e^{-i\varphi \gamma} \, d\varphi \quad \text{for } x \in \mathbb{R}, \ \gamma \in \mathbb{N}_0, \]
we easily obtain
\[ |D^\gamma F_\zeta(x)| \leq e^{\gamma} \exp\{(|\eta| + 1)^2 - (|\zeta - x| - 1)^2\} \quad \text{where } \zeta = \xi + i\eta. \]

Now, by calculations in the spirit of those in the proof of Lemma 2 we get
\[ \sup_{x \in \mathbb{R}} k_{a,b}(x)|D^\gamma F_\zeta(x)| \leq C_{a,b} \gamma! |k_{a,b}(\text{Re } \zeta)| \exp\{(|\text{Im } \zeta| + 1)^2\} \]
with some \( C_{a,b} < \infty \). Finally,
\[ \sup_{\alpha \in \mathbb{N}_0} \sum_{\beta \leq \alpha} \frac{2^{\alpha - \beta} \alpha! (\alpha - \beta)!}{h^{2-\beta} \alpha!} < \infty \]
and this ends the proof.

**Proposition 1.** Assume (M.1), (M.2') and (M.3'). Let \( S \in L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) \) and put
\[ \Lambda A S(\zeta) = \frac{-1}{2\pi i} S[A(\zeta, \cdot)] \quad \text{for } \zeta \in C \setminus \mathbb{R}. \]
Then \( \Lambda A S \in O(C \setminus \mathbb{R}) \) and for any \( a > \nu \) and \( b < \omega \) one can find \( C = C(a, b) < \infty \) and \( L = L(a, b) < \infty \) such that
\[ |\Lambda A S(\zeta)| \leq \begin{cases} C_{a,b}(\text{Re } \zeta) \exp\{M^*(L/|\text{Im } \zeta|)\} & \text{for } 0 < |\text{Im } \zeta| \leq 1, \\ C_{a,b}(\text{Re } \zeta) \exp\{(|\text{Im } \zeta| + 1)^2\} & \text{for } |\text{Im } \zeta| \geq 1. \end{cases} \]

Thus, \( \Lambda A S \in L^{(M_p)}_{(\nu,\omega)}(C \setminus \mathbb{R}) \).

We call the map
\[ L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) \ni S \mapsto \Lambda A S \in L^{(M_p)}_{(\nu,\omega)}(C \setminus \mathbb{R}) \]
the \( \Lambda \)-Cauchy transformation.

**Proof.** Since the holomorphy of \( \Lambda A S \) follows from the continuity of \( S \) by standard arguments we only need to prove (15). To this end fix \( a > \nu \) and \( b < \omega \). Then by the continuity of \( S \) one can find \( C < \infty \) and \( A > 0 \) such that
\[ |S(\varphi)| \leq C|\varphi|^{M_p}_{a,b} \quad \text{for } \varphi \in L^{(M_p)}_{a,b}(\mathbb{R}). \]

So (15) is a consequence of Lemma 6.

**Proposition 2.** Assume (M.1), (M.2) and (M.3). Let \( H \in O(\{0 < \text{Im } z < \mathbb{R}\}) \) be such that for every \( a > \nu \) and \( b < \omega \) one can find \( L < \infty \)
such that
\[ |H(z)| \leq Ck_{a,b}(\text{Re } z) \exp\{M^*(L/|\text{Im } z|)\} \quad \text{for } 0 < \text{Im } z \leq \mathbb{R}. \]

Then \( H \) has the boundary value \( b(H) \in L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) \).

**Proof.** Fix \( a > \nu \) and \( b < \omega \). Choose \( a' > \nu \), \( b < \omega \) and \( \delta, \sigma \in \mathbb{R} \) such that \( a' + \delta > 0 \), \( b - \sigma < 0 \) and \( \delta + \sigma \geq 0 \). Consider the function
\[ H_{\delta,\sigma}(z) := \text{ch}_{\delta,\sigma}(z) \cdot H(z) \quad \text{for } 0 < \text{Im } z < \mathbb{R}. \]

Then we can find \( \mathbb{R}' > 0 \) and \( L < \infty \) such that
\[ |H_{\delta,\sigma}(z)| \leq Ck_{a'+\delta,\sigma}(\text{Re } z) \exp\{M^*(L/|\text{Im } z|)\} \quad \text{for } 0 < \text{Im } z \leq \mathbb{R}'. \]

