[Q4] T. Qian, *A holomorphic extension result*, Complex Variables Theory Appl. 32 (1996), 59–77.

[Q5] —, *Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space and generalizations to* $\mathbb{R}^n$, in: Proc. Conf. on Clifford and Quaternionic Analysis and Numerical Methods, June 1996, to appear.

[S] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[V] G. Verchota, *Layer potentials and regularity for the Dirichlet problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal. 59 (1984), 572–611.

[Z] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, London and New York, 1968.

Department of Mathematics
The New England University
Armidale, New South Wales 2351
Australia
E-mail: tao@neumann.une.edu.au

Received May 26, 1994 (3278)
Revised version November 8, 1996

STUDIA MATHEMATICA 123 (3) (1997)

A Phragmén–Lindelöf type quasi-analyticity principle

by

GRZEGORZ ŁYSIK (Warszawa)

Abstract. Quasi-analyticity theorems of Phragmén–Lindelöf type for holomorphic functions of exponential type on a half plane are stated and proved. Spaces of Laplace distributions (ultradistributions) on $\mathbb{R}$ are studied and their boundary value representation is given. A generalization of the Painlevé theorem is proved.

1. Introduction and statement of the main results. The well-known Phragmén–Lindelöf theorem consists of two parts. The first one ([H]), called the maximum principle, says that a function holomorphic and of exponential type on a sector S of opening less than π is bounded if it is bounded on the boundary of S. The second one ([T]), called the quasi-analyticity principle, says that a holomorphic function F on a sector S vanishes if the opening of S is greater than π and F is exponentially decreasing in S.

In the present paper we study the quasi-analyticity principle in the critical case of a half plane Π. To ensure vanishing of F in that case we assume that F is of exponential type in Π and decreases exponentially along the boundary of Π. More precisely, we have

THEOREM 1 (Quasi-analyticity principle, continuous version). *Let* $F \in \mathcal{O}(\{\operatorname{Re} z > 0\}) \cap C^0(\{\operatorname{Re} z \geq 0\})$ *be of exponential type, i.e.*

(1) $$|F(z)| \leq Ce^{c|z|} \quad \text{for } \operatorname{Re} z \geq 0 \text{ with some } C < \infty \text{ and } c < \infty.$$

If

(2) $$|F(\pm ir)| \leq Ce^{c^{\pm}r} \quad \text{for } r \geq 0$$

with some $c^{\pm} \in \mathbb{R}$ *such that* $c^+ + c^- < 0$ *then* $F \equiv 0$.

The elementary proof of Theorem 1 is based on the Laplace integral representation of holomorphic functions of exponential type.

1991 *Mathematics Subject Classification*: 30D15, 44A15, 46F12, 46F20.
Key words and phrases: quasi-analyticity, Laplace distributions, Laplace ultradistributions, boundary values.
Partially supported by KBN grant No 2 PO3A 006 08.

Next we give the distributional version of Theorem 1. In that case it is more convenient to assume that F is holomorphic in the upper half plane $\{\operatorname{Im} z > 0\}$ and of exponential type. The condition (2) is replaced by the assumption that the boundary value $b(F)$ of F is a Laplace distribution on $\mathbb{R}$ (see Section 3). More precisely, we have

THEOREM 2 (Quasi-analyticity principle, distributional version). *Let* $F \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ *be of exponential type in* $\{\operatorname{Im} z \geq \varepsilon\}$ *for all* $\varepsilon > 0$. *If* $b(F) \in L'_{(\nu,\omega)}(\mathbb{R})$ *with some* $\nu < \omega$ *then* $F \equiv 0$.

In fact, Theorem 2 is a consequence of Theorem 1 and

THEOREM 3. *Let* $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ *with some* $R > 0$. *If* H *has a boundary value* $b(H) \in L'_{(\nu,\omega)}(\mathbb{R})$ *with some* $\nu, \omega \in \mathbb{R}$ *then for every* $a > \nu$ *and* $b < \omega$ *there exist* $0 < R' < R$ *and* $k \in \mathbb{N}$ *such that*

$$(3)\qquad |H(z)| \leq \begin{cases} Ce^{-a \operatorname{Re} z}/(\operatorname{Im} z)^k & \text{for } \operatorname{Re} z \leq 0,\ 0 < \operatorname{Im} z \leq R', \\ Ce^{-b \operatorname{Re} z}/(\operatorname{Im} z)^k & \text{for } \operatorname{Re} z \geq 0,\ 0 < \operatorname{Im} z \leq R'. \end{cases}$$

In the proof of Theorem 3 we follow an idea presented by Z. Szmydt and B. Ziemian in [SZ].

In Section 4 we give the ultradistributional versions of Theorems 2 and 3; namely

THEOREM 4 (Quasianalyticity principle, ultradistributional version). *Let* (M_p) *be a sequence of positive numbers satisfying the conditions* (M.1) *and* (M.3′) (see Section 4). *Let* $F \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ *be of exponential type in* $\{\operatorname{Im} z \geq \varepsilon\}$ *for all* $\varepsilon > 0$. *If* $b(F) \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ *with some* $\nu < \omega$ *then* $F \equiv 0$.

THEOREM 5. *Let* (M_p) *be a sequence of positive numbers satisfying the conditions* (M.1), (M.2) *and* (M.3) (see Section 4). *Let* $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ *with some* $R > 0$. *If* H *has a boundary value* $b(H) \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ *with some* $\nu \in \mathbb{R} \cup \{-\infty\}$ *and* $\omega \in \mathbb{R} \cup \{\infty\}$ *then for every* $a > \nu$ *and* $b < \omega$ *there exist* $0 < R' < R$ *and* $L < \infty$ *such that*

$$(4)\qquad |H(z)| \leq \begin{cases} C \exp\{-a \operatorname{Re} z + M^*(L/\operatorname{Im} z)\} \\ \qquad\qquad \text{for } \operatorname{Re} z \leq 0,\ 0 < \operatorname{Im} z \leq R', \\ C \exp\{-b \operatorname{Re} z + M^*(L/\operatorname{Im} z)\} \\ \qquad\qquad \text{for } \operatorname{Re} z \geq 0,\ 0 < \operatorname{Im} z \leq R'. \end{cases}$$

In the above M^* is the growth function of the sequence (M_p) defined by

$$M^*(\varrho) = \sup_{p \in \mathbb{N}_0} \ln \frac{M_0 p! \varrho^p}{M_p} \quad \text{for } \varrho > 0.$$

One can try to prove Theorem 5 by the method used in the proof of Theorem 3, replacing smooth functions by ultradifferentiable ones. In this way we get a slightly worse estimate than (4) (see Theorem 8), but still good enough to deduce Theorem 4 and only under the assumptions (M.1) and

(M.3′) on the sequence (M_p). To get the precise estimate (4) we represent the space of Laplace ultradistributions on $\mathbb{R}$ as a suitable quotient space of holomorphic functions on a tubular neighbourhood of $\mathbb{R}$ with some growth restrictions (see Theorem 9).

Theorems 1, 2 and 4 allow us to prove, in the final section, some generalizations of the Painlevé theorem.

2. Proof of the continuous version.

Let $F \in \mathcal{O}(\{\operatorname{Re} z > 0\}) \cap C^0(\{\operatorname{Re} z \geq 0\})$ satisfy (1) and (2). Let

$$\Psi_0(\zeta) = \int_0^\infty F(x) e^{x\zeta}\, dx, \qquad \operatorname{Re} \zeta < -c,$$

be the Laplace transform of $F_{|\mathbb{R}_+}$. By the uniqueness theorem for the Laplace transformation ([W], Theorem 6.3) it is sufficient to show that $\Psi_0 \equiv 0$. To this end define for $|\varphi| \leq \pi/2$,

$$\Psi_\varphi(\zeta) = \int_{l_\varphi} F(z) e^{z\zeta}\, dz \qquad \text{for } \zeta \in \Omega_\varphi,$$

where $l_\varphi = \{z \in \mathbb{C} : z = re^{i\varphi},\ 0 \leq r < \infty\}$ and Ω_φ is the set of all $\zeta \in \mathbb{C}$ such that

$$\begin{cases} c + \operatorname{Re} \zeta \cos\varphi - \operatorname{Im} \zeta \sin\varphi < 0 & \text{if } |\varphi| < \pi/2, \\ \pm \operatorname{Im} \zeta > c^{\pm} & \text{if } \varphi = \pm\pi/2. \end{cases}$$

Then $\Psi_\varphi \in \mathcal{O}(\Omega_\varphi)$ and since $\Psi_{\varphi_1}(\zeta) = \Psi_{\varphi_2}(\zeta)$ for $\zeta \in \Omega_{\varphi_1} \cap \Omega_{\varphi_2}$, where $\varphi_1, \varphi_2 \in [-\pi/2, \pi/2]$ with $|\varphi_1 - \varphi_2| < \pi$, it follows that Ψ_0 extends holomorphically to a function $\Psi \in \mathcal{O}(\Omega)$, where $\Omega = \bigcup_{|\varphi| \leq \pi/2} \Omega_\varphi$. Now the assumption $c^+ + c^- < 0$ implies that $\Omega = \mathbb{C}$ and to end the proof it is sufficient to note that for all $\varepsilon > 0$, Ψ is bounded on $\{\pm \operatorname{Im} \zeta \geq c^{\pm} + \varepsilon\}$ and $\Psi(i\eta) \to 0$ as $\eta \to \infty$.

