Contents of Volume 123, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Subscription information (1997): Vols. 122–126 (15 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

Singular integrals with holomorphic kernels and Fourier multipliers on star-shaped closed Lipschitz curves

by

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Dedicated to Professor Alan McIntosh

Abstract. The paper presents a theory of Fourier transforms of bounded holomorphic functions defined in sectors. The theory is then used to study singular integral operators on star-shaped Lipschitz curves, which extends the result of Coifman–McIntosh–Meyer on the L^2-boundedness of the Cauchy integral operator on Lipschitz curves. The operator theory has a counterpart in Fourier multiplier theory, as well as a counterpart in functional calculus of the differential operator $\frac{1}{i}\frac{d}{dz}$ on the curves.

1. Introduction. Let γ be a Lipschitz graph with the parameterization

$$\gamma = \{\gamma(x) = x + ia(x) : -\infty < x < \infty\},$$

where a is a bounded Lipschitz function, $m = \min_{-\infty<x<\infty}\{a(x)\}$, $M = \max_{-\infty<x<\infty}\{a(x)\}$. In [CM1] the authors introduce the dense subclass

$$\mathcal{A}(\gamma) = \Big\{ f : f \text{ is holomorphic in } m-\varepsilon < \mathrm{Im}(z) < M+\varepsilon \text{ for some } \varepsilon \text{ and } \int\limits_{m-\varepsilon<y<M+\varepsilon} |f(x+iy)|^2\,dx < c \text{ for some } c < \infty \Big\}$$

of $L^2(\gamma)$ and define the Fourier transform of functions in $\mathcal{A}(\gamma)$. They establish that the H^∞-functions, i.e. the bounded and holomorphic functions, on the double sectors that contain the difference set $\mathbf{D} = \{0 \neq z - \eta : z, \eta \in \gamma\}$ give rise to L^2-bounded Fourier multiplier operators. The Lipschitz graphs they consider are restricted to be of small Lipschitz constants because of the use of a result of A. P. Calderón. The restriction can be eliminated owing to the later result of Coifman, McIntosh and Meyer ([CMcM]).

1991 *Mathematics Subject Classification*: Primary 42B15, 42B20, 47A60; Secondary 42A16.

In [McQ1] the authors deal with an analogous theory on infinite Lipschitz graphs. They prove that the singular integral kernels associated with the above-mentioned H^∞-functions are those which are holomorphic, of the Calderón–Zygmund type, and satisfy a kind of weak-boundedness condition (see (ii) of Theorem A below). In [McQ2] the converse result is proved. Another version of the theory in [McQ1], [McQ2] is the H^∞-functional calculus of the differential operator $\frac{1}{i}\frac{d}{dz}$, which has also been considered for instance in [DJS] and [Mc].

In [LMcS] and [LMcQ] the authors develop a high-dimensional theory using Clifford algebras and several complex variables which, in view of the non-commutativity of Clifford algebras, is by no means a parallel generalization of the one-dimensional case.

It is now natural to ask: Is there an analogous theory for closed curves? In this paper we shall answer this question for the star-shaped Lipschitz curves given by the parameterization $\widetilde{\Gamma} = \{\exp(iz) : z \in \Gamma\}$, where $\Gamma = \{x + iA(x) : A' \in L^\infty([-\pi,\pi]),\ A(-\pi) = A(\pi)\}$. It may be shown that star-shaped Lipschitz curves defined using this parameterization are the same as those defined as star-shaped and Lipschitz in the ordinary sense (see e.g. [Q3]).

One can define, in the same pattern as in the standard case, the Fourier series of L^2 functions on Γ, and the question can now be specified into the following two. The first, what kind of holomorphic kernels give rise to L^2-bounded operators on the curves? The second, is there a corresponding Fourier multiplier theory? In other words, what complex number sequences act as L^p-bounded Fourier multipliers on those curves? Note that the question is non-trivial even for the case $p = 2$, as the Plancherel theorem does not hold in this case. On the other hand, the case $p = 2$ is essential, as the boundedness for $1 < p < \infty$ can be deduced from the L^2 theory using the standard Calderón–Zygmund techniques (see [S], for example).

The basic method of the paper is the use of the Poisson summation formula in a sense closely related to the properties of the function pairs (ϕ, ϕ_1) obtained in [McQ1]. Section 2 recalls the related previously known results together with an account of notation and terminology. §3 contains the theory of Fourier transform between the H^∞-functions defined on sectors and the corresponding holomorphic kernels on truncated sectors. §4 contains the singular integral theory and its version for the H^∞-functional calculus of the operator $z\frac{d}{dz}$. We also include some results on holomorphic extension, which is an interpretation of the theory in terms of complex analysis. §5 contains some results on general $L^p(\Gamma)$ Fourier multipliers. In the whole paper we shall omit the proofs which are similar to the infinite Lipschitz graph case, and devote ourselves to those which reveal new features of the theory.

The reason why we restrict ourselves to the star-shaped Lipschitz curves is as follows. Firstly, if a closed curve $\widetilde{\Gamma}$ is not star-shaped, then the corresponding difference set $\mathbf{D} = \{0 \neq z - \eta : z, \eta \in \Gamma\}$, where

$$\Gamma = \left\{\zeta = \frac{1}{i}\ln z : \operatorname{Re}(\zeta) \in [-\pi,\pi],\ z \in \widetilde{\Gamma}\right\},$$

is not contained in any double sector defined in §2, and it may eventually spread over a region $0 < |z| < a$, in case $\widetilde{\Gamma}$ is winding enough. We are considering holomorphic kernels on Γ, and they should be defined on open connected sets containing $\mathbf{D}$. However, such holomorphic kernels with the 2π-periodicity satisfying the standard Calderón–Zygmund size conditions at $z = 0$ are of the form $A\cot(z/2) + \psi(z)$, where A is a constant and ψ is a bounded holomorphic function on a bounded neighbourhood of $\mathbf{D}$. The Fourier transform of $\cot(z/2)$ is a constant multiple of the signum function (see Example (i) of §4) and the corresponding singular integral theory can be deduced, for example, from Coifman–McIntosh–Meyer's theorem ([CMcM]) using a partition of unity or Guy David's theory ([D]).

The second reason is related to the potential solutions of the Dirichlet and the Neumann boundary value problems on Lipschitz domains (see [FJR] and [V], for example). The star-shaped Lipschitz domains are general enough to serve this purpose, owing to the fact that every simply connected Lipschitz domain of the complex plane is the image of a star-shaped Lipschitz domain under a conformal mapping, and the fact that conformal mappings preserve harmonic functions.

The subjects presented in this paper have been further developed to various higher-dimensional cases including Lipschitz perturbations of the m-torus, the unit spheres of quaternionic and Clifford spaces and of the euclidean spaces. They are not trivial generalizations. For instance, there is no Poisson summation formula for the higher-dimensional spheres. The references are [Q1–5].

Special thanks are due to Alan McIntosh and Carlos E. Kenig for their suggestions on this topic. The author also wishes to thank Garth Gaudry, Gengkai Zhang, Joachim Hempel and Zhongming Guo for their comments, suggestions and their interest in this topic during my stay at the Flinders University of South Australia, Mittag-Leffler Institute, and the New England University, respectively. This work was partly supported by the Australian Research Council in the form of Research Fellowship for the period when this study was being carried out.

2. Preliminaries. Let Γ be a Lipschitz curve defined on the interval $[-\pi,\pi]$ with the parameterization $\Gamma(x) = x + iA(x)$, $A : [-\pi,\pi] \to \mathbb{R}$, where $\mathbb{R}$ denotes the real number field, $A(-\pi) = A(\pi)$, $A' \in L^\infty([-\pi,\pi])$, and $\|A'\|_\infty = N < \infty$. Denote by $p\Gamma$ the 2π-periodic extension of Γ to $-\infty < x < \infty$, and by $\widetilde{\Gamma}$ the closed curve

$$\widetilde{\Gamma} = \{\exp(iz) : z \in \Gamma\} = \{\exp(i(x + iA(x)) : -\pi \le x \le \pi\}.$$

We will call $\widetilde{\Gamma}$ the *star-shaped Lipschitz curve associated with* Γ.