Now for \( z \in C \) with \( 0 < \text{Im } z \leq \mathbb{R}' := \mathbb{R}'/2 \) put
\[ \tilde{H}_{\delta,\sigma}(z) = i \int_{\gamma} G(t(z - w))H_{\delta,\sigma}(w) \, dw, \]
where \( \gamma \) is a closed curve starting from \( \tilde{z} = \frac{3}{2}\mathbb{R}' \cdot i \), encircling \( z \) once in the positive direction and such that \(|\arg(z - \tilde{z})| < \pi/2 \) for \( z \in \gamma \); \( G \) is the Green kernel for
\[ P(\zeta) = (1 + \zeta)^2 \prod_{p=1}^{\infty} \left(1 + \frac{L \zeta}{m_p}\right) \]
and holomorphically continued to the Riemann domain \(-\pi/2 < \arg z < 5\pi/2\). Then by Lemma 11.4 of [K] and the observation made in the proof of Proposition 2 of [L2] we have
\[ P(D)\tilde{H}_{\delta,\sigma}(z) = \tilde{H}_{\delta,\sigma}(z) \quad \text{for } 0 < \text{Im } z \leq \mathbb{R}' \]
and
\[ |\tilde{H}_{\delta,\sigma}(z)| \leq Ck_{a'+\delta,\sigma}(\text{Re } z) \quad \text{for } 0 \leq \text{Im } z \leq \mathbb{R}' \]
exists. Now observe that for \( \varphi \in L^{(M_x)}_{\alpha}(\mathbb{R}) \) and \( y > 0 \) small enough,

\[
H(\cdot + iy)[\varphi] = \int_\mathbb{R} H_{\delta,\sigma}(x + iy) \frac{\varphi(x)}{c_{\delta,\sigma}(x + iy)} \, dx
= \int_\mathbb{R} P(D) \tilde{H}_{\delta,\sigma}(x + iy) \frac{\varphi(x)}{c_{\delta,\sigma}(x + iy)} \, dx
= \tilde{H}_{\delta,\sigma} \left[ P^*(D) \frac{\varphi}{c_{\delta,\sigma}(\cdot + iy)} \right].
\]

Thus \( H(\cdot + iy) \in L^{(M_x)}_{\alpha}(\mathbb{R}) \) and if we put

\[
h = \frac{1}{c_{\delta,\sigma}} P(D) \tilde{H}_{\delta,\sigma} \in L^{(M_x)}_{\alpha}(\mathbb{R})
\]

then

\[
\lim_{y \to 0} H(\cdot + iy)[\varphi] = \tilde{H}_{\delta,\sigma} \left[ P^*(D) \frac{\varphi}{c_{\delta,\sigma}} \right] = h[\varphi] \quad \text{for} \varphi \in L^{(M_x)}_{\alpha}(\mathbb{R}).
\]

Since \( a > \nu \) and \( b < \omega \) were arbitrary this ends the proof.

In the proof of Theorem 9 we shall use the lemma stated below. To formulate it let us define for \( \nu \in \mathbb{R} \cup \{-\infty\} \) and \( \omega \in \mathbb{R} \cup \{\infty\} \),

\[
L^{(M_x)}_{\nu,\omega}(\mathbb{R}) = \lim_{W \to \mathbb{R}} \lim_{\nu \to \omega, b \to \omega} L_{a,b}(W),
\]

where for any \( a, b \in \mathbb{R} \) and any tubular neighbourhood \( W \) of \( \mathbb{R} \),

\[
L_{a,b}(W) = \{ F \in O(W) \cap C^0(\overline{W}) : p_{a,b}(F) := \sup_{z \in \overline{W}} |F(z)|_{\kappa_{a,b}(\mathbb{R} \cap z)} < \infty \}.
\]

**Lemma 7.** Assume (M.1), (M.2') and (M.3'). The space \( L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \) is contained in \( L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \). Therefore it is a dense subset of \( L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \).