3. Laplace distributions on $\mathbb{R}$.

Let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. The space $L'_{(\nu,\omega)}(\mathbb{R})$ of *Laplace distributions* on $\mathbb{R}$ is defined ([Z]) as the dual space of

$$L_{(\nu,\omega)}(\mathbb{R}) = \varinjlim_{a>\nu,\, b<\omega} L_{a,b}(\mathbb{R})$$

where for any $a, b \in \mathbb{R}$,

$$L_{a,b}(\mathbb{R}) = \varprojlim_{m \in \mathbb{N}_0} L_{a,b,m}(\mathbb{R})$$

with

$$L_{a,b,m}(\mathbb{R}) = \{\varphi \in C^m(\mathbb{R}) : \|\varphi\|_{a,b,m} = \sup_{\alpha \leq m} \sup_{x \in \mathbb{R}} |D^\alpha \varphi(x)| \kappa_{a,b}(x) < \infty\}$$

and

$$\kappa_{a,b}(x) = \begin{cases} e^{-ax} & \text{for } x \le 0, \\ e^{-bx} & \text{for } x \ge 0. \end{cases} \tag{5}$$

The spaces $L'_{(\nu,\emptyset)}(\overline{\mathbb{R}}_-)$ and $L'_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ of Laplace distributions on $\overline{\mathbb{R}}_-$ and $\overline{\mathbb{R}}_+$ are defined in an analogous way replacing $L_{a,b,m}(\mathbb{R})$ respectively by

$$L_{a,\emptyset,m}(\overline{\mathbb{R}}_-) = \{\varphi \in C^m(\overline{\mathbb{R}}_-) : \sup_{\alpha \le m} \sup_{x \in \overline{\mathbb{R}}_-} |D^\alpha \varphi(x)| e^{-ax} < \infty\}$$

and

$$L_{\emptyset,b,m}(\overline{\mathbb{R}}_+) = \{\varphi \in C^m(\overline{\mathbb{R}}_+) : \sup_{\alpha \le m} \sup_{x \in \overline{\mathbb{R}}_+} |D^\alpha \varphi(x)| e^{-bx} < \infty\}.$$

We have the topological inclusions

$$L'_{(\nu,\emptyset)}(\overline{\mathbb{R}}_-) \hookrightarrow L'_{(\nu,\omega)}(\mathbb{R}) \quad \text{for any } \omega \in \mathbb{R} \cup \{\infty\},$$
$$L'_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+) \hookrightarrow L'_{(\nu,\omega)}(\mathbb{R}) \quad \text{for any } \nu \in \mathbb{R} \cup \{-\infty\}.$$

Since $C_0^\infty(\mathbb{R})$ is dense in $L_{(\nu,\omega)}(\mathbb{R})$ we also have the topological inclusion $L'_{(\nu,\omega)}(\mathbb{R}) \hookrightarrow D'(\mathbb{R})$. The following theorem characterizes the image of $L'_{(\nu,\omega)}(\mathbb{R})$ under the above inclusion.

THEOREM 6 (Structure theorem). *In order that a distribution $S \in D'(\mathbb{R})$ belong to $L'_{(\nu,\omega)}(\mathbb{R})$ it is necessary and sufficient that for any $a > \nu$ and $b < \omega$ there are differential operators $P_a(D)$ and $P_b(D)$ and functions $S_a, S_b \in C^0(\mathbb{R})$ such that $\operatorname{supp} S_a \subset \overline{\mathbb{R}}_-$, $|S_a(x)| \le Ce^{-ax}$ for $x \le 0$, $\operatorname{supp} S_b \subset \overline{\mathbb{R}}_+$, $|S_b(x)| \le Ce^{-bx}$ for $x \ge 0$ and*

$$S = P_a(D)S_a + P_b(D)S_b \quad \text{in } L'_{(a,b)}(\mathbb{R}).$$

The proof of Theorem 6 follows easily from the structure theorem for the Laplace distributions on $\overline{\mathbb{R}}_-$ and $\overline{\mathbb{R}}_+$ (cf. [Ł1], Theorem 2), and

LEMMA 1. *Let $S \in L'_{(\nu,\omega)}(\mathbb{R})$. Then one can find $S^- \in L'_{(\nu,\emptyset)}(\overline{\mathbb{R}}_-)$ and $S^+ \in L'_{(\emptyset,\omega)}(\overline{\mathbb{R}}_+)$ such that $S = S^- + S^+$ in $L'_{(\nu,\omega)}(\mathbb{R})$. The decomposition is unique modulo a distribution with support at zero.*

Proof. Take a cut-off function $\chi \in D(\mathbb{R})$ equal to one in a neighbourhood of zero. Then $\chi S \in E'(\mathbb{R})$ and so one can find a differential operator $J(D)$ and a function $g \in C^0(\mathbb{R})$ such that $\chi S = J(D)g$. Put $g^+ = g \cdot H$, H the Heaviside function, $g^- = g - g^+$ and $S^+ = J(D)g^+ + (1-\chi)H \cdot S$, $S^- = J(D)g^- + (1-\chi)H^\vee \cdot S$, $H^\vee(x) = H(-x)$. Then S^+ and S^- satisfy the conclusion of Lemma 1.

Let $\delta, \sigma \in \mathbb{R}$. Define

$$\operatorname{ch}_{\delta,\sigma}(z) = \tfrac{1}{2}(e^{-\delta z} + e^{\sigma z}) \quad \text{for } z \in \mathbb{C}. \tag{6}$$

Note that the function $\operatorname{ch}_{\delta,\sigma}$ does not vanish on a tubular neighbourhood of $\mathbb{R}$. If $\delta + \sigma \ge 0$ then multiplication by $\operatorname{ch}_{\delta,\sigma}$ gives an isomorphism of $L_{a,b}(\mathbb{R})$ onto $L_{a-\delta,b+\sigma}(\mathbb{R})$ and consequently an isomorphism of $L_{(\nu,\omega)}(\mathbb{R})$ onto $L_{(\nu-\delta,\omega+\sigma)}(\mathbb{R})$, and of $L'_{(\nu,\omega)}(\mathbb{R})$ onto $L'_{(\nu+\delta,\omega-\sigma)}(\mathbb{R})$ with the inverse being multiplication by $1/\operatorname{ch}_{\delta,\sigma}$.

The next two lemmas will be useful in the proof of Theorem 3.

LEMMA 2. *Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-1,1]$ and let $a, b \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Then for any $\mathring{x} \in \mathbb{R}$ and $0 < r \le 1$,*

$$\left\|\varphi\left(\frac{\cdot - \mathring{x}}{r}\right)\right\|_{a,b,l} \le \frac{C_{a,b}}{r^l} \kappa_{a,b}(\mathring{x}) \|\varphi\|_{a,b,l}$$

where $C_{a,b} = \max(e^{|a|}, e^{|b|}, e^{|a-b|})$.

Proof. Assume that $\mathring{x} \le -1$. Since $\operatorname{supp}\varphi \subset [-1,1]$ we derive

$$\begin{aligned}
\left\|\varphi\left(\frac{\cdot - \mathring{x}}{r}\right)\right\|_{a,b,l}
&= \sup_{\alpha \le l} \frac{1}{r^\alpha} \sup_{\xi \in \overline{\mathbb{R}}_-} \left| D^\alpha \varphi\left(\frac{\xi - \mathring{x}}{r}\right)\right| e^{-a\xi} \\
&\le \frac{1}{r^l} \sup_{\alpha \le l} \sup_{|x - \mathring{x}| \le 1} |D^\alpha \varphi(x - \mathring{x})| e^{-a(r(x-\mathring{x})+\mathring{x})} \\
&= \frac{1}{r^l} \max\Big(\sup_{\alpha \le l} \sup_{-1 \le x - \mathring{x} \le 0} |D^\alpha \varphi(x - \mathring{x})| e^{-a(x-\mathring{x})+a(1-r)(x-\mathring{x})-a\mathring{x}}, \\
&\qquad \sup_{\alpha \le l} \sup_{0 \le x - \mathring{x} \le 1} |D^\alpha \varphi(x - \mathring{x})| e^{-b(x-\mathring{x})+(b-ar)(x-\mathring{x})-a\mathring{x}}\Big) \\
&\le \frac{\max(1, e^{-a}, e^b, e^{b-a})}{r^l} \kappa_{a,b}(\mathring{x}) \|\varphi\|_{a,b,l}.
\end{aligned}$$

The other cases $-1 \le \mathring{x} \le 0$, $0 \le \mathring{x} \le 1$ and $\mathring{x} \ge 1$ can be treated in the same way.