We will use f, F and $\widetilde{F}$, etc., to denote functions defined on $p\Gamma, \Gamma$ and $\widetilde{\Gamma}$, respectively. For $\widetilde{F} \in L^2(\widetilde{\Gamma})$, define

$$\widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n) = \frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}} z^{-n}\widetilde{F}(z)\,\frac{dz}{z},$$

the nth Fourier coefficient of $\widetilde{F}$ with respect to $\widetilde{\Gamma}$. We will sometimes suppress the subscript and write $\widehat{\widetilde{F}}(n)$ if no confusion can occur.

Set

$$\sigma = \exp(-\max A(x)), \qquad \tau = \exp(-\min A(x)).$$

Similarly to [CM1] we consider the following dense subclass of $L^2(\widetilde{\Gamma})$ (see also [GQW]):

$$\mathcal{A}(\widetilde{\Gamma}) = \{\widetilde{F}(z) : \widetilde{F}(z) \text{ is holomorphic in } \sigma - \eta < |z| < \tau + \eta \text{ for some } \eta > 0\}.$$

Without loss of generality, we assume that $\min A(x) < 0$, $\max A(x) > 0$. In this case the domains of the functions in $\mathcal{A}(\widetilde{\Gamma})$ always contain the unit circle $\mathbb{T}$, and owing to Cauchy's theorem we have $\widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n) = \widehat{\widetilde{F}}_{\mathbb{T}}(n)$. If $\widetilde{F}$ and $\widetilde{G}$ belong to $\mathcal{A}(\widetilde{\Gamma})$, this remark, together with Laurent series theory, implies the Fourier inversion formula

$$\widetilde{F}(z) = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n)z^n, \tag{1}$$

where z is in the annulus in which $\widetilde{F}$ is defined; and a use of Cauchy's theorem gives the Parseval formula

$$\frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}} \widetilde{F}(z)\widetilde{G}(z)\,\frac{dz}{z} = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n)\widehat{\widetilde{G}}_{\widetilde{\Gamma}}(-n). \tag{2}$$

We shall use the following half and double sectors in the complex plane $\mathbb{C}$: for $\omega \in (0, \pi/2]$,

$$\mathbf{S}^0_{\omega,+} = \{z \in \mathbb{C} : |\arg(z)| < \omega,\ z \ne 0\},$$
$$\mathbf{S}^0_{\omega,-} = -\mathbf{S}^0_{\omega,+}, \qquad \mathbf{S}^0_{\omega} = \mathbf{S}^0_{\omega,+} \cup \mathbf{S}^0_{\omega,-},$$

and the sets

$$\mathbf{C}^0_{\omega,+} = \mathbf{S}^0_{\omega} \cup \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\},$$
$$\mathbf{C}^0_{\omega,-} = \mathbf{S}^0_{\omega} \cup \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}.$$

Let $\mathbf{X}$ be a set defined above. Denote by

$$\mathbf{X}(\pi) = \mathbf{X} \cap \{z \in \mathbb{C} : |\operatorname{Re}(z)| \le \pi\}$$

the truncated set, and by

$$\mathbf{pX}(\pi) = \bigcup_{k=-\infty}^{\infty} \{\mathbf{X}(\pi) + 2k\pi\}$$

the periodic set associated with the truncated one. We shall use sets of the form $\exp(i\mathbf{O}) = \{\exp(iz) : z \in \mathbf{O}\}$, where $\mathbf{O}$ will be the truncated sets defined above. In the sequel $H^\infty(\mathbf{Q})$ denotes the function space $\{f : \mathbf{Q} \to \mathbb{C} : f$ is holomorphic and bounded in $\mathbf{Q}\}$, where $\mathbf{Q}$ will be a double or half sector defined above. We will use $\|\ \|_\infty$ to denote $\|\ \|_{H^\infty(\mathbf{Q})}$ if no confusion can occur.

Let $b \in H^\infty(\mathbf{S}^0_\omega)$, $\omega \in (0, \pi/2]$. Then b can be decomposed into two parts: $b = b^+ + b^-$, where

$$b^+ = b\chi_{\{z:\operatorname{Re}(z)>0\}}, \qquad b^- = b\chi_{\{z:\operatorname{Re}(z)<0\}},$$

and so $b^\pm \in H^\infty(\mathbf{S}^0_{\omega,\pm})$, respectively.

In each of the following statements "$\pm$" should be read as either all "+" or all "$-$".

The following transforms are used in [McQ1]:

$$G^\pm(b^\pm)(z) = \phi^\pm(z) = \frac{1}{2\pi}\int\limits_{\varrho^\pm_\theta} \exp(iz\zeta)b(\zeta)\,d\zeta, \qquad z \in \mathbf{C}^0_{\omega,\pm},$$

where $\varrho^\pm_\theta$ is the ray $s\exp(i\theta)$, $0 < s < \infty$, and θ is chosen, depending on $z \in \mathbf{C}^0_{\omega,\pm}$, so that $\varrho^\pm_\theta \subset \mathbf{S}^0_{\omega,\pm}$, and $\exp(iz\zeta)$ is exponentially decaying as $\varrho^\pm_\theta \ni \zeta \to \infty$; and

$$G^\pm_1(b^\pm)(z) = \phi^\pm_1(z) = \int\limits_{\delta^\pm(z)} \phi^\pm(\zeta)\,d\zeta, \qquad z \in \mathbf{S}^0_{\omega,+},$$

where the integral is along any path $\delta^\pm(z)$ from $-z$ to z in $\mathbf{C}^0_{\omega,\pm}$.

In what follows, c_0, c_1, and C will denote universal constants and $C_{\omega,\mu}$ will denote constants that depend on ω, μ, and so on, and they may vary from one occurrence to another.

Our theory is based on the main results in [McQ1], which we now reformulate for the reader's convenience.

THEOREM A. *Let $\omega \in (0,\pi/2]$ and $b^{\pm} \in H^\infty(\mathbf{S}^0_{\omega,\pm})$. Then $\phi^{\pm} = G^{\pm}(b^{\pm})$ and $\phi_1^{\pm} = G_1^{\pm}(b^{\pm})$ defined as above are holomorphic functions in their domains, and for every $\mu \in (0,\omega)$,*

(i) $$|\phi^{\pm}(z)| \le \frac{C_{\omega,\mu}\|b^{\pm}\|_\infty}{|z|}, \quad z \in \mathbf{C}^0_{\mu,\pm};$$

(ii) $$\phi_1^{\pm} \in H^\infty(\mathbf{S}^0_{\mu,+}), \quad \|\phi_1^{\pm}\|_{H^\infty(\mathbf{S}^0_{\mu,+})} \le C_{\omega,\mu}\|b^{\pm}\|_\infty, \quad \text{and}$$
$$\phi_1^{\pm\prime}(z) = \phi^{\pm}(z) + \phi^{\pm}(-z), \quad z \in \mathbf{S}^0_{\omega,+};$$

(iii) $$(2\pi)^{-1}\int_{-\infty}^{\infty} b^{\pm}(\zeta)\widehat{f}(-\zeta)\,d\zeta = \lim_{\varepsilon\to 0}\Big\{\int_{|x|\ge\varepsilon} \phi^{\pm}(x)f(x)\,dx + \phi_1^{\pm}(\varepsilon)f(0)\Big\}$$

for all f in the Schwartz class $\mathcal{S}(\mathbb{R})$, where $\widehat{f}$ stands for the standard Fourier transform of f.