**Proof.** Since functions from \( L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \) are holomorphic on a tubular neighbourhood of \( \mathbb{R} \) the first assertion is clear. To prove the second one observe that it is sufficient to show that for all \( a, b \in \mathbb{R} \) and \( h > 0 \), each function \( \varphi \in L^{(M_x)}_{a,b}(\mathbb{R}) \) can be approximated by elements of \( L^{(M_x)}_{a,b}(W) \) with some \( W \supset \mathbb{R} \) in the topology of \( L^{(M_x)}_{a,b}(\mathbb{R}) \). So fix \( a, b \in \mathbb{R} \), \( h > 0 \) and \( \varphi \in L^{(M_x)}_{a,b}(\mathbb{R}) \). Since the spaces \( L^{(M_x)}_{a,b}(\mathbb{R}) \) (resp. \( L^{(M_x)}_{a,b}(W) \)) with \( W \) narrow enough) and \( L^{(M_x)}_{0,0}(\mathbb{R}) \) (resp. \( L^{(M_x)}_{0,0}(W) \)) are isomorphic with the isomorphism being multiplication by \( c_{\delta,\sigma} \) if \( \delta \) and by \( 1/c_{\delta,\sigma} \) if \( \alpha < \delta \) (see (6)), we can assume \( a = b = 0 \). Put

\[
F_{\varepsilon}(z) = \frac{j}{\sqrt{\pi}} \int_\mathbb{R} \exp(-j^2(z-t)^2) \varphi(t) \, dt \quad \text{for} \ z \in \mathbb{C}, \ j \in \mathbb{N}.
\]

If \( W \) is contained in \( \mathbb{R} + i[-R, R] \) with some \( R \in \mathbb{R} \), then \( \varphi \) is bounded by \( C \) then \( p_{\varepsilon,\delta,\sigma}(F_{\varepsilon}) \leq C \exp\{j^2R^2\} \) for \( j \in \mathbb{N} \). Next observe that for \( x \in \mathbb{R} \) and \( h \in \mathbb{N}_0 \),

\[
D^h(\varphi(x) - \varphi(x)) = \frac{j}{\sqrt{\pi}} \int_\mathbb{R} \exp(-j^2r^2) D^h(\varphi(x + r) - \varphi(r)) \, dr.
\]

Thus, since by (M.2'),

\[
\sup_{x \in \mathbb{R}} \|D^h \varphi(x)\| \leq C_h(h \alpha)^{\alpha} \quad \text{for} \alpha \in \mathbb{N}_0
\]

with some \( C_h < \infty \) and \( H < \infty \), we obtain

\[
\|F_{\varepsilon} - \varphi\|_{0,\varepsilon} \leq C \|\varphi\|_{0,\varepsilon}/H
\]

proving the lemma.

Let \( h \in H^{(M_x)}_{(\nu,\omega)}(\mathbb{R}) \). Then \( h = [H] \) with some \( H \in L^{(M_x)}_{(\nu,\omega)}(W \setminus \mathbb{R}) \). By Proposition 2, \( H \) admits boundary values from above \( b^+(H) \) and from below \( b^-(H) \). Since \( G \in L^{(M_x)}_{(\nu,\omega)}(W) \) then \( b^+(G) = b^-(G) \), the difference \( b^+(H) - b^-(H) \) does not depend on the choice of a defining function \( H \). Thus, the boundary value map

\[
b : H^{(M_x)}_{(\nu,\omega)}(W, \mathbb{R}) \to L^{(M_x)}_{(\nu,\omega)}(\mathbb{R}),
\]

\[
b(h) := b^{+}(H) - b^{−}(H), \quad \text{where} \quad h = [H] \text{ mod } L^{(M_x)}_{(\nu,\omega)}(W),
\]

is well defined.

**Theorem 9.** Assume (M.1), (M.2) and (M.3). Let \( W \) be a tubular neighbourhood of \( \mathbb{R} \) and \( \nu \in \mathbb{R} \cup \{-\infty\} \), \( \omega \in \mathbb{R} \cup \{\infty\} \). Then the mapping

\[
C : L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \to H^{(M_x)}_{\nu,\omega}(W, \mathbb{R}), \quad L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \supset S \mapsto [C_S] \text{ mod } L^{(M_x)}_{\nu,\omega}(W),
\]

with \( CS \) given by (14) is a topological isomorphism with inverse \( C^{-1} = b \), where \( b \) is the boundary value map.