LEMMA 3. *Let $\varrho \in C_0^\infty(\mathbb{C})$ be a radial function such that $\operatorname{supp}\varrho \subset \{z \in \mathbb{C} : |z| < 1\}$ and $\int \varrho(x,y)\,dx\,dy = (2\pi)^{-1}$. Then for any $F \in \mathcal{O}(\{z \in \mathbb{C} : |z| < 1\})$,*

$$\int \varrho(x+iy)F(x+iy)\,dx\,dy = F(0).$$

DEFINITION. Let $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ with some $R > 0$ and let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. Assume that for any $a > \nu$ and $b < \omega$ there exists $R' > 0$ such that $H_y \in L'_{a,b}(\mathbb{R})$ for $0 < y \le R'$, where

$$H_y[\varphi] := \int_{\mathbb{R}} H(x+iy)\varphi(x)\,dx \quad \text{for } \varphi \in L_{a,b}(\mathbb{R}).$$

If for every $\varphi \in L_{(\nu,\omega)}(\mathbb{R})$ the limit $\lim_{y\to 0} H_y[\varphi] =: u[\varphi]$ exists then, by the Banach–Steinhaus theorem, $u \in L'_{(\nu,\omega)}(\mathbb{R})$ and we say that H has *boundary value* $u = b(H)$ in $L'_{(\nu,\omega)}(\mathbb{R})$.

Proof of Theorem 3. Suppose $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ has a boundary value $u = b(H) \in L'_{(\nu,\omega)}(\mathbb{R})$. Fix $a > \nu$ and $b < \omega$. By definition $H_y \in L'_{a,b}(\mathbb{R})$ for $0 < y \le R''$ with some $R'' > 0$ and the limit $\lim_{y\to 0} H_y = u \in L'_{a,b}(\mathbb{R})$ exists. So one can find $\widetilde{C} < \infty$ and $l \in \mathbb{N}_0$ such that

$$(7) \qquad |H_y[\varphi]|, |u[\varphi]| \le \widetilde{C}\|\varphi\|_{a,b,l} \quad \text{for } \varphi \in L_{a,b,l}(\mathbb{R}) \text{ and } 0 < y \le R''.$$

Take $0 < R' < \min(1, R'')$ and $0 < c < \min(1, R''/R' - 1)$. Fix $\mathring{x} \in \mathbb{R}$ and $\mathring{y} \in (0, R']$ and set $\mathring{z} = \mathring{x} + i\mathring{y}$. Then $H \in \mathcal{O}(\mathbb{R} + i\{y : |y - \mathring{y}| \le c\mathring{y}\})$. Let $\psi \in C_0^\infty(\{x : |x - \mathring{x}| \le c\mathring{y}\})$ and define $\mu(x) = \psi(x + \mathring{x})$ and $g(z) = H(z + \mathring{z})$. Note that $\mu \in C_0^\infty(\{|x| \le c\mathring{y}\})$, $g \in \mathcal{O}(\mathbb{R} + i\{|y| \le c\mathring{y}\})$ and $H(\mathring{z}) = g(0)$. So if we take ψ with $\|\psi\|_{a,b,l} \le 1/\widetilde{C}$ then by (7),

$$(8) \qquad \Big|\int g(x+iy)\mu(x)\,dx\Big| = \Big|\int H(x + \mathring{x} + i(y + \mathring{y}))\psi(x + \mathring{x})\,dx\Big| = \Big|\int H(x + i(y+\mathring{y}))\psi(x)\,dx\Big| = |H_{y+\mathring{y}}[\psi]| \le 1 \quad \text{for } y \text{ with } |y| \le c\mathring{y}.$$

Define $g_{c\mathring{y}}(x + iy) = g(c\mathring{y}x + ic\mathring{y}y)$. Then $g_{c\mathring{y}} \in \mathcal{O}(\mathbb{R} + i\{|y| \le 1\})$ and by Lemma 3 we get

$$(9) \qquad H(\mathring{z}) = g_{c\mathring{y}}(0) = \int \varrho(x+iy) g(c\mathring{y}x + ic\mathring{y}y)\,dx\,dy = \frac{1}{(c\mathring{y})^2}\int \varrho\Big(\frac{\xi}{c\mathring{y}} + i\frac{\eta}{c\mathring{y}}\Big) g(\xi + i\eta)\,d\xi\,d\eta,$$

where ϱ is given in Lemma 3. Now fix η such that $|\eta| \le c\mathring{y}$. We shall apply (8) to the function $\xi \mapsto \sigma(\xi,\eta)$, where

$$\sigma(\xi,\eta) = \frac{(c\mathring{y})^l}{M_l\widetilde{C}C_{a,b}}\varrho\Big(\frac{\xi - \mathring{x}}{c\mathring{y}} + i\frac{\eta}{c\mathring{y}}\Big)\kappa_{-a,-b}(\mathring{x})$$

with $M_l = \sup_{|y|\le 1}\|\varrho(\cdot + iy)\|_{a,b,l}$ and $C_{a,b}$ given in Lemma 2. Then $\|\sigma(\cdot,\eta)\|_{a,b,l} \le 1/\widetilde{C}$. So by (9) and (8) with $\widetilde{\sigma}(\xi,\eta) = \sigma(\xi + \mathring{x}, \eta)$ in place of μ we derive

$$|H(\mathring{z})| = \frac{1}{(c\mathring{y})^2}\Big|\iint \varrho\Big(\frac{\xi}{c\mathring{y}} + i\frac{\eta}{c\mathring{y}}\Big) g(\xi + i\eta)\,d\xi\,d\eta\Big| \le \frac{M_l\widetilde{C}C_{a,b}}{(c\mathring{y})^{2+l}}\kappa_{a,b}(\mathring{x})\Big|\int d\eta\int g(\xi+i\eta)\widetilde{\sigma}(\xi,\eta)\,d\xi\Big| \le \frac{M_l\widetilde{C}C_{a,b}}{(c\mathring{y})^{2+l}}\kappa_{a,b}(\mathring{x})\Big|\int_{|\eta|\le c\mathring{y}} d\eta\Big| \le C\frac{\kappa_{a,b}(\mathring{x})}{\mathring{y}^k}$$

with $k = l + 1$.

Proof of Theorem 2. Let $F \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ be of exponential type in $\{\operatorname{Im} z > 0\}$ and have a boundary value $b(F) \in L'_{(\nu,\omega)}(\mathbb{R})$ with some $\nu < \omega$. Fix $\nu < a < b < \omega$. By Theorem 3 we can find $R > 0$ and $k \in \mathbb{N}$ such that

$$|F(z)| \le \begin{cases} Ce^{-a\operatorname{Re} z}/(\operatorname{Im} z)^k & \text{if } \operatorname{Re} z \le 0,\ 0 < \operatorname{Im} z \le R, \\ Ce^{-b\operatorname{Re} z}/(\operatorname{Im} z)^k & \text{if } \operatorname{Re} z \ge 0,\ 0 < \operatorname{Im} z \le R, \\ Ce^{c_R|z|} & \text{if } \operatorname{Im} z \ge R. \end{cases}$$

Thus, the function $H(z) = F(i(z + R))$, $\operatorname{Re} z \ge 0$, satisfies the assumptions of Theorem 1 with $c^+ = a$ and $c^- = -b$. Since $c^+ + c^- = a - b < 0$ we get $H \equiv 0$ and consequently $F \equiv 0$.

4. Laplace ultradistributions on $\mathbb{R}$. Let $(M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers. We consider the conditions

(M.1) $\quad M_p^2 \le M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$;

(M.2′) $\quad M_p \le H^p M_{p-1}$ for $p \in \mathbb{N}$ with some $H < \infty$;

(M.2) $\quad M_p \le H^p M_q M_{p-q}$ for $p \in \mathbb{N}$, $0 < q \le l$ with some $H < \infty$;

(M.3′) $\quad \displaystyle\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p} < \infty$;

(M.3) $\quad \displaystyle\sum_{p=q}^{\infty}\frac{M_{p-1}}{M_p} \le Aq\frac{M_q}{M_{q+1}}$ for $q \in \mathbb{N}$ with some $A < \infty$.

We refer to [K] or [M] for the significance of these conditions. We always assume (M.1), (M.3′) and $M_0 = 1$.

Let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. The space $L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ of *Laplace ultradistributions* on $\mathbb{R}$ is defined as the dual space of

$$L^{(M_p)}_{(\nu,\omega)}(\mathbb{R}) = \varinjlim_{a>\nu,\, b<\omega} L^{(M_p)}_{a,b}(\mathbb{R})$$

where for any $a, b \in \mathbb{R}$,

$$L_{a,b}^{(M_p)}(\mathbb{R}) = \varinjlim_{h>0} L_{a,b,h}^{(M_p)}(\mathbb{R})$$

with

$$L_{a,b,h}^{(M_p)}(\mathbb{R}) = \left\{ \varphi \in C^\infty(\mathbb{R}) : \|\varphi\|_{a,b,h}^{(M_p)} = \sup_{x\in\mathbb{R}} \sup_{\alpha\in\mathbb{N}_0} \frac{|D^\alpha\varphi(x)|\kappa_{a,b}(x)}{h^\alpha M_\alpha} < \infty \right\}$$

and $\kappa_{a,b}$ given by (5).