For $b \in H^\infty(\mathbf{S}^0_\omega)$ using the decomposition $b = b^+ + b^-$ and Theorem A, and then letting $\phi = \phi^+ + \phi^-$, $\phi_1 = \phi_1^+ + \phi_1^-$, we obtain the following

COROLLARY 1. *Let $\omega \in (0,\pi/2]$ and $b \in H^\infty(\mathbf{S}^0_\omega)$. Then there exists a pair of holomorphic functions (ϕ,ϕ_1) defined in $\mathbf{S}^0_\omega$ and $\mathbf{S}^0_{\omega,+}$, respectively, satisfying, for every $\mu \in (0,\omega)$,*

(i) $$|\phi(z)| \le \frac{C_{\omega,\mu}\|b\|_\infty}{|z|}, \quad z \in \mathbf{S}^0_\mu;$$

(ii) $$\phi_1 \in H^\infty(\mathbf{S}^0_{\mu,+}), \quad \|\phi_1\|_{H^\infty(\mathbf{S}^0_{\mu,+})} \le C_{\omega,\mu}\|b\|_\infty, \text{ and}$$
$$\phi_1'(z) = \phi(z) + \phi(-z), \quad z \in \mathbf{S}^0_{\omega,+};$$

(iii) $$(2\pi)^{-1}\int_{-\infty}^{\infty} b(\zeta)\widehat{f}(-\zeta)\,d\zeta = \lim_{\varepsilon\to 0}\Big\{\int_{|x|\ge\varepsilon} \phi(x)f(x)\,dx + \phi_1(\varepsilon)f(0)\Big\}$$

for all f in $\mathcal{S}(\mathbb{R})$.

THEOREM B. *Let $\omega \in (0,\pi/2]$ and (ϕ,ϕ_1) be a pair of holomorphic functions defined in $\mathbf{S}^0_\omega$ and $\mathbf{S}^0_{\omega,+}$, respectively, satisfying*

(i) *there is a constant c_0 such that*
$$|\phi(z)| \le \frac{c_0}{|z|}, \quad z \in \mathbf{S}^0_\omega;$$

(ii) *there is a constant c_1 such that $\|\phi_1\|_{H^\infty(\mathbf{S}^0_{\omega,+})} < c_1$, and*
$$\phi_1'(z) = \phi(z) + \phi(-z), \quad z \in \mathbf{S}^0_{\omega,+}.$$

Then there exists a unique function $b \in H^\infty(\mathbf{S}^0_\mu)$ for every $\mu \in (0,\omega)$ such that
$$\|b\|_{H^\infty(\mathbf{S}^0_\mu)} \le C_{\omega,\mu}(c_0 + c_1),$$

and the function pair determined by b according to Theorem A is identical to (ϕ,ϕ_1). Moreover, for all complex numbers $\zeta \in \mathbf{S}^0_\omega$ the function b is given by

$$b(\zeta) = \lim_{\varepsilon\to 0}\lim_{N\to\infty}\Big\{\int_{\varepsilon<|x|<N} \exp(-i\zeta x)\phi(x)\,dx + \phi_1(\varepsilon)\Big\}. \tag{3}$$

Remark. If $\phi|_{\mathbb{R}}$, the restriction of ϕ to $\mathbb{R}$, is a good enough function, say, for instance, if $\phi|_{\mathbb{R}}$ is in $L^2(\mathbb{R})\cap L^1_{\text{loc}}(\mathbb{R})$, then $b|_{\mathbb{R}}$ is the standard Fourier transform of $\phi|_{\mathbb{R}}$, $\lim_{\varepsilon\to 0}\phi_1(\varepsilon z) = 0$, for $z \in \mathbf{S}^0_{\omega,+}$, and (iii) of Theorem A reduces to the standard Parseval identity.

3. Fourier transforms between $\mathbf{S}^0_\omega$ and $\mathbf{pS}^0_\omega(\pi)$

THEOREM 1. *Let $\omega \in (0,\pi/2]$ and $b \in H^\infty(\mathbf{S}^0_\omega)$, and (ϕ,ϕ_1) be the function pair associated with b in the pattern of Corollary 1. Then there exists a pair of holomorphic functions (Φ,Φ_1) defined in $\mathbf{S}^0_\omega(\pi)$ and $\mathbf{S}^0_{\omega,+}(\pi)$, respectively, satisfying, for every $\mu \in (0,\omega)$,*

(i) *Φ can be periodically and holomorphically extended to $\mathbf{pS}^0_\omega(\pi)$ and*
$$|\Phi(z)| \le \frac{C_{\omega,\mu}\|b\|_\infty}{|z|}, \quad z \in \mathbf{S}^0_\mu(\pi).$$

Moreover,
$$\Phi(z) = \phi(z) + \phi_0(z), \quad z \in \mathbf{S}^0_\mu(\pi),$$
where ϕ_0 is a bounded holomorphic function in $\mathbf{S}^0_\mu(\pi)$;

(ii) *$\Phi_1 \in H^\infty(\mathbf{S}^0_{\mu,+}(\pi))$, $\|\Phi_1\|_{H^\infty(\mathbf{S}^0_{\mu,+})} \le C_{\omega,\mu}\|b\|_\infty$, and*
$$\Phi_1'(z) = \Phi(z) + \Phi(-z), \quad z \in \mathbf{S}^0_\omega(\pi);$$

(iii) *Φ and Φ_1 are uniquely determined (modulo constants) by the Parseval formula*
$$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{F}(-n) = \lim_{\varepsilon\to 0}\Big\{\int_{\varepsilon\le|x|\le\pi} \Phi(x)F(x)\,dx + \Phi_1(\varepsilon)F(0)\Big\}$$

for any 2π-periodic and smooth functions F defined on $\mathbb{R}$, where $\widehat{F}(n)$ stands for the nth Fourier coefficient of F, and $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$.

Proof. Define Φ by the Poisson summation formula:

$$\Phi(z) = 2\pi\sum_{k=-\infty}^{\infty} \phi(z + 2k\pi), \quad z \in \mathbf{pS}^0_\omega(\pi), \tag{4}$$

where the summation takes the following sense: there is a subsequence (n_l) of (n) such that for all $z \in \mathbf{S}^0_\omega(\pi)$ the partial sum

$$s_{n_l}(z) = 2\pi\sum_{k=-n_l}^{n_l} \phi(z + 2k\pi)$$

locally uniformly converges, as $l \to \infty$, to a 2π-periodic and holomorphic function satisfying the assertion (i). In the sequel we shall call such sequences *applicable sequences*. Moreover, we shall show that limit functions defined through different applicable sequences differ from one another by constants bounded by $c\|b\|_\infty$.

To proceed, we use the decomposition

$$\sum_{k=-n}^{n} \phi(z+2k\pi) = \phi(z) + \sum_{k\neq 0}^{\pm n}(\phi(z+2k\pi)-\phi(2k\pi)) + \sum_{k=1}^{n}\phi_1'(2k\pi)$$
$$= \phi(z) + \sum\nolimits_1 + \sum\nolimits_2.$$

We shall show that the series $\sum_1$ locally uniformly converges to a bounded holomorphic function in $\mathbf{S}^0_\mu(\pi)$, and some subsequence of the partial sums of $\sum_2$ converges to a constant dominated by $C_\mu\|b\|_\infty$.

The convergence of $\sum_1$ follows from the estimate

$$|\phi'(z)| \le \frac{C_\mu}{|z|^2}, \quad z \in \mathbf{S}^0_\mu,$$

deduced from the estimate in Corollary 1(i), the fact that ϕ is holomorphic in the sectors and Cauchy's theorem. To deal with $\sum_2$ we use the mean value theorem for integrals and we have

$$\sum_{k=1}^{n}\phi_1'(2k\pi) = \int_{2\pi}^{2(n+1)\pi}\phi_1'(r)\,dr + \sum_{k=1}^{n}(\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\,\mathrm{Im}(\phi_1'(\eta_k)))$$
$$= \phi_1(2(n+1)\pi) - \phi_1(2\pi)$$
$$+ \sum_{k=1}^{n}(\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\,\mathrm{Im}(\phi_1'(\eta_k))),$$

where $\xi_k, \eta_k \in (2k\pi, 2(k+1)\pi)$. Owing to the estimate of ϕ' again, the series in the above expression converges absolutely. The boundedness of ϕ_1 then guarantees the existence of an applicable sequence (n_l) such that $\phi_1(2(n_l+1)\pi)$ converges to a constant c_0 with the desired bounds. We therefore have

$$\frac{1}{2\pi}\Phi(z) = \phi(z) + \sum_{k\neq 0}(\phi(z+2k\pi)-\phi(2k\pi)) + \lim_{l\to\infty}\sum_{n=1}^{n_l}\phi_1'(2n\pi)$$
$$= \phi(z) + \phi_0(z) + c_0,$$

where ϕ_0 is a bounded holomorphic function in $\mathbf{S}^0_\mu(\pi)$, and c_0 is a constant depending on the subsequence (n_l) chosen. The argument also shows that Φ can be holomorphically extended to $\mathbf{pS}^0_\omega(\pi)$, and the different Φ's asso-

ciated with different applicable sequences may differ from one another by constants dominated by $c\|b\|_\infty$.