**Proof.** Let \( S \in L^{(M_x)}_{\nu,\omega}(\mathbb{R}) \) and let \( CS = h \in H^{(M_x)}_{\nu,\omega}(\mathbb{R}) \). Then by definition \( h = [H] \) with \( H(\zeta) = \frac{1}{2\pi j} \int \overline{S(\zeta)} \, d\zeta \) for \( \zeta \in \mathbb{C} \setminus \mathbb{R} \). Put \( \overline{S} = b(h) \) and observe that to prove the equality \( \overline{S} = S \), by Lemma 7, it is sufficient to show that \( \overline{S}(\varphi) = S(\varphi) \) for \( \varphi \in L^{(M_x)}_{\nu,\omega}(W) \). To this end fix \( \varphi \in L^{(M_x)}_{\nu,\omega}(W) \) and let \( a > \nu, \ b < \omega \) and \( W \supset \mathbb{R} \) be such that \( \varphi \in L^{(M_x)}_{a,b}(W) \). Note that

\[
\lim_{\varepsilon \to 0} \int_\mathbb{R} H(\xi \pm i\varepsilon)(\varphi(\xi \pm i\varepsilon) - \varphi(\xi)) \, d\xi = 0
\]

and that the integral \( \int_\mathbb{R} H(\xi \pm i\varepsilon) \varphi(\xi \pm i\varepsilon) \, d\xi \) does not depend on \( \varepsilon \) for \( \varepsilon > 0 \) small enough. Thus, choosing \( \varepsilon > 0 \) small enough we derive
\[ S[\varphi] = b^+(H)[\varphi] - b^-(H)[\varphi] \]
\[ = \int_{R} H(\xi + i\delta)\varphi(\xi + i\delta) d\xi - \int_{R} H(\xi - i\delta)\varphi(\xi - i\delta) d\xi \]
\[ = -\frac{1}{2\pi i} \int_{R} S[A(\xi + i\delta, \cdot)]\varphi(\xi + i\delta) d\xi + \frac{1}{2\pi i} \int_{R} S[A(\xi - i\delta, \cdot)]\varphi(\xi - i\delta) d\xi \]
\[ = -\frac{1}{2\pi i} \int_{R} S[A(\xi + i\delta, \cdot)]\varphi(\xi + i\delta) d\xi - \frac{1}{2\pi i} \int_{R} A(\xi - i\delta, \cdot)\varphi(\xi - i\delta) d\xi = S[\varphi], \]

since for \( \varphi \in L_{a,b}(W) \), \( \delta > 0 \) small enough and \( z \in \mathbb{R} \),
\[ -\frac{1}{2\pi i} \int_{R} A(\xi + i\delta, \cdot)\varphi(\xi + i\delta) d\xi + \frac{1}{2\pi i} \int_{R} A(\xi - i\delta, \cdot)\varphi(\xi - i\delta) d\xi = \varphi(z). \]

To prove the second part of the theorem take \( h = [H] \in H_{(M^\nu)}(W \setminus \mathbb{R}) \), where \( H \in C_{(M^\nu)}(W \setminus \mathbb{R}) \) and let \( F = C_{4\delta} S \in L_{(M^\nu)}(C \setminus \mathbb{R}) \). We have to show that \( G := H - F \in L_{(M^\nu)}(W \setminus \mathbb{R}) \) extends holomorphically to a function \( \tilde{G} \in L_{(\nu, \omega)}(W) \). To this end fix \( a > \nu, b < \omega \) and a closed tubular subset \( \tilde{W} \) of \( W \). By the proof of the first part of the theorem, for \( \delta > 0 \) small enough and \( \varphi \in L_{a,b}(\tilde{W}) \), we have
\[ S[\varphi] = \int_{R} H(\xi + i\delta)\varphi(\xi + i\delta) d\xi - \int_{R} H(\xi - i\delta)\varphi(\xi - i\delta) d\xi \]
and
\[ S[\varphi] = \frac{1}{2\pi i} \int_{R} S[A(\xi + i\delta, \cdot)]\varphi(\xi + i\delta) d\xi + \frac{1}{2\pi i} \int_{R} S[A(\xi - i\delta, \cdot)]\varphi(\xi - i\delta) d\xi. \]

So for \( \varphi \in L_{a,b}(\tilde{W}) \),
\[ G(\xi + i\delta)\varphi(\xi + i\delta) d\xi - G(\xi - i\delta)\varphi(\xi - i\delta) d\xi = 0. \]

Now, for \( R > \delta \) close to \( \delta \) put
\[ \Psi(z) = -\frac{1}{2\pi i} \int_{R} G(\xi + iR)A(\xi + iR, z) d\xi + \frac{1}{2\pi i} \int_{R} G(\xi - iR)A(\xi - iR, z) d\xi \]
for \( z \in \mathbb{C} \) with \( |\text{Im } z| < R \).