Multiplication by the function $\mathrm{ch}_{\delta,\sigma}$ with $\delta+\sigma \geq 0$ gives an isomorphism of $L_{a,b}^{(M_p)}(\mathbb{R})$ onto $L_{a-\delta,b+\sigma}^{(M_p)}(\mathbb{R})$ and consequently an isomorphism of $L_{(\nu,\omega)}^{(M_p)}(\mathbb{R})$ onto $L_{(\nu-\delta,\omega+\sigma)}^{(M_p)}(\mathbb{R})$, and of $L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$ onto $L_{(\nu+\delta,\omega-\sigma)}^{(M_p)\prime}(\mathbb{R})$ with the inverse being multiplication by $1/\mathrm{ch}_{\delta,\sigma}$.

As in the case of Laplace distributions we can also define the spaces $L_{(\nu,\emptyset)}^{(M_p)\prime}(\overline{\mathbb{R}}_-)$ and $L_{(\emptyset,\omega)}^{(M_p)\prime}(\overline{\mathbb{R}}_+)$ of Laplace ultradistributions on $\overline{\mathbb{R}}_-$ and $\overline{\mathbb{R}}_+$. We have the topological inclusions

$$L_{(\nu,\emptyset)}^{(M_p)\prime}(\overline{\mathbb{R}}_-) \hookrightarrow L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R}) \quad \text{for any } \omega \in \mathbb{R}\cup\{\infty\},$$

$$L_{(\emptyset,\omega)}^{(M_p)\prime}(\overline{\mathbb{R}}_+) \hookrightarrow L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R}) \quad \text{for any } \nu \in \mathbb{R}\cup\{-\infty\}.$$

Since $D^{(M_p)}(\mathbb{R})$ is dense in $L_{(\nu,\omega)}^{(M_p)}(\mathbb{R})$ the space $L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$ is imbedded into $D^{(M_p)\prime}(\mathbb{R})$ and we have a counterpart of Theorem 6.

THEOREM 7 (Structure theorem). *Assume* (M.1), (M.2) *and* (M.3). *In order that an ultradistribution* $S \in D^{(M_p)\prime}(\mathbb{R})$ *belong to* $L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$ *it is necessary and sufficient that for any* $a > \nu$ *and* $b < \omega$ *there are ultradifferential operators* $J_a(D)$ *and* $J_b(D)$ *of class* (M_p) (cf. [K]) *and functions* $S_a, S_b \in C^0(\mathbb{R})$ *such that* $\operatorname{supp} S_a \subset \overline{\mathbb{R}}_-, |S_a(x)| \leq Ce^{-ax}$ *for* $x \leq 0$, $\operatorname{supp} S_b \subset \overline{\mathbb{R}}_+, |S_b(x)| \leq Ce^{-bx}$ *for* $x \geq 0$ *and*

$$S = J_a(D)S_a + J_b(D)S_b \quad \text{in } L_{(a,b)}^{(M_p)\prime}(\mathbb{R}).$$

The proof of Theorem 7 is based on the structure theorem for Laplace ultradistributions on $\overline{\mathbb{R}}_-$ and $\overline{\mathbb{R}}_+$ (cf. [Ł2], Theorem 4), and on the following analogue of Lemma 1.

LEMMA 4. *Assume* (M.1), (M.2) *and* (M.3). *Let* $S \in L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$. *Then one can find* $S^- \in L_{(\nu,\emptyset)}^{(M_p)\prime}(\overline{\mathbb{R}}_-)$ *and* $S^+ \in L_{(\emptyset,\omega)}^{(M_p)\prime}(\overline{\mathbb{R}}_+)$ *such that* $S = S^- + S^+$ *in* $L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$. *The decomposition is unique modulo an ultradistribution of class* (M_p) *with support at zero.*

Proof. We follow the proof of Lemma 1, this time taking $\chi \in D^{(M_p)}(\mathbb{R})$, an ultradifferential operator $J(D)$ of class (M_p) and a function $g \in C^0(\mathbb{R})$ such that $\chi S = J(D)g$.

DEFINITION. Let $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ with some $R > 0$ and let $\nu \in \mathbb{R}\cup\{-\infty\}$ and $\omega \in \mathbb{R}\cup\{\infty\}$. Assume that for any $a > \nu$ and $b < \omega$ there exists $R' > 0$ such that $H_y \in L_{a,b}^{(M_p)\prime}(\mathbb{R})$ for $0 < y \leq R'$. If for every $\varphi \in L_{(\nu,\omega)}^{(M_p)}(\mathbb{R})$ the limit $\lim_{y\to 0} H_y[\varphi] =: u[\varphi]$ exists then, by the Banach–Steinhaus theorem, $u \in L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$ and we say that H has *boundary value* $u = b(H)$ in $L_{(\nu,\omega)}^{(M_p)\prime}(\mathbb{R})$.

One would like to prove Theorem 5 by the method used in the proof of Theorem 3. To this end we should state analogues of Lemmas 2 and 3. Since there is no problem with Lemma 3 (we only have to replace a smooth function ϱ by that of class $D^{(M_p)}(\{z \in \mathbb{C} : |z| < 1\})$), we shall concentrate on Lemma 2.

So take $\varphi \in D^{(M_p)}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-1,1]$ and fix $\mathring{x} \in \mathbb{R}$, $0 < r \leq 1$ and $a, b \in \mathbb{R}$. Following the proof of Lemma 2 we easily arrive at the estimate

$$\left\|\varphi\left(\frac{\cdot - \mathring{x}}{r}\right)\right\|_{a,b,h}^{(M_p)} \leq C_{a,b}\kappa_{a,b}(\mathring{x})\|\varphi\|_{a,b,rh}^{(M_p)}, \tag{10}$$

where $\kappa_{a,b}$ is given by (5). Now, in general, we cannot estimate $\|\varphi\|_{a,b,rh}^{(M_p)}$ by $C\|\varphi\|_{a,b,ch}^{(M_p)}$ with some $c > 0$ independent of $0 < r \leq 1$.

But we have

LEMMA 5. *Let* (M_p) *satisfy* (M.1) *and* (M.3′). *Then there exists a sequence* (Q_p) *satisfying* (M.1) *and* (M.3′) *such that* $(Q_p) \prec (M_p)$ (*i.e. for any* $h > 0$ *there exists* $C < \infty$ *such that* $Q_p \leq Ch^pM_p$ *for* $p \in \mathbb{N}_0$). *Moreover, given such sequences* (M_p) *and* (Q_p) *one can find a sequence* $(N_p) \succ (p!)$ *such that*

$$N_pQ_p \leq p!M_p \quad \text{for } p \in \mathbb{N}_0. \tag{11}$$

Furthermore, for $\varphi \in D^{(Q_p)}(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset [-1,1]$ *and for* $\mathring{x} \in \mathbb{R}$, $a, b \in \mathbb{R}$, $h > 0$ *and* $0 < r \leq 1$ *we have*

$$\left\|\varphi\left(\frac{\cdot - \mathring{x}}{r}\right)\right\|_{a,b,h}^{(M_p)} \leq C_{a,b}\kappa_{a,b}(\mathring{x})\exp N^*\left(\frac{1}{r}\right)\|\varphi\|_{a,b,h}^{(Q_p)}. \tag{12}$$

Note that the condition $(p!) \prec (N_p)$ is equivalent to the finiteness of the growth function N^* for (N_p).

Proof. The existence of a sequence (Q_p) satisfying (M.1), (M.3′) and $(Q_p) \prec (M_p)$ is well known (cf. [R], pp. 66–67). Now if we put

$$N_p = \frac{p!M_p}{Q_p} \quad \text{for } p \in \mathbb{N}_0$$

then $(p!) \prec (N_p)$ and (11) holds. Thus, we only have to prove (12), and this follows by (10) and (11), since

$$\|\varphi\|^{(M_p)}_{a,b,rh} \le \exp N^*\left(\frac{1}{r}\right)\|\varphi\|^{(Q_p)}_{a,b,h}.$$

Now following the proof of Theorem 3 with Lemma 5 in place of Lemma 2 we obtain

THEOREM 8. *Let (M_p) satisfy* (M.1) *and* (M.3′). *Let $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ with some $R > 0$ and assume that H has a boundary value $b(H) \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ with some $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. Then there exists a sequence (N_p) such that $(M_p) \succ (N_p) \succ (p!)$ and for every $a > \nu$ and $b < \omega$ one can find $0 < R' < R$ and $L < \infty$ such that*

$$(13)\quad |H(z)| \le \begin{cases} C\exp\{-a\operatorname{Re} z + N^*(L/\operatorname{Im} z)\} & \text{for } \operatorname{Re} z \le 0,\ 0 < \operatorname{Im} z \le R',\\ C\exp\{-b\operatorname{Re} z + N^*(L/\operatorname{Im} z)\} & \text{for } \operatorname{Re} z \ge 0,\ 0 < \operatorname{Im} z \le R'.\end{cases}$$

Proof of Theorem 4. We follow the proof of Theorem 2, this time using (13) instead of (3).