Now we prove (ii) and (iii). We use the decomposition $b = b^+ + b^-$ indicated in §2. Define $b^{\pm,\alpha}(z) = \exp(\mp\alpha z)b^\pm(z)$, $\alpha > 0$. Let $\phi^\pm$ and $\phi^{\pm,\alpha}$ be associated, according to Theorem A, with $b^\pm$ and $b^{\pm,\alpha}$, respectively. Owing to the remark made after Theorem B, $\phi^{\pm,\alpha}(\cdot) = \phi^\pm(\cdot \pm i\alpha)$, and the latter are the inverse Fourier transforms of $b^{\pm,\alpha}$. We now define the corresponding holomorphic and periodic functions $\Phi^\pm$ and $\Phi^{\pm,\alpha}$ in $\mathbf{pC}^0_{\omega,\pm}(\pi)$, respectively, which satisfy the size condition in the assertion (i). It is to be noted that for all $\Phi^{\pm,\alpha}$ we may, and we actually do, choose the same applicable sequence (n_l) for $\Phi^{\pm,\alpha}$ as we have chosen for $\Phi^\pm$. Using the estimate in Corollary 1(i) and the fact that ϕ is holomorphic, we can show that the convergence of $\sum_1$ is locally (in z) uniform for $\alpha \to 0$, and is absolute. Let

$$\frac{1}{2\pi}\Phi^{\pm,\alpha}(z) = \phi^{\pm,\alpha}(z) + \phi_0^{\pm,\alpha}(z) + c_0^{\pm,\alpha},$$
$$\frac{1}{2\pi}\Phi^{\pm}(z) = \phi^{\pm}(z) + \phi_0^{\pm}(z) + c_0^{\pm},$$

where $\phi_0^{\pm,\alpha}$ and $\phi_0^\pm$ are holomorphic and uniformly (for $\alpha \to 0$) bounded in $\mathbf{C}^0_{\mu,\pm}(\pi)$. Since the convergence as $n_l \to \infty$ is uniform for $\alpha \to 0$, we can exchange the order of taking the limits as $n_l \to \infty$ and $\alpha \to 0$, and conclude that $\phi^{\pm,\alpha}, \phi_0^{\pm,\alpha}$ and $c_0^{\pm,\alpha}$ converge to $\phi^\pm, \phi_0^\pm$ and $c_0^\pm$, respectively, locally uniformly in $\mathbf{C}^0_{\omega,\pm}(\pi)$. Therefore, $\lim_{\alpha\to 0}\Phi^{\pm,\alpha}(z) = \Phi^\pm(z)$. Since for a fixed α, $\Phi^{\pm,\alpha} \in L^\infty([-\pi,\pi])$, and the series which defines $\Phi^{\pm,\alpha}$ converges uniformly in $x \in [-\pi,\pi]$ as $n_l \to \infty$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-i\xi x)\Phi^{\pm,\alpha}(x)\,dx = \int_{-\pi}^{\pi}\exp(-i\xi x)\lim_{l\to\infty}\sum_{k=-n_l}^{n_l}\phi^{\pm,\alpha}(x+2k\pi)\,dx$$
$$= \int_{-\infty}^{\infty}\exp(-i\xi x)\phi^{\pm,\alpha}(x)\,dx = b^{\pm,\alpha}(\xi)$$

for all non-zero real ξ in the sense of (3) in Theorem B. In particular, $\{b^{\pm,\alpha}(n)\}$, $n \neq 0$, are the standard Fourier coefficients of $\Phi^{\pm,\alpha}$. If F is any smooth and periodic function on $[-\pi,\pi]$, then the standard Parseval identity holds:

$$2\pi\sum_{n=-\infty}^{\infty} b^{\pm,\alpha}(n)\widehat{F}(-n) = \int_{-\pi}^{\pi}\Phi^{\pm,\alpha}(x)F(x)\,dx,$$

where $b^{\pm,\alpha}(0) = (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi^\pm(x \pm i\alpha)\,dx$. Now we proceed as in [McQ1]. Let $\varepsilon > 0$. Since $\widehat{F}(n)$ decays rapidly as $n \to \pm\infty$, on letting $\alpha \to 0+$, we have

$$2\pi\sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\alpha\to 0+}\Big\{\int\limits_{[-\pi,\pi]\setminus(-\varepsilon,\varepsilon)} \Phi^{\pm}(x\pm i\alpha)F(x)\,dx + \int\limits_{|x|\le\varepsilon} \Phi^{\pm}(x\pm i\alpha)(F(x)-F(0))\,dx + \int\limits_{|x|\le\varepsilon} \Phi^{\pm}(x\pm i\alpha)F(0)\,dx\Big\}.$$

Now

$$\lim_{\alpha\to 0+}\int\limits_{[-\pi,\pi]\setminus(-\varepsilon,\varepsilon)} \Phi^{\pm}(x\pm i\alpha)F(x)\,dx = \int\limits_{[-\pi,\pi]\setminus(-\varepsilon,\varepsilon)} \Phi^{\pm}(x)F(x)\,dx,$$

and

$$\limsup_{\alpha\to 0+}\int\limits_{|x|\le\varepsilon} |\Phi^{\pm}(x\pm i\alpha)|\cdot|F(x)-F(0)|\,dx \le \limsup_{\alpha\to 0+}\int\limits_{|x|\le\varepsilon}\frac{1}{|x|}\cdot|x|\,dx \le C\varepsilon.$$

Define

$$\Phi_1^{\pm}(z) = \int\limits_{\delta^{\pm}(z)} \Phi^{\pm}(\eta)\,d\eta,$$

where $\delta^{\pm}(z)$ is any path in $\mathbf{C}^0_{\omega,\pm}(\pi)$ from $-z$ to z. The assertion (ii) for $\Phi_1^{\pm}$ then follows, and

$$\lim_{\alpha\to 0+}\int\limits_{|x|\le\varepsilon} \Phi^{\pm}(x\pm i\alpha)F(0)\,dx = \Phi_1^{\pm}(\varepsilon)F(0).$$

We have therefore established the Parseval identities for $b^{\pm}$:

$$2\pi\sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\varepsilon\to 0}\Big\{\int\limits_{[-\pi,\pi]\setminus(-\varepsilon,\varepsilon)} \Phi^{\pm}(x)F(x)\,dx + \Phi_1^{\pm}(\varepsilon)F(0)\Big\},$$

where $b^{\pm}(0) = \frac{1}{2\pi}\Phi_1^{\pm}(\pi)$. Note that if in the above formulas we replace $\Phi^{\pm}$ by $\Phi^{\pm}+c^{\pm}$, then correspondingly we need to replace $b^{\pm}(0)$ by $b^{\pm}(0)+c^{\pm}$ in order to make the formulas still hold. Since $\Phi = \Phi^{+}+\Phi^{-}$, on letting $\Phi_1 = \Phi_1^{+}+\Phi_1^{-}$, we see that (ii) and (iii) hold. The proof is complete.