Then \( \Psi \in C_{(\text{Im } z > 0)} \cap L_{a,b}(\{ \text{Im } z \leq \delta \}) \) and by (16), \( \Psi(z) = G(z) \) for \( \delta \leq |\text{Im } z| < R \). Thus, if we define
\[ \tilde{G} = \begin{cases} G(z) & \text{for } z \in W \setminus \mathbb{R}, \\ \Psi(z) & \text{for } |\text{Im } z| < R, \end{cases} \]
then \( \tilde{G} \in C(W) \cap L_{a,b}(\tilde{W}) \). Since \( a > \nu, b < \omega \) and \( \tilde{W} \subset W \) were arbitrary this ends the proof.

**Proof of Theorem 5.** Suppose \( H \in C_{(\nu, \omega)}(W) \) with some \( R > 0 \) has a boundary value \( u := b(H) \in L_{(\nu, \omega)}(R) \). Then by Theorem 8, \( H \) satisfies (13). Next as in the proof of the second part of Theorem 9 we show that the difference \( H - C_{4\delta}u \) belongs to \( L_{(\nu, \omega)}(W) \), where \( W = \{ Re z < R' \} \) with some 0 < \( R' \) < \( R \) and \( H \) is extended by zero to \( \{ \text{Im } z \leq 0 \} \). Since \( C_{4\delta}u \in L_{(\nu, \omega)}(W \setminus R) \) this ends the proof.

**6. Final remarks.** Let \( U \) be a complex neighbourhood of \( \mathbb{R} \) and set \( U^\pm = U \cap \{ \pm \text{Im } z > 0 \} \). Then the well known Painlevé theorem states that if \( F^\pm \in C(U^\pm) \) and \( b^+(F^+) = b^-(F^-) \) (in \( C^0(W) \) or \( D(W) \) or \( D(M^\nu)(W) \)) then there exists an \( F \in C(U) \) such that \( F^\pm = F_{0^\pm} \). The results of Theorems 1, 2 and 4 allow us to formulate some generalization of the Painlevé theorem. Namely, if we assume that \( F^\pm \in C(\{ \pm \text{Im } z > 0 \}) \) is of exponential type and the difference of the boundary values \( b^+(F^+) - b^-(F^-) \) is in a sense small then \( F^- \) determines \( F^+ \) and conversely. More precisely, we have

**Corollary 1.** Let \( F^\pm \in C(\{ \pm \text{Im } z > 0 \}) \cap C^0(\{ \pm \text{Im } z \geq 0 \}) \) be of exponential type. If
\[ |F^+(z) - F^-(z)| \leq C_{a,b}(z) \quad \text{for } z \in \mathbb{R} \]
with some \( a < b \) then \( F^+ \) determines \( F^- \) and conversely.

**Proof.** Fix \( F^- \in C(\{ \text{Im } z < 0 \}) \cap C^0(\{ \text{Im } z \leq 0 \}) \) and let \( F^+ \in C(\{ \text{Im } z > 0 \}) \cap C^0(\{ \text{Im } z \geq 0 \}) \) be of exponential type and satisfy (17). Then \( F^+ - F^- \in C(\{ \text{Im } z > 0 \}) \cap C^0(\{ \text{Im } z \geq 0 \}) \) is of exponential type and
\[ |F^+(z) - F^-(z)| \leq C_{a,b}(z) \quad \text{for } z \in \mathbb{R}. \]

Thus, by Theorem 1, \( F^+ = F^- \). Note that until now we have not used the fact that \( F^- \) is of exponential type. The proof that \( F^+ \) determines \( F^- \) is the same.

**Remark.** Under the assumption of Corollary 1 there need not exist an \( F \in C(U) \) such that \( F^\pm = F_{0^\pm} \). The counter-example is given by the pair of functions \( \{ F^+, F^- \} \), where
\[ F^\pm(z) = \begin{cases} \pm \text{Im } z & \text{for } z \neq 0, \pm \text{Im } z \geq 0, \\ 0 & \text{for } z = 0. \end{cases} \]