5. Boundary value representation. Throughout this section W denotes a tubular neighbourhood of $\mathbb{R}$, i.e. a set of the type $\mathbb{R} + iU$, where U is a bounded neighbourhood of zero in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ and $h > 0$. We define the spaces

$$\mathfrak{L}^{(M_p)}_{a,b,h}(W \setminus \mathbb{R}) = \{F \in \mathcal{O}(W \setminus \mathbb{R}) :$$
$$q_{a,b,h,\widetilde{W}}(F) = \sup_{z\in\widetilde{W}\setminus\mathbb{R}} |F(z)\kappa_{-a,-b}(\operatorname{Re} z)\exp\{-M^*(h/|\operatorname{Im} z|)\}| < \infty$$
$$\text{for any closed tubular subset } \widetilde{W} \subset W\};$$

$$\mathfrak{L}^{(M_p)}_{a,b}(W \setminus \mathbb{R}) = \varinjlim_{h>0} \mathfrak{L}^{(M_p)}_{a,b,h}(W \setminus \mathbb{R});$$

$$\mathfrak{L}_{a,b}(W) = \{G \in \mathcal{O}(W) : q_{a,b,\widetilde{W}}(G) = \sup_{z\in\widetilde{W}} |G(z)\kappa_{-a,-b}(\operatorname{Re} z)| < \infty$$
$$\text{for any closed tubular subset } \widetilde{W} \subset W\}.$$

It follows by the 3-line theorem that $\mathfrak{L}_{a,b}(W)$ is a closed subspace of

$\mathfrak{L}^{(M_p)}_{a,b}(W \setminus \mathbb{R})$. Thus, we can define the quotient space

$$H^{(M_p)}_{a,b}(W, \mathbb{R}) = \mathfrak{L}^{(M_p)}_{a,b}(W \setminus \mathbb{R})/\mathfrak{L}_{a,b}(W).$$

Now, let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. We define

$$\mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R}) = \varprojlim_{a>\nu,b<\omega} \mathfrak{L}^{(M_p)}_{a,b}(W \setminus \mathbb{R}),$$
$$\mathfrak{L}_{(\nu,\omega)}(W) = \varprojlim_{a>\nu,b<\omega} \mathfrak{L}_{a,b}(W)$$

and

$$H^{(M_p)}_{(\nu,\omega)}(W, \mathbb{R}) = \varprojlim_{a>\nu,b<\omega} H^{(M_p)}_{a,b}(W, \mathbb{R}).$$

By the Mittag-Leffler lemma (cf. [K]) we also have

$$H^{(M_p)}_{(\nu,\omega)}(W, \mathbb{R}) \simeq \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W \setminus \mathbb{R})/\mathfrak{L}_{(\nu,\omega)}(W).$$

In Theorem 9 we shall prove that the space $H^{(M_p)}_{(\nu,\omega)}(W, \mathbb{R})$ is isomorphic to $L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and consequently, it does not depend on the choice of a tubular neighbourhood W of $\mathbb{R}$. In the proof we shall use, as a Cauchy kernel, the function

$$\Lambda(\zeta, z) = \frac{\exp\{-(\zeta - z)^2\}}{\zeta - z}.$$

LEMMA 6. *Assume* (M.1), (M.2′) *and* (M.3′). *Fix $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Then for any $a, b \in \mathbb{R}$ the function $\mathbb{R} \ni x \mapsto \Lambda(\zeta, x)$ belongs to $L^{(M_p)}_{a,b}(\mathbb{R})$. In particular, for any $h > 0$ one can find $C = C(a, b, h) < \infty$ and $L = L(h) < \infty$ such that*

$$\|\Lambda(\zeta, \cdot)\|^{(M_p)}_{a,b,h} \le \begin{cases} C\kappa_{a,b}(\operatorname{Re}\zeta)\exp\{M^*(L/|\operatorname{Im}\zeta|)\} & \text{for } 0 < |\operatorname{Im}\zeta| \le 1,\\ C\kappa_{a,b}(\operatorname{Re}\zeta)\exp\{(|\operatorname{Im}\zeta| + 1)^2\} & \text{for } |\operatorname{Im}\zeta| \ge 1.\end{cases}$$

Proof. Applying the Leibniz formula we get (by (M.1))

$$\|\Lambda(\zeta, \cdot)\|^{(M_p)}_{a,b,h} = \sup_{\alpha\in\mathbb{N}_0}\sup_{x\in\mathbb{R}} \frac{\kappa_{a,b}(x)\left|D^\alpha\left(\frac{\exp\{-(\zeta-x)^2\}}{\zeta-x}\right)\right|}{h^\alpha M_\alpha}$$
$$\le \sup_{\alpha\in\mathbb{N}_0}\sup_{x\in\mathbb{R}} \kappa_{a,b}(x)$$
$$\times \sum_{\beta\le\alpha} \frac{2^{\alpha-\beta}|D^{\alpha-\beta}(\exp\{-(\zeta - x)^2\})|}{h^{\alpha-\beta}M_{\alpha-\beta}} \cdot \frac{2^\beta\left|D^\beta\left(\frac{1}{\zeta-x}\right)\right|}{h^\beta M_\beta}.$$

By (M.2′) the second factor under the sum sign can be estimated by $C\exp\{M^*(L/|\operatorname{Im}\zeta|)\}$ with some $L < \infty$. To estimate the first factor define

$$F_\zeta(z)=\exp\{-(\zeta-z)^2\}\quad \text{for } z\in\mathbb{C},\ \zeta\in\mathbb{C}.$$

Using the formula

$$D^\gamma F_\zeta(x)=\frac{\gamma!}{2\pi}\int_0^{2\pi}F_\zeta(x+e^{i\varphi})e^{-i\varphi\gamma}\,d\varphi\quad \text{for } x\in\mathbb{R},\ \gamma\in\mathbb{N}_0$$

we easily obtain

$$|D^\gamma F_\zeta(x)|\le e\gamma!\exp\{(|\eta|+1)^2-(|\xi-x|-1)^2\}\quad \text{where } \zeta=\xi+i\eta.$$

Now, by calculations in the spirit of those in the proof of Lemma 2 we get

$$\sup_{x\in\mathbb{R}}\kappa_{a,b}(x)|D^\gamma F_\zeta(x)|\le C_{a,b}\gamma!\kappa_{a,b}(\operatorname{Re}\zeta)\exp\{(|\operatorname{Im}\zeta|+1)^2\}$$

with some $C_{a,b}<\infty$. Finally,

$$\sup_{\alpha\in\mathbb{N}_0}\sum_{\beta\le\alpha}\frac{2^{\alpha-\beta}(\alpha-\beta)!}{h^{\alpha-\beta}M_{\alpha-\beta}}<\infty$$

since $(p!)\prec(M_p)$ (by (M.1) and (M.3′)), and this ends the proof.

PROPOSITION 1. *Assume* (M.1), (M.2′) *and* (M.3′). *Let* $S\in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ *and put*

$$\mathcal{C}_\Lambda S(\zeta)=\frac{-1}{2\pi i}S[\Lambda(\zeta,\cdot)]\quad \textit{for } \zeta\in\mathbb{C}\setminus\mathbb{R}. \tag{14}$$

Then $\mathcal{C}_\Lambda S\in\mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ *and for any* $a>\nu$ *and* $b<\omega$ *one can find* $C=C(a,b)<\infty$ *and* $L=L(a,b)<\infty$ *such that*

$$|\mathcal{C}_\Lambda S(\zeta)|\le\begin{cases}C\kappa_{a,b}(\operatorname{Re}\zeta)\exp\{M^*(L/|\operatorname{Im}\zeta|)\} & \textit{for } 0<|\operatorname{Im}\zeta|\le1,\\ C\kappa_{a,b}(\operatorname{Re}\zeta)\exp\{(|\operatorname{Im}\zeta|+1)^2\} & \textit{for } |\operatorname{Im}\zeta|\ge1.\end{cases} \tag{15}$$

Thus, $\mathcal{C}_\Lambda S\in\mathfrak{L}^{(M_p)}_{(\nu,\omega)}(\mathbb{C}\setminus\mathbb{R})$.

We call the map

$$L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})\ni S\mapsto\mathcal{C}_\Lambda S\in\mathfrak{L}^{(M_p)}_{(\nu,\omega)}(\mathbb{C}\setminus\mathbb{R})$$

the *Λ-Cauchy transformation.*

Proof. Since the holomorphy of $\mathcal{C}_\Lambda S$ follows from the continuity of S by standard arguments we only need to prove (15). To this end fix $a>\nu$ and $b<\omega$. Then by the continuity of S one can find $C<\infty$ and $h>0$ such that

$$|S[\varphi]|\le C\|\varphi\|^{(M_p)}_{a,b,h}\quad \text{for } \varphi\in L^{(M_p)}_{a,b}(\mathbb{R}).$$

So (15) is a consequence of Lemma 6.