Remark. When we consider the Parseval formula associated with the given function $b\in H^{\infty}(\mathbf{S}^0_{\omega})$, the value of b at the origin is naturally involved. The convention for this value should be $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$ in consistency with the formula as shown in the theorem. The proof of the theorem indicates that adding a constant to Φ (which does not change the Fourier coefficients $\widehat{\Phi}(n) = b(n)$ for $n\ne 0$) results in adding the same constant to $b(0)$. The special Φ defined through the formula (6) below has the property $b(0)=0$.

THEOREM 2. *Let $\omega\in(0,\pi/2]$ and (Φ,Φ_1) be a pair of holomorphic functions defined in $\mathbf{pS}^0_{\omega}(\pi)$ and $\mathbf{S}^0_{\omega,+}(\pi)$, respectively, satisfying*

(i) *Φ is 2π-periodic, and there is a constant c_0 such that*

$$|\Phi(z)|\le\frac{c_0}{|z|},\qquad z\in\mathbf{S}^0_{\omega}(\pi);$$

(ii) *there is a constant c_1 such that $\|\Phi_1\|_{H^{\infty}(\mathbf{S}^0_{\omega,+}(\pi))} < c_1$, and*

$$\Phi_1'(z) = \Phi(z)+\Phi(-z),\qquad z\in\mathbf{S}^0_{\omega,+}(\pi).$$

Then for every $\mu\in(0,\omega)$ there exists a function b^{μ} such that $b^{\mu}\in H^{\infty}(\mathbf{S}^0_{\mu})$, and

$$\|b^{\mu}\|_{H^{\infty}(\mathbf{S}^0_{\mu})}\le C_{\mu}(c_0+c_1),$$

and the function pair determined by the function b^{μ} according to Theorem 1 is identical to (Φ,Φ_1) (modulo constants). Moreover, $b^{\mu} = b^{\mu,+}+b^{\mu,-}$,

$$(5)\qquad b^{\mu,\pm}(\eta) = \frac{1}{2\pi}\lim_{\varepsilon\to 0+}\Big\{\int\limits_{\Lambda^{\pm}(\varepsilon,\theta,|\eta|^{-1})} \exp(-i\eta z)\Phi(z)\,dz + \Phi_1(\varepsilon)\Big\},\qquad \eta\in\mathbf{S}^0_{\mu,\pm},$$

where $\theta = (\mu+\omega)/2$, $\Lambda^{\pm}(\varepsilon,\theta,\varrho) = l(\varepsilon,\varrho)\cup c^{\pm}(\theta,\varrho)\cup\Lambda^{\pm}(\theta,\varrho)$, and for $\varrho\le\pi$,

$$l(\varepsilon,\varrho) = \{z = x+iy : y=0,\ \varepsilon\le|x|\le\varrho\},$$

$$c^{\pm}(\theta,\varrho) = \{z = \varrho\exp(i\alpha) : \alpha \text{ goes from } \pi\pm\theta \text{ to } \pi, \text{ and then from } 0 \text{ to } \mp\theta\},$$

$$\Lambda^{\pm}(\theta,\varrho) = \{z\in\mathbf{C}^0_{\omega,\pm}(\pi) : z = r\exp(i(\pi\pm\theta)),\ r \text{ goes from } \pi\sec\theta \text{ to } \varrho, \text{ and then } z = r\exp(\mp i\theta),\ r \text{ goes from } \varrho \text{ to } \pi\sec\theta\},$$

and, for $\varrho>\pi$,

$$l(\varepsilon,\varrho) = l(\varepsilon,\pi),\qquad c^{\pm}(\theta,\varrho) = c^{\pm}(\theta,\pi),\qquad \Lambda^{\pm}(\theta,\varrho) = \Lambda^{\pm}(\theta,\pi).$$

Proof. Fix $\mu\in(0,\omega)$, and write b^{μ} as b in the rest of the proof. For all $\varepsilon\in(0,\pi)$ and $\eta\in\mathbf{S}^0_{\mu}\cup\{0\}$, define $b_{\varepsilon}(\eta) = b_{\varepsilon}^{+}(\eta)+b_{\varepsilon}^{-}(\eta)$, where $b_{\varepsilon}^{\pm}$ are the functions in the definition of the functions $b^{\pm}$ in the theorem before taking the limit as $\varepsilon\to 0$. We observe that $b_{\varepsilon}(0) = \frac{1}{2\pi}\Phi_1(\pi)$ for all ε. For $|\eta|^{-1}\le\pi$, we can show, using the estimates in [McQ1], that $b_{\varepsilon}(\eta)$ is uniformly bounded with the bounds indicated in the theorem, and that $\lim_{\varepsilon\to 0+} b_{\varepsilon}(\eta) = b(\eta)$ exists. For $|\eta|^{-1}>\pi$, to the integral over the contour $l(\varepsilon,\pi)$ we use the same argument and estimates as to the integral over $l(\varepsilon,|\eta|^{-1})$ for the case $|\eta|^{-1}\le\pi$. To estimate the integrals over $c^{\pm}(\theta,\pi)$ and $\Lambda^{\pm}(\theta,\pi)$ we use Cauchy's theorem to change the contour of integration and so to integrate over the set $\{z = x+iy : x=-\pi,\ y \text{ goes from } -(\pm\pi)\tan\theta \text{ to } 0, \text{ and then } x=\pi,\ y \text{ goes from } 0 \text{ to } -(\pm\pi)\tan\theta\}$. It is easy, however, to show that the integral over the last mentioned contour is bounded, using only the fact that $\pm\mathrm{Re}(z) > 0$. Therefore b is well defined with the desired bounds. We leave the details to the interested reader (or refer to [Q4]).

Let F be any 2π-periodic smooth function on $[-\pi,\pi]$. Expanding F in a Fourier series and using the definition of b_ε, we have

$$2\pi\sum_{n=-\infty}^{\infty} b_\varepsilon(n)\widehat{F}_{[-\pi,\pi]}(-n) = \int_{\varepsilon<|x|\le\pi} \Phi(x)F(x)\,dx + \Phi_1(\varepsilon)F(0).$$

On letting $\varepsilon\to 0$, we get

$$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{F}_{[-\pi,\pi]}(-n) = \lim_{\varepsilon\to 0}\Big\{\int_{\varepsilon<|x|\le\pi} \Phi(x)F(x)\,dx + \Phi_1(\varepsilon)F(0)\Big\}.$$

Denoting by $(\mathbf{G}(b),\mathbf{G}_1(b))$ a pair of holomorphic functions associated with b in the pattern of Theorem 1, from the Parseval identity it follows that

$$\lim_{\varepsilon\to 0}\Big\{\int_{\varepsilon<|x|<\pi} (\mathbf{G}(b)(x)-\Phi(x))F(x)\,dx + (\mathbf{G}_1(\varepsilon)-\Phi_1(\varepsilon))F(0)\Big\}$$
$$= 2\pi(b_1(0)-b(0))\widehat{F}_{[-\pi,\pi]}(0),$$

where $b_1(0)$ is associated with $(\mathbf{G}(b),\mathbf{G}_1(b))$ in the Parseval identity (iii) of Theorem 1. According to Theorem 1 (see also the argument at the end of its proof), we can add any constant to $\mathbf{G}(b)$ and accordingly adjust the value of $b_1(0)$ in order to make (iii) of Theorem 3 still hold. In particular, we can choose a constant such that $b_1(0)-b(0)=0$. The right hand side of the last displayed equality then becomes zero. Using an approximation to identity (F_n) with the property $F_n(0)=0$ for all n, we conclude that $\mathbf{G}(b)(x)=\Phi(x)$ for $x\ne 0$, which implies $\mathbf{G}(b)(z)=\Phi(z)$ for all $z\in\mathbf{S}^0_\omega(\pi)$ owing to analyticity. Using the assertion (ii) of Theorem 1 on $\mathbf{G}_1(b)$ and the assumption (ii) on the function Φ_1, we have $\Phi_1'=\mathbf{G}_1'(b)$ and so $\Phi_1-\mathbf{G}_1$ is a constant. Together with the property $\lim_{\varepsilon\to0}(\mathbf{G}_1(b)(\varepsilon)-\Phi_1(\varepsilon))=0$, this implies that $\Phi_1=\mathbf{G}_1(b)$. The uniqueness of b can be proved similarly. The proof is complete.