**Corollary 2.** Let \( (M^\nu) \) satisfy (M.1) and (M.3). Let \( F^\pm \in C(\{ \text{Im } z > 0 \}) \) be of exponential type in \( \{ \pm \text{Im } z \geq \varepsilon \} \) for all \( \varepsilon > 0 \). Assume that \( F^\pm \) has a boundary value \( u^\pm \) in \( L_{(\nu, \omega)}(\mathbb{R}) \) (resp. \( L_{(\nu, \omega)}(\mathbb{R}) \)) with some \( \nu^\pm \in \mathbb{R} \cup \{ -\infty \} \) and \( \omega^\pm \in \mathbb{R} \cup \{ \infty \} \). If \( u^+ - u^- \in L_{(\nu, \omega)}(\mathbb{R}) \) (resp. \( L_{(\nu, \omega)}(\mathbb{R}) \)) with some \( \nu < \omega \) then \( F^- \) determines \( F^+ \) and conversely.
Proof. The proof goes along the same lines as the one of Corollary 1, with Theorem 2 (resp. Theorem 4) in place of Theorem 1.

Analogously we get

**Corollary 3.** Let \( (M_p) \) satisfy (M1) and (M3'), and let \( \hat{u} \in L^1(\mathbb{R}) \) (resp. \( L^1(\mathbb{R}) \)) with some \( \hat{v} \in \mathbb{R} \cup \{-\infty\} \) and \( \hat{\omega} \in \mathbb{R} \cup \{\infty\} \). Then there exists at most one \( F^\pm \in \mathcal{O}(\{\text{Im} z > 0\}) \) of exponential type in \( \{\pm \text{Im} z \geq \epsilon\} \) for all \( \epsilon > 0 \) such that \( b(F^\pm) \in L^1(\nu^\pm, c\omega^\pm)(\mathbb{R}) \) (resp. \( L^1(\nu^\pm, c\omega^\pm)(\mathbb{R}) \)) with some \( \nu^\pm \in \mathbb{R} \cup \{-\infty\} \) and \( \omega^\pm \in \mathbb{R} \cup \{\infty\} \), and \( b(F^\pm) - \hat{u} \in L^1(\nu^\pm, c\omega^\pm)(\mathbb{R}) \) (resp. \( L^1(\nu^\pm, c\omega^\pm)(\mathbb{R}) \)) with some \( \nu < \omega \). Furthermore, if \( \nu < \omega \) then \( F^\pm \equiv 0 \).

We remark that in the case \( \hat{v} \geq \hat{\omega} \), in general, the problem of existence of such an \( F^\pm \) remains open.

**Acknowledgements.** The author would like to thank the referees for their helpful remarks.

**References**


Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: lysik@impan.gov.pl

Received December 5, 1995
Revised version March 29, 1996 and December 6, 1996

**STUDIA MATHEMATICA 123 (3) (1997)**

**Compact homomorphisms between algebras of analytic functions**

by

**RICHARD ARON** (Kent, Ohio), **PABLO GALINDO** (Valencia),

and **MIKAEL LINDBRÖM** (Äbo)

**Abstract.** We prove that every weakly compact multiplicative linear continuous map from \( H^\infty(D) \) into \( H^\infty(D) \) is compact. We also give an example which shows that this is not generally true for uniform algebras. Finally, we characterize the spectra of compact composition operators acting on the uniform algebra \( H^\infty(B_E) \), where \( B_E \) is the open unit ball of an infinite-dimensional Banach space \( E \).

Let \( E \) denote a complex Banach space with open unit ball \( B_E \) and let \( \phi : B_E \to B_E \) be an analytic map. We will consider the composition operator \( C_\phi \) defined by \( C_\phi(f) = f \circ \phi \), acting on the uniform algebra \( H^\infty(B_E) \) of all bounded analytic functions on \( B_E \). This operator may also be regarded as acting on the smaller uniform algebra \( A_1(B_E) \) of all analytic functions on \( B_E \) which are uniformly continuous, in which case we assume that \( f \circ \phi \in A_1(B_E) \) whenever \( f \) is in \( A_1(B_E) \). These algebras, which are natural generalizations of the classical algebras \( H^\infty(D) \) and \( A(D) \) of analytic functions on the complex open disc \( D \), have been studied in [AGC].

Several results automatically yielding compactness of composition operators from weak compactness have appeared recently. For instance, D. Sarason in [Sa] proved that every weakly compact composition operator on \( H^1(D) \) is compact, and K. Madigan and A. Matheson [MM] obtained the analogue for the little Bloch space \( B_0 \). In the first section we study compactness of \( C_\phi \) and prove that every weakly compact homomorphism from \( H^\infty(D) \) into \( H^\infty(D) \) is automatically compact. This result has also inde-