PROPOSITION 2. *Assume* (M.1), (M.2) *and* (M.3). *Let* $H\in\mathcal{O}(\{0<\operatorname{Im}z<R\})$ *be such that for every* $a>\nu$ *and* $b<\omega$ *one can find* $L<\infty$ *such that*

$$|H(z)|\le C\kappa_{a,b}(\operatorname{Re}z)\exp\{M^*(L/|\operatorname{Im}z|)\}\quad \textit{for } 0<\operatorname{Im}z\le R'.$$

Then H *has the boundary value* $b(H)\in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$.

Proof. Fix $a>\nu$ and $b<\omega$. Choose $a>a'>\nu$, $b<b'<\omega$ and $\delta,\sigma\in\mathbb{R}$ such that $a'+\delta>0$, $b'-\sigma<0$ and $\delta+\sigma\ge0$ (if $\nu\ge0$ and $\omega\le0$ we can take $\delta=\sigma=0$). Consider the function

$$H_{\delta,\sigma}(z)=\operatorname{ch}_{\delta,\sigma}(z)\cdot H(z)\quad \text{for } 0<\operatorname{Im}z<R.$$

Then we can find $R'>0$ and $L<\infty$ such that

$$|H_{\delta,\sigma}(z)|\le C\kappa_{a'+\delta,b'-\sigma}(\operatorname{Re}z)\exp\{M^*(L/|\operatorname{Im}z|)\}\quad \text{for } 0<\operatorname{Im}z\le R'.$$

Now for $z\in\mathbb{C}$ with $0<\operatorname{Im}z\le R'':=R'/2$ put

$$\widetilde{H}_{\delta,\sigma}(z)=i\int_\gamma G(i(z-w))H_{\delta,\sigma}(w)\,dw,$$

where γ is a closed curve starting from $\mathring{z}=\frac{3}{4}R'\cdot i$, encircling z once in the positive direction and such that $|\arg i(z-\mathring{z})|<\pi/2$ for $z\in\gamma$; G is the Green kernel for

$$P(\zeta)=(1+\zeta)^2\prod_{p=1}^\infty\left(1+\frac{L\zeta}{m_p}\right)$$

($m_p=M_p/M_{p-1}$ for $p\in\mathbb{N}$) defined for $\operatorname{Re}z<0$ by

$$G(z)=\frac{1}{2\pi i}\int_0^\infty\frac{e^{z\zeta}}{P(\zeta)}\,d\zeta$$

and holomorphically continued to the Riemann domain $\{-\pi/2<\arg z<5\pi/2\}$. Then by Lemma 11.4 of [K] and the observation made in the proof of Proposition 2 of [Ł2] we have

$$P(D)\widetilde{H}_{\delta,\sigma}(z)=H_{\delta,\sigma}(z)\quad \text{for } 0<\operatorname{Im}z\le R'',$$
$$\widetilde{H}_{\delta,\sigma}\in\mathcal{O}(\{0<\operatorname{Im}z<R''\})\cap C^0(\{0\le\operatorname{Im}z\le R''\})$$

and

$$|\widetilde{H}_{\delta,\sigma}(z)|\le C_\varepsilon\kappa_{a'+\delta+\varepsilon,b'-\sigma-\varepsilon}(\operatorname{Re}z)\quad \text{for } 0\le\operatorname{Im}z\le R'' \text{ with any } \varepsilon>0.$$

Thus, choosing $\varepsilon>0$ small enough, we find that

$$\widetilde{H}_{\delta,\sigma}(\cdot+iy)\in L^{(M_p)\prime}_{(a+\delta,b-\sigma)}(\mathbb{R})\quad \text{for } 0<y\le R''$$

and the limit

$$\lim_{y\to0}\widetilde{H}_{\delta,\sigma}(\cdot+iy)=:\widetilde{h}_{\delta,\sigma}\in L^{(M_p)\prime}_{(a+\delta,b-\sigma)}(\mathbb{R})$$

exists. Now observe that for $\varphi \in L^{(M_p)}_{(a,b)}(\mathbb{R})$ and $y > 0$ small enough,

$$\begin{aligned}H(\cdot + iy)[\varphi] &= \int_{\mathbb{R}} H_{\delta,\sigma}(x+iy)\frac{\varphi(x)}{\operatorname{ch}_{\delta,\sigma}(x+iy)}\,dx\\ &= \int_{\mathbb{R}} P(D)\widetilde{H}_{\delta,\sigma}(x+iy)\frac{\varphi(x)}{\operatorname{ch}_{\delta,\sigma}(x+iy)}\,dx\\ &= \widetilde{H}_{\delta,\sigma}\Big[P^*(D)\frac{\varphi}{\operatorname{ch}_{\delta,\sigma}(\cdot+iy)}\Big].\end{aligned}$$

Thus $H(\cdot+iy) \in L^{(M_p)\prime}_{(a,b)}(\mathbb{R})$ and if we put

$$h = \frac{1}{\operatorname{ch}_{\delta,\sigma}}P(D)\widetilde{h}_{\delta,\sigma} \in L^{(M_p)\prime}_{(a,b)}(\mathbb{R})$$

then

$$\lim_{y\to 0} H(\cdot+iy)[\varphi] = \widetilde{h}_{\delta,\sigma}\Big[P^*(D)\frac{\varphi}{\operatorname{ch}_{\delta,\sigma}}\Big] = h[\varphi] \quad \text{for } \varphi \in L^{(M_p)}_{(a,b)}(\mathbb{R}).$$

Since $a > \nu$ and $b < \omega$ were arbitrary this ends the proof.

In the proof of Theorem 9 we shall use the lemma stated below. To formulate it let us define for $\nu \in \mathbb{R}\cup\{-\infty\}$ and $\omega \in \mathbb{R}\cup\{\infty\}$,

$$\underset{\sim}{L}_{(\nu,\omega)}(\mathbb{R}) = \varinjlim_{W\supset\mathbb{R}}\ \varinjlim_{a>\nu,\,b<\omega} \underset{\sim}{L}_{a,b}(W),$$

where for any $a,b \in \mathbb{R}$ and any tubular neighbourhood W of $\mathbb{R}$,

$$\underset{\sim}{L}_{a,b}(W) = \{F \in \mathcal{O}(W)\cap C^0(\overline{W}) : p_{a,b,W}(F) := \sup_{z\in\overline{W}}|F(z)\kappa_{a,b}(\operatorname{Re} z)| < \infty\}.$$

LEMMA 7. *Assume* (M.1), (M.2′) *and* (M.3′). *The space* $\underset{\sim}{L}_{(\nu,\omega)}(\mathbb{R})$ *is contained in* $L^{(M_p)}_{(\nu,\omega)}(\mathbb{R})$. *Therefore it is a dense subset of* $L^{(M_p)}_{(\nu,\omega)}(\mathbb{R})$.

Proof. Since functions from $\underset{\sim}{L}_{(\nu,\omega)}(\mathbb{R})$ are holomorphic on a tubular neighbourhood of $\mathbb{R}$ the first assertion is clear. To prove the second one observe that it is sufficient to show that for all $a,b \in \mathbb{R}$ and $h > 0$, each function $\varphi \in L^{(M_p)}_{a,b}(\mathbb{R})$ can be approximated by elements of $\underset{\sim}{L}_{a,b}(W)$ with some $W \supset \mathbb{R}$ in the topology of $L^{(M_p)}_{a,b,h}(\mathbb{R})$. So fix $a,b \in \mathbb{R}$, $h > 0$ and $\varphi \in L^{(M_p)}_{a,b}(\mathbb{R})$. Since the spaces $L^{(M_p)}_{a,b}(\mathbb{R})$ (resp. $\underset{\sim}{L}_{a,b}(W)$ with W narrow enough) and $L^{(M_p)}_{0,0}(\mathbb{R})$ (resp. $\underset{\sim}{L}_{0,0}(W)$) are isomorphic with the isomorphism being multiplication by $\operatorname{ch}_{a,-b}$ if $a \ge b$ and by $1/\operatorname{ch}_{-a,b}$ if $a < b$ (see (6)), we can assume $a = b = 0$. Put

$$F_j(z) = \frac{j}{\sqrt{\pi}}\int_{\mathbb{R}} \exp\{-j^2(z-t)^2\}\varphi(t)\,dt \quad \text{for } z \in \mathbb{C},\ j \in \mathbb{N}.$$

If W is contained in $\mathbb{R}+i[-R,R]$ with some $R < \infty$ and φ is bounded by C then $p_{0,0,W}(F_j) \le C\exp\{j^2R^2\}$ for $j \in \mathbb{N}$. Next observe that for $x \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0$,

$$D^\alpha(F_j(x) - \varphi(x)) = \frac{j}{\sqrt{\pi}}\int_{\mathbb{R}} \exp\{-j^2r^2\}D^\alpha(\varphi(x+r) - \varphi(r))\,dr.$$

Thus, since by (M.2′),

$$\sup_{x\in\mathbb{R}}|D^{\alpha+1}\varphi(x)| \le C_h(Hh)^\alpha M_\alpha \quad \text{for } \alpha \in \mathbb{N}_0$$

with some $C_h < \infty$ and $H < \infty$, we obtain

$$\|F_j - \varphi\|_{0,0,h} \le \frac{C}{j}\|\varphi\|_{0,0,h/H}$$

proving the lemma.