4. Singular integrals on star-shaped Lipschitz curves. The results obtained in §3 can be used to study the relations between singular integrals and multiplier transforms on periodic Lipschitz curves. Alternatively we can consider the closed star-shaped Lipschitz curves defined in §2. By performing the change of variable $z\to\exp(iz)$ and substituting $\widetilde{\Phi}=\Phi\circ(\frac{1}{i}\ln)$ and $\widetilde{\Phi}_1=\Phi_1\circ(\frac{1}{i}\ln)$ in Theorems 1 and 2, we obtain the following theorems.

THEOREM 3. *Let $\omega\in(0,\pi/2]$ and $b\in H^\infty(\mathbf{S}^0_\omega)$. Then there exists a pair of functions $(\widetilde{\Phi},\widetilde{\Phi}_1)$ such that $\widetilde{\Phi}$ and $\widetilde{\Phi}_1$ are holomorphic in $\exp(i\mathbf{S}^0_\omega(\pi))$ and $\exp(i\mathbf{S}^0_{\omega,+}(\pi))$, respectively and for every $\mu\in(0,\omega)$,*

(i) $$|\widetilde{\Phi}(z)|\le\frac{C_{\omega,\mu}\|b\|_\infty}{|1-z|},\qquad z\in\exp(i\mathbf{S}^0_\mu(\pi));$$

(ii) $$\widetilde{\Phi}_1\in H^\infty(\exp(i\mathbf{S}^0_\mu(\pi))),\quad \|\widetilde{\Phi}_1\|_{H^\infty(\exp(i\mathbf{S}^0_\mu(\pi)))}<C_{\omega,\mu}\|b\|_\infty,\quad \text{and}$$
$$\widetilde{\Phi}_1'(z)=\frac{1}{iz}(\widetilde{\Phi}(z)+\widetilde{\Phi}(z^{-1})),\quad z\in\exp(i\mathbf{S}^0_{\omega,+}(\pi));$$

(iii) $$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(-n)$$
$$=\lim_{\varepsilon\to0}\Big\{\int_{|\ln z|>\varepsilon,\,z\in\mathbb{T}}\widetilde{\Phi}(z)\widetilde{F}(z)\frac{dz}{z}+\widetilde{\Phi}_1(\exp(i\varepsilon))\widetilde{F}(1)\Big\}$$

for all smooth functions $\widetilde{F}$ defined on $\mathbb{T}$, where $\widehat{\widetilde{F}}_{\mathbb{T}}(n)$ is the nth Fourier coefficient of $\widetilde{F}$ and $b(0)=\frac{1}{2\pi}\widetilde{\Phi}_1(\exp(i\pi))$.

THEOREM 4. *Let $\omega\in(0,\pi/2]$ and $(\widetilde{\Phi},\widetilde{\Phi}_1)$ be a pair of holomorphic functions defined in $\exp(i\mathbf{S}^0_\omega(\pi))$ and $\exp(i\mathbf{S}^0_{\omega,+}(\pi))$, respectively, satisfying*

(i) *there is a constant c_0 such that*
$$|\widetilde{\Phi}(z)|\le\frac{c_0}{|1-z|},\qquad z\in\exp(i\mathbf{S}^0_\omega(\pi));$$

(ii) *there is a constant c_1 such that $\|\widetilde{\Phi}_1\|_{H^\infty(\exp(i\mathbf{S}^0_{\omega,+}(\pi)))}<c_1$, and*
$$\widetilde{\Phi}_1'(z)=\frac{1}{iz}(\widetilde{\Phi}(z)+\widetilde{\Phi}(z^{-1})),\qquad z\in\exp(i\mathbf{S}^0_{\omega,+}(\pi)).$$

Then for every $\mu\in(0,\omega)$, there exists a function b^μ in $H^\infty(\mathbf{S}^0_\mu)$,
$$\|b^\mu\|_{H^\infty(\mathbf{S}^0_\mu)}\le C_\mu(c_0+c_1),$$
and the function pair determined by b^μ according to Theorem 3 is identical to $(\widetilde{\Phi},\widetilde{\Phi}_1)$ modulo additive constants. Moreover, $b^\mu=b^{\mu,+}+b^{\mu,-}$,
$$b^\pm(\eta)=\frac{1}{2\pi}\lim_{\varepsilon\to0}\Big\{\int_{-i\ln z\in A^\pm(\varepsilon,\theta,\varrho)} z^{-\eta}\widetilde{\Phi}(z)\frac{dz}{z}+\widetilde{\Phi}_1(\exp(i\varepsilon))\Big\},\qquad \eta\in\mathbf{S}^0_{\mu,\pm},$$
where the contour $A^\pm(\varepsilon,\theta,\varrho)$ is defined in Theorem 2, and
$$\widetilde{\Phi}_1(\exp(i\varepsilon))=\int_{l(\varepsilon)}\widetilde{\Phi}(\exp(iz))\,dz,$$
where $l(\varepsilon)$ is any path from $-\varepsilon$ to ε lying in $\mathbf{C}^0_{\omega,\pm}$.

The following corollaries are versions of Theorems 3 and 4 in terms of holomorphic extension of series of positive and negative powers (see also the paragraph following the statement of Theorem 6 below).

COROLLARY 2. *Let* $(b_n)_{n=\pm1}^{\pm\infty} \in l^\infty$ *and* $\widetilde{\Phi}(z) = \sum_{n=\pm1}^{\pm\infty} b_n z^n$, $|z^{\pm1}| < 1$, *and* $\omega \in (0, \pi/2)$. *If there exist* $\delta > 0$ *such that* $\omega + \delta \le \pi/2$, *and a function* $b \in H^\infty(\mathbf{S}^0_{\omega+\delta,\pm})$ *such that* $b(n) = b_n$ *for all* $\pm n = \pm1, \pm2, \ldots$, *then the function* $\widetilde{\Phi}$ *can be holomorphically extended to the region* $\exp(i\mathbf{C}^0_{\omega+\delta,\pm}(\pi))$. *Moreover, we have*

$$|\widetilde{\Phi}(z)| \le \frac{C_{\omega,\delta}}{|1-z|}, \qquad z \in \exp(i\mathbf{C}^0_{\omega,\pm}(\pi)).$$

COROLLARY 3. *Let* $\omega \in (0, \pi/2]$, *and* $\widetilde{\Phi}$ *be holomorphic and satisfy*

$$|\widetilde{\Phi}(z)| \le \frac{C}{|1-z|}, \qquad z \in \exp(i\mathbf{C}^0_{\omega,\pm}(\pi)).$$

Then for every $\mu \in (0, \omega)$ *there exists a function* b^μ *such* $b^\mu \in H^\infty(\mathbf{S}^0_{\mu,\pm})$ *and* $\widetilde{\Phi}(z) = \sum_{n=\pm1}^{\pm\infty} b^\mu(n) z^n$. *Moreover*, $b^\mu = b^{\mu,+} + b^{\mu,-}$,

$$b^{\mu,\pm}(\eta) = \frac{1}{2\pi} \lim_{\varepsilon\to0+} \Big\{ \int\limits_{-i\ln z \in A^\pm(\varepsilon,\theta,\varrho)} \exp(-i\eta z)\widetilde{\Phi}(\exp(iz))\,dz + \widetilde{\Phi}_1(\exp(i\varepsilon)) \Big\},$$

$$\eta \in \mathbf{S}^0_{\mu,\pm},$$

where $A^\pm(\varepsilon, \theta, \varrho)$ *is defined as in Theorem* 2, *and*

$$\widetilde{\Phi}_1(\exp(i\varepsilon)) = \int\limits_{l(\varepsilon)} \widetilde{\Phi}(\exp(iz))\,dz,$$

where $l(\varepsilon)$ *is any path from* $-\varepsilon$ *to* ε *lying in* $\mathbf{C}^0_{\omega,\pm}$.