Let $h \in H^{(M_p)}_{(\nu,\omega)}(W,\mathbb{R})$. Then $h = [H]$ with some $H \in \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W\setminus\mathbb{R})$. By Proposition 2, H admits boundary values from above $b^+(H)$ and from below $b^-(H)$. Since if $G \in \mathfrak{L}_{(\nu,\omega)}(W)$ then $b^+(G) = b^-(G)$, the difference $b^+(H) - b^-(H)$ does not depend on the choice of a defining function H for h. Thus, the *boundary value map*

$$b : H^{(M_p)}_{(\nu,\omega)}(W,\mathbb{R}) \to L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R}),$$
$$b(h) := b^+(H) - b^-(H), \quad \text{where} \quad h = [H] \bmod \mathfrak{L}_{(\nu,\omega)}(W),$$

is well defined.

THEOREM 9. *Assume* (M.1), (M.2) *and* (M.3). *Let W be a tubular neighbourhood of $\mathbb{R}$ and $\nu \in \mathbb{R}\cup\{-\infty\}$, $\omega \in \mathbb{R}\cup\{\infty\}$. Then the mapping*

$$\mathcal{C} : L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R}) \to H^{(M_p)}_{(\nu,\omega)}(W,\mathbb{R}), \quad L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R}) \ni S \mapsto [\mathcal{C}_\Lambda S] \bmod \mathfrak{L}_{(\nu,\omega)}(W),$$

with $\mathcal{C}_\Lambda S$ given by (14) *is a topological isomorphism with inverse $\mathcal{C}^{-1} = b$, where b is the boundary value map.*

Proof. Let $S \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and let $\mathcal{C}S = h \in H^{(M_p)}_{(\nu,\omega)}(\mathbb{R})$. Then by definition $h = [H]$ with $H(\zeta) = \frac{1}{2\pi i}S[\Lambda(\zeta,\cdot)]$ for $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Put $\widetilde{S} = b(h)$ and observe that to prove the equality $\widetilde{S} = S$, by Lemma 7, it is sufficient to show that $\widetilde{S}[\varphi] = S[\varphi]$ for $\varphi \in \underset{\sim}{L}_{(\nu,\omega)}(\mathbb{R})$. To this end fix $\varphi \in \underset{\sim}{L}_{(\nu,\omega)}(\mathbb{R})$ and let $a > \nu$, $b < \omega$ and $W \supset \mathbb{R}$ be such that $\varphi \in \underset{\sim}{L}_{a,b}(W)$. Note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} H(\xi\pm i\varepsilon)(\varphi(\xi\pm i\varepsilon) - \varphi(\xi))\,d\xi = 0$$

and that the integral $\int_{\mathbb{R}} H(\xi\pm i\varepsilon)\varphi(\xi\pm i\varepsilon)\,d\xi$ does not depend on ε for $\varepsilon > 0$ small enough. Thus, choosing $\mathring{\varepsilon} > 0$ small enough we derive

$$\begin{aligned}\widetilde{S}[\varphi] &= b^+(H)[\varphi] - b^-(H)[\varphi]\\ &= \int_{\mathbb{R}} H(\xi+i\mathring{\varepsilon})\varphi(\xi+i\mathring{\varepsilon})\,d\xi - \int_{\mathbb{R}} H(\xi-i\mathring{\varepsilon})\varphi(\xi-i\mathring{\varepsilon})\,d\xi\\ &= \frac{-1}{2\pi i}\int_{\mathbb{R}} S[\Lambda(\xi+i\mathring{\varepsilon},\cdot)]\varphi(\xi+i\mathring{\varepsilon})\,d\xi + \frac{1}{2\pi i}\int_{\mathbb{R}} S[\Lambda(\xi-i\mathring{\varepsilon},\cdot)]\varphi(\xi-i\mathring{\varepsilon})\,d\xi\\ &= \frac{-1}{2\pi i} S\Big[\int_{\mathbb{R}} \Lambda(\xi+i\mathring{\varepsilon},\cdot)\varphi(\xi+i\mathring{\varepsilon})\,d\xi - \int_{\mathbb{R}} \Lambda(\xi-i\mathring{\varepsilon},\cdot)\varphi(\xi-i\mathring{\varepsilon})\,d\xi\Big] = S[\varphi],\end{aligned}$$

since for $\varphi \in \underset{\sim}{L}_{a,b}(W)$, $\mathring{\varepsilon} > 0$ small enough and $x \in \mathbb{R}$,

$$\frac{-1}{2\pi i}\int_{\mathbb{R}} \Lambda(\xi+i\mathring{\varepsilon},x)\varphi(\xi+i\mathring{\varepsilon})\,d\xi + \frac{1}{2\pi i}\int_{\mathbb{R}} \Lambda(\xi-i\mathring{\varepsilon},x)\varphi(\xi-i\mathring{\varepsilon})\,d\xi = \varphi(x).$$

To prove the second part of the theorem take $h = [H] \in H^{(M_p)}_{(\nu,\omega)}(W,\mathbb{R})$, where $H \in \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W\setminus\mathbb{R})$. Put $S = b(H) \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$ and let $F = \mathcal{C}_\Lambda S \in \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(\mathbb{C}\setminus\mathbb{R})$. We have to show that $G := H - F \in \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W\setminus\mathbb{R})$ extends holomorphically to a function $\widetilde{G} \in \mathfrak{L}_{(\nu,\omega)}(W)$. To this end fix $a > \nu$, $b < \omega$ and a closed tubular subset $\widetilde{W}$ of W. By the proof of the first part of the theorem, for $\mathring{\varepsilon} > 0$ small enough and $\varphi \in \underset{\sim}{L}_{a,b}(\widetilde{W})$, we have

$$S[\varphi] = \int_{\mathbb{R}} H(\xi+i\mathring{\varepsilon})\varphi(\xi+i\mathring{\varepsilon})\,d\xi - \int_{\mathbb{R}} H(\xi-i\mathring{\varepsilon})\varphi(\xi-i\mathring{\varepsilon})\,d\xi$$

and

$$S[\varphi] = \frac{1}{2\pi i}\int_{\mathbb{R}} S[\Lambda(\xi+i\mathring{\varepsilon},\cdot)]\varphi(\xi+i\mathring{\varepsilon})\,d\xi - \frac{1}{2\pi i}\int_{\mathbb{R}} S[\Lambda(\xi-i\mathring{\varepsilon},\cdot)]\varphi(\xi-i\mathring{\varepsilon})\,d\xi.$$

So for $\varphi \in \underset{\sim}{L}_{a,b}(\widetilde{W})$,

$$\int_{\mathbb{R}} G(\xi+i\mathring{\varepsilon})\varphi(\xi+i\mathring{\varepsilon})\,d\xi - \int_{\mathbb{R}} G(\xi-i\mathring{\varepsilon})\varphi(\xi-i\mathring{\varepsilon})\,d\xi = 0. \tag{16}$$

Now, for $R > \mathring{\varepsilon}$ close to $\mathring{\varepsilon}$ put

$$\Psi(z) = \frac{-1}{2\pi i}\int_{\mathbb{R}} G(\xi+iR)\Lambda(\xi+iR,z)\,d\xi + \frac{1}{2\pi i}\int_{\mathbb{R}} G(\xi-iR)\Lambda(\xi-iR,z)\,d\xi \quad \text{for } z\in\mathbb{C} \text{ with } |\operatorname{Im} z| < R.$$

Then $\Psi \in \mathcal{O}(\{|\operatorname{Im} z| < R\}) \cap \mathfrak{L}_{a,b}(\{|\operatorname{Im} z| \le \mathring{\varepsilon}\})$ and by (16), $\Psi(z) = G(z)$ for $\mathring{\varepsilon} \le |\operatorname{Im} z| < R$. Thus, if we define

$$\widetilde{G} = \begin{cases} G(z) & \text{for } z \in W\setminus\mathbb{R},\\ \Psi(z) & \text{for } |\operatorname{Im} z| < R,\end{cases}$$

then $\widetilde{G} \in \mathcal{O}(W) \cap \mathfrak{L}_{a,b}(\widetilde{W})$. Since $a > \nu$, $b < \omega$ and $\widetilde{W} \subset W$ were arbitrary this ends the proof.

Proof of Theorem 5. Suppose $H \in \mathcal{O}(\{0 < \operatorname{Im} z < R\})$ with some $R > 0$ has a boundary value $u := b(H) \in L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$. Then by Theorem 8, H satisfies (13). Next as in the proof of the second part of Theorem 9 we show that the difference $H - \mathcal{C}_\Lambda u$ belongs to $\mathfrak{L}_{(\nu,\omega)}(W)$, where $W = \{-R' < \operatorname{Im} z < R'\}$ with some $0 < R' < R$ and H is extended by zero to $\{\operatorname{Im} z \le 0\}$. Since $\mathcal{C}_\Lambda u \in \mathfrak{L}^{(M_p)}_{(\nu,\omega)}(W\setminus\mathbb{R})$ this ends the proof.