Remark. As indicated in Corollary 3 the mapping $\widetilde{\Phi} \to b$ satisfying $\widetilde{\Phi}(z) = \sum b(n) z^n$ is not single-valued. In fact, if $\mu_1 \ne \mu_2$, then both b^{μ_1} and b^{μ_2} satisfy the requirement, but $b^{\mu_1} \ne b^{\mu_2}$ in general. This can be verified by using $\widetilde{\Phi}(z) = z^n$, $n \in \mathbb{Z}^+$, for example (see also [Q4]).

Joachim Hempel observed the following application of Corollary 2.

COROLLARY 4. *For any* $\omega \in (0, \pi/2]$ *there does not exist any function* b *satisfying* $b \in H^\infty(\mathbf{S}_{\omega,+})$ *and* $b(n) = 1$ *for* $n = 2^k$, $k = 1, 2, \ldots$, *and* $b(n) = 0$ *for any other positive integers.*

Proof. Consider the function

$$\widetilde{\Phi}(z) = z + z^2 + z^{2^2} + \ldots + z^{2^k} + \ldots$$

It is well known that $\widetilde{\Phi}$ does not have any holomorphic extension across intervals on the unit circle, and hence, according to Corollary 2, it is not induced by a function b in $H^\infty(\mathbf{S}^0_{\omega,+})$ in the pattern of Corollary 2.

For a function b and a function $\widetilde{F}$ defined in Theorem 3, by Laurent series theory, the series

$$\sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n) z^n$$

locally uniformly converges to a holomorphic function in the annulus on which $\widetilde{F}$ is defined. Recalling the fact that $\widehat{\widetilde{F}}_{\mathbb{T}}(n) = \widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n)$, we can define an operator $\widetilde{M}_b : \mathcal{A}(\widetilde{\Gamma}) \to \mathcal{A}(\widetilde{\Gamma})$ by

$$\widetilde{M}_b \widetilde{F}(z) = 2\pi \sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n) z^n.$$

On the other hand, for a pair of functions $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ specified in Theorem 4 we can formally write

$$T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}\widetilde{F}(z)$$

$$= \lim_{\varepsilon\to0} \Big\{ \int\limits_{\pi \ge |\mathrm{Re}(i^{-1}\ln(\eta z^{-1}))| > \varepsilon,\ \eta\in\widetilde{\Gamma}} \widetilde{\Phi}(z\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta} + \widetilde{\Phi}_1(\exp(i\varepsilon t(z)))\widetilde{F}(z) \Big\},$$

where $t(z)$ is the normalized tangent vector to Γ at z lying inside $\mathbf{S}^0_{\omega,+}(\pi)$.

We have the following theorem:

THEOREM 5. *Let* $\omega \in (\arctan N, \pi/2]$, $b \in H^\infty(\mathbf{S}^0_\omega)$ *and* $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ *be the function pair associated with* b *in the pattern of Theorem* 3. *Then the following hold:*

(i) $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$ *is a well-defined operator from* $\mathcal{A}(\widetilde{\Gamma})$ *to* $\mathcal{A}(\widetilde{\Gamma})$, *and* $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)} = \widetilde{M}_b$ *modulo a constant multiple of the identity operator*;

(ii) $\widetilde{M}_b$ *extends to a bounded operator on* $L^2(\widetilde{\Gamma})$ *whose operator norm is dominated by* $c\|b\|_\infty$.

Proof. (i) For any $\alpha > 0$, define $b_z^{\pm,\alpha}(\zeta) = -z^{-\zeta} b^{\pm,\alpha}(-\zeta)$, where $b^{\pm,\alpha}$ are the functions defined in the proof of Theorem 1. Let $(\widetilde{\Phi}_z^{\pm,\alpha}, (\widetilde{\Phi}_z^{\pm,\alpha})_1)$ be the function pair associated with $b_z^{\pm,\alpha}$ in the pattern of Theorem 3. Owing to (iii) of Theorem 3 and Cauchy's theorem, we have

$$\widetilde{M}_{b_z^{\pm,\alpha}}\widetilde{F}(z) = 2\pi \sum_{n=-\infty}^{\infty} b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\widetilde{\Gamma}}(n) = 2\pi \sum_{n=-\infty}^{\infty} b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n)$$

$$= \int\limits_{\mathbb{T}} \widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta} = \int\limits_{\widetilde{\Gamma}} \widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}.$$

Taking the limit $\alpha \to 0$ as in the proof of Theorem 1 and noticing that

$$\widetilde{\Phi}_z^{\pm}(\eta^{-1}) = \widetilde{\Phi}^{\pm}(z\eta^{-1}),$$

we obtain the desired equality for $b^\pm$, and hence for b.

(ii) One can alternatively prove the boundedness of the operator

$$T_{(\Phi,\Phi_1)}F(z)=\lim_{\varepsilon\to 0}\Big\{\int\limits_{\pi\geq|\mathrm{Re}(z-\eta)|>\varepsilon}\Phi(z-\eta)F(\eta)\,d\eta+\Phi_1(\varepsilon t(z))F(z)\Big\},$$

$$F\in\mathcal{A}(\Gamma),$$

where $t(z)$ is the normalized tangent vector of Γ at z lying inside $\mathbf{S}^0_{\omega,+}(\pi)$, and $\mathcal{A}(\Gamma)$ is the class of 2π-periodic and holomorphic functions defined by the condition $F\in\mathcal{A}(\Gamma)$ if and only if $\widetilde{F}=F\circ(i^{-1}\ln)\in\mathcal{A}(\widetilde{\Gamma})$. Owing to the decomposition of Φ in the assertion (i) of Theorem 1, we have

$$\begin{aligned}T_{(\Phi,\Phi_1)}F(z)=\lim_{\varepsilon_n\to 0}\Big\{&\int\limits_{\pi\geq|\mathrm{Re}(z-\eta)|>\varepsilon_n}\phi(z-\eta)F(\eta)\,d\eta\\&+\int\limits_{\pi\geq|\mathrm{Re}(z-\eta)|>\varepsilon_n}\phi_0(z-\eta)F(\eta)\,d\eta\Big\}\\&+c_1\int\limits_{-\pi}^{\pi}F(\eta)\,d\eta+c_2F(z),\end{aligned}$$

where $\varepsilon_n\to 0$ is an appropriate subsequence of $\varepsilon\to 0$, and c_1 and c_2 are constants.

The second and the third integrals are dominated by the L^2-norm of F, while the first integral is dominated by

$$\sup_{\varepsilon>0}\Big|\int\limits_{|\mathrm{Re}(z-\eta)|>\varepsilon}\phi(z-\eta)F_1(\eta)\,d\eta\Big|+c\mathcal{M}F_1(z),\qquad\mathrm{Re}(z)\in[-\pi,\pi],$$

where $F_1(\eta)=F(\eta)$ for $|\mathrm{Re}(\eta)|\leq 2\pi$, and $F_1(\eta)=0$ otherwise, and $\mathcal{M}F_1$ is the Hardy–Littlewood maximal operator of F_1 on the curve. Owing to the boundedness results for the operator introduced by (ϕ,ϕ_1) (see [McQ1]) and for the Hardy–Littlewood maximal function, we obtain the desired boundedness.

Remark. There is some interest in direct proofs of Theorem 5(ii), for which we refer to [GQW] and [Q2].

We state without proof the following theorem. For a proof we refer the reader to [McQ2].