6. Final remarks. Let U be a complex neighbourhood of $\mathbb{R}$ and set $U^\pm = U \cap \{\pm\operatorname{Im} z > 0\}$. Then the well known Painlevé theorem states that if $F^\pm \in \mathcal{O}(U^\pm)$ and $b^+(F^+) = b^-(F^-)$ (in $C^0(W)$ or $D'(W)$ or $D^{(M_p)\prime}(W)$) then there exists an $F \in \mathcal{O}(U)$ such that $F^\pm = F_{|U^\pm}$. The results of Theorems 1, 2 and 4 allow us to formulate some generalization of the Painlevé theorem. Namely, if we assume that $F^\pm \in \mathcal{O}(\{\pm\operatorname{Im} z > 0\})$ is of exponential type and the difference of the boundary values $b^+(F^+) - b^-(F^-)$ is in a sense small then F^- determines F^+ and conversely. More precisely, we have

COROLLARY 1. *Let $F^\pm \in \mathcal{O}(\{\pm\operatorname{Im} z > 0\}) \cap C^0(\{\pm\operatorname{Im} z \ge 0\})$ be of exponential type. If*

$$|F^+(x) - F^-(x)| \le C\kappa_{a,b}(x) \quad \text{for } x \in \mathbb{R} \tag{17}$$

with some $a < b$ then F^- determines F^+ and conversely.

Proof. Fix $F^- \in \mathcal{O}(\{\operatorname{Im} z < 0\}) \cap C^0(\{\operatorname{Im} z \le 0\})$ and let $F^+, \widetilde{F}^+ \in \mathcal{O}(\{\operatorname{Im} z > 0\}) \cap C^0(\{\operatorname{Im} z \ge 0\})$ be of exponential type and satisfy (17). Then $F^+ - \widetilde{F}^+ \in \mathcal{O}(\{\operatorname{Im} z > 0\}) \cap C^0(\{\operatorname{Im} z \ge 0\})$ is of exponential type and

$$|F^+(x) - \widetilde{F}^+(x)| \le C\kappa_{a,b}(x) \quad \text{for } x \in \mathbb{R}.$$

Thus, by Theorem 1, $F^+ \equiv \widetilde{F}^+$. Note that until now we have not used the fact that F^- is of exponential type. The proof that F^+ determines F^- is the same.

Remark. Under the assumption of Corollary 1 there need not exist an $F \in \mathcal{O}(\mathbb{C})$ such that $F^\pm = F_{|\{\pm\operatorname{Im} z > 0\}}$. The counter-example is given by the pair of functions $\{F^+, F^-\}$, where

$$F^\pm(z) = \begin{cases} z\ln z & \text{for } z \ne 0,\ \pm\operatorname{Im} z \ge 0,\\ 0 & \text{for } z = 0.\end{cases}$$

COROLLARY 2. *Let (M_p) satisfy* (M.1) *and* (M.3′). *Let $F^\pm \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ be of exponential type in $\{\pm\operatorname{Im} z \ge \varepsilon\}$ for all $\varepsilon > 0$. Assume that $F^\pm$ has a boundary value $u^\pm$ in $L'_{(\nu^\pm,\omega^\pm)}(\mathbb{R})$ (resp. $L^{(M_p)\prime}_{(\nu^\pm,\omega^\pm)}(\mathbb{R})$) with some $\nu^\pm \in \mathbb{R}\cup\{-\infty\}$ and $\omega^\pm \in \mathbb{R}\cup\{\infty\}$. If $u^+ - u^- \in L'_{(\nu,\omega)}(\mathbb{R})$ (resp. $L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$) with some $\nu < \omega$ then F^- determines F^+ and conversely.*

Proof. The proof goes along the same lines as the one of Corollary 1, with Theorem 2 (resp. Theorem 4) in place of Theorem 1.

Analogously we get

COROLLARY 3. *Let* (M_p) *satisfy* (M.1) *and* (M.3′), *and let* $\mathring{u} \in L'_{(\mathring{\nu},\mathring{\omega})}(\mathbb{R})$ (*resp.* $L^{(M_p)\prime}_{(\mathring{\nu},\mathring{\omega})}(\mathbb{R})$) *with some* $\mathring{\nu} \in \mathbb{R}\cup\{-\infty\}$ *and* $\mathring{\omega} \in \mathbb{R}\cup\{\infty\}$. *Then there exists at most one* $F^{\pm} \in \mathcal{O}(\{\operatorname{Im} z > 0\})$ *of exponential type in* $\{\pm \operatorname{Im} z \geq \varepsilon\}$ *for all* $\varepsilon > 0$ *such that* $b(F^{\pm}) \in L'_{(\nu^{\pm},\omega^{\pm})}(\mathbb{R})$ (*resp.* $L^{(M_p)\prime}_{(\nu^{\pm},\omega^{\pm})}(\mathbb{R})$) *with some* $\nu^{\pm} \in \mathbb{R}\cup\{-\infty\}$ *and* $\omega^{\pm} \in \mathbb{R}\cup\{\infty\}$, *and* $b(F^{\pm}) - \mathring{u} \in L'_{(\nu,\omega)}(\mathbb{R})$ (*resp.* $L^{(M_p)\prime}_{(\nu,\omega)}(\mathbb{R})$) *with some* $\nu < \omega$. *Furthermore, if* $\mathring{\nu} < \mathring{\omega}$ *then* $F^{\pm} \equiv 0$.

We remark that in the case $\mathring{\nu} \geq \mathring{\omega}$, in general, the problem of existence of such an $F^{\pm}$ remains open.

Acknowledgements. The author would like to thank the referees for their helpful remarks.

References

[H] E. Hille, *Analytic Function Theory*, Vol. 2, Chelsea, New York, 1962.

[K] H. Komatsu, *Ultradistributions, I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo 20 (1973), 25–105.

[Ł1] G. Łysik, *Generalized analytic functions and a strong quasi-analyticity principle*, Dissertationes Math. 340 (1995), 195–200.

[Ł2] —, *Laplace ultradistributions on a half line and a strong quasi-analyticity principle*, Ann. Polon. Math. 63 (1996), 13–33.

[M] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.

[R] C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. 77 (1960), 41–121.

[SZ] Z. Szmydt and B. Ziemian, *The Laplace distributions on* $\mathbb{R}^n_+$, submitted to J. Math. Sci. Univ. Tokyo.

[T] J. C. Tougeron, *Gevrey expansions and applications*, preprint, University of Toronto, 1991.

[W] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, N.J., 1946.

[Z] A. H. Zemanian, *Generalized Integral Transformations*, Interscience, 1969.

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: lysik@impan.gov.pl

Received December 5, 1995 (3579)
Revised version March 29, 1996 and December 6, 1996

STUDIA MATHEMATICA 123 (3) (1997)

Compact homomorphisms between algebras of analytic functions

by

RICHARD ARON (Kent, Ohio), PABLO GALINDO (Valencia), and MIKAEL LINDSTRÖM (Åbo)

Abstract. We prove that every weakly compact multiplicative linear continuous map from $H^\infty(D)$ into $H^\infty(D)$ is compact. We also give an example which shows that this is not generally true for uniform algebras. Finally, we characterize the spectra of compact composition operators acting on the uniform algebra $H^\infty(B_E)$, where B_E is the open unit ball of an infinite-dimensional Banach space E.

Let E denote a complex Banach space with open unit ball B_E and let $\phi : B_E \to B_E$ be an analytic map. We will consider the composition operator C_ϕ defined by $C_\phi(f) = f \circ \phi$, acting on the uniform algebra $H^\infty(B_E)$ of all bounded analytic functions on B_E. This operator may also be regarded as acting on the smaller uniform algebra $A_{\mathrm{u}}(B_E)$ of all analytic functions on B_E which are uniformly continuous, in which case we assume that $f \circ \phi \in A_{\mathrm{u}}(B_E)$ whenever f is in $A_{\mathrm{u}}(B_E)$. These algebras, which are natural generalizations of the classical algebras $H^\infty(D)$ and $A(D)$ of analytic functions on the complex open disc D, have been studied in [ACG].

Several results automatically yielding compactness of composition operators from weak compactness have appeared recently. For instance, D. Sarason in [Sa] proved that every weakly compact composition operator on $H^1(D)$ is compact, and K. Madigan and A. Matheson [MM] obtained the analogue for the little Bloch space $\mathcal{B}_0$. In the first section we study compactness of C_ϕ and prove that every weakly compact homomorphism from $H^\infty(D)$ into $H^\infty(D)$ is automatically compact. This result has also inde-

1991 *Mathematics Subject Classification*: Primary 46J15; Secondary 46E15, 46G20.

Research of the first author was partially supported by NSF grant INT-9023951.

Research of the second author was partially supported by DGICYT (Spain) pr. 91-0326 and pr. 93-081.

The research of Mikael Lindström was supported by a grant from the Foundation of Åbo Akademi University Research Institute; the research of this paper was carried out during the spring semester 1995 while this author was visiting Kent State University, whose hospitality is acknowledged with thanks.