THEOREM 6. *Let $\omega\in(\arctan N,\pi/2]$, $\widetilde{\Phi}$ be holomorphic in $\exp(i\mathbf{S}^0_\omega)$ and satisfy* (i) *of Theorem* 4 *with respect to ω. If T is a bounded operator on $L^2(\widetilde{\Gamma})$ and*

$$T(\widetilde{F})(z)=\int\limits_{\widetilde{\Gamma}}\widetilde{\Phi}(z\zeta^{-1})\widetilde{F}(\zeta)\frac{d\zeta}{\zeta},\qquad z\notin\mathrm{supp}(\widetilde{F}),$$

for all $\widetilde{F}\in C_0(\widetilde{\Gamma})$, the class of continuous functions, then there exists a unique function $\widetilde{\Phi}_1\in H^\infty(\exp(i\mathbf{S}^0_{\mu,+}))$, $\mu\in(0,\omega)$, such that

$$\widetilde{\Phi}_1'(z)=\frac{1}{iz}(\widetilde{\Phi}(z)+\widetilde{\Phi}(z^{-1})),\qquad z\in\exp(i\mathbf{S}^0_{\omega,+}(\pi)),$$

and

$$T(\widetilde{F})=T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}(\widetilde{F})$$

for all $\widetilde{F}\in C_0(\widetilde{\Gamma})$.

As stated in Corollary 2, for $b\in\mathbf{S}^0_\omega$ the function $\widetilde{\Phi}^+(z)=\sum_{n=1}^{\infty}b(n)z^n$, $|z|<1$, can be holomorphically extended to $\exp(i\mathbf{C}^0_{\omega,+}(\pi))$, and $\widetilde{\Phi}^-(z)=\sum_{n=-\infty}^{-1}b(n)z^n$, $|z|>1$, can be holomorphically extended to $\exp(\mathbf{C}^0_{\omega,-}(\pi))$. So, we have the expression

$$\widetilde{\Phi}(z)=\sum_{n=-\infty}^{\infty}{}'\,b(n)z^n,\qquad z\in\exp(i\mathbf{S}^0_\omega(\pi)).\tag{6}$$

In many cases using (6) is more convenient than using (4) in finding an explicit formula for $\widetilde{\Phi}$, and hence for Φ.

EXAMPLE (i). If $b(z)=-i\,\mathrm{sgn}(z)$, then from [McQ1] we get $\phi(z)=\frac{1}{\pi}\cdot\frac{1}{z}$, $\phi_1=0$, which corresponds to the Hilbert transform with kernel $\phi(z)$. Using the expression (6) we obtain $\Phi(z)=\cot\frac{z}{2}$, $\Phi_1=0$, $\widetilde{\Phi}(z)=-i\frac{1+z}{1-z}$, and $\widetilde{\Phi}_1=0$. From the assertion (i) of Theorem 5, the Fourier multiplier $-i\,\mathrm{sgn}$ corresponds to the kernels $\frac{-i}{2\pi}\cdot\frac{1+z}{1-z}$ and $\frac{1}{2\pi}\cot\frac{z}{2}$ on $\widetilde{\Gamma}$ and Γ, respectively.

EXAMPLE (ii). Let $\lambda\notin\mathbf{S}^0_\omega$. Then $b(z)=\frac{1}{z-\lambda}$ corresponds to the resolvent of the surface Dirac operator on every star-shaped Lipschitz curve. If $\mathrm{Im}(\lambda)>0$, then from [McQ1] we have

$$\phi_\lambda(z)=\begin{cases}i\exp(i\lambda z)&\text{if }\mathrm{Re}(z)>0,\\0&\text{if }\mathrm{Re}(z)<0.\end{cases}$$

If $\mathrm{Im}(\lambda)<0$, then we have

$$\phi_\lambda(z)=\begin{cases}0&\text{if }\mathrm{Re}(z)>0,\\-i\exp(i\lambda z)&\text{if }\mathrm{Re}(z)<0.\end{cases}$$

It is easy to see that in each of the two cases ϕ_λ is in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, and so the remark we made after Theorem B applies to both cases.

From the definition, for $\mathrm{Im}(\lambda)>0$, we have

$$\Phi_\lambda(z)=\begin{cases}\dfrac{i\exp(i\lambda(z+2\pi))}{1-\exp(i\lambda 2\pi)}&\text{if }-\pi<\mathrm{Re}(z)<0,\\[2ex]\dfrac{i\exp(i\lambda z)}{1-\exp(i\lambda 2\pi)}&\text{if }0<\mathrm{Re}(z)<\pi.\end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\Phi_\lambda(z) = \begin{cases} \dfrac{-i\exp(i\lambda(z-2\pi))}{1-\exp(-i\lambda 2\pi)} & \text{if } 0 < \mathrm{Re}(z) < \pi, \\ \dfrac{-i\exp(i\lambda z)}{1-\exp(-i\lambda 2\pi)} & \text{if } -\pi < \mathrm{Re}(z) < 0. \end{cases}$$

For $\mathrm{Im}(\lambda) > 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \dfrac{i\exp(i\lambda 2\pi)z^\lambda}{1-\exp(i\lambda 2\pi)} & \text{if } -\pi < \mathrm{Re}\left(\dfrac{\ln z}{i}\right) < 0, \\ \dfrac{iz^\lambda}{1-\exp(i\lambda 2\pi)} & \text{if } 0 < \mathrm{Re}\left(\dfrac{\ln z}{i}\right) < \pi. \end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \dfrac{-i\exp(-i\lambda 2\pi)z^\lambda}{1-\exp(-i\lambda 2\pi)} & \text{if } 0 < \mathrm{Re}\left(\dfrac{\ln z}{i}\right) < \pi, \\ \dfrac{-iz^\lambda}{1-\exp(-i\lambda 2\pi)} & \text{if } -\pi < \mathrm{Re}\left(\dfrac{\ln z}{i}\right) < 0. \end{cases}$$

As in the above example, $\frac{1}{2\pi}$ times the above functions will be the kernels of the resolvents on the curves Γ and $\widetilde{\Gamma}$, respectively.

We now outline how the H^∞-functional calculus developed in [Mc] can be applied to the present case (see [McQ1], [McQ3] for the infinite Lipschitz graph case), and indicate the relation between this functional calculus and the operator classes $\widetilde{M}_b$ and $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$.

For a function $\widetilde{F} \in \mathcal{A}(\widetilde{\Gamma})$ we define the differential operator $\frac{d}{dz}\big|_{\widetilde{\Gamma}}$ by

$$\frac{d}{dz}\bigg|_{\widetilde{\Gamma}} \widetilde{F}(z) = \lim_{\substack{h\to 0 \\ z+h\in\widetilde{\Gamma}}} \frac{\widetilde{F}(z+h)-\widetilde{F}(z)}{h} \quad \text{for } z \in \widetilde{\Gamma}.$$

For $1 < p < \infty$, $\langle L^p(\widetilde{\Gamma}), L^{p'}(\widetilde{\Gamma})\rangle$ is a pairing of Banach spaces given by $\langle \widetilde{F}, \widetilde{G}\rangle = \int_{\widetilde{\Gamma}} \widetilde{F}(z)\widetilde{G}(z)\,dz$, where $p' = (1-p^{-1})^{-1}$. Now use duality to define $D_{\widetilde{\Gamma},p}$ as the closed operator with the largest domain in $L^p(\widetilde{\Gamma})$ which satisfies

$$\langle D_{\widetilde{\Gamma},p}\widetilde{F}, \widetilde{G}\rangle = \left\langle \widetilde{F}, -z\frac{d}{dz}\bigg|_{\widetilde{\Gamma}}\widetilde{G}\right\rangle$$

for all $\widetilde{F}$ and $\widetilde{G}$ in $\mathcal{A}(\widetilde{\Gamma})$.

Let $\omega \in (\arctan N, \pi/2]$ and $\lambda \notin \mathbf{S}^0_\omega$. It is easy to verify that $D_{\widetilde{\Gamma},p}$ is the surface Dirac operator on $\widetilde{\Gamma}$ and the function $\frac{1}{2\pi}\widetilde{\Phi}_\lambda$ given in Example (ii) is the convolution kernel of the resolvent operator $(D_{\widetilde{\Gamma},p} - \lambda)^{-1}$ in the sense

of Theorem 5. Moreover,

$$\begin{aligned} \|(D_{\widetilde{\Gamma},p} - \lambda)^{-1}\| &\le \left\|\frac{1}{2\pi}\widetilde{\Phi}_\lambda\right\|_{L^1(\widetilde{\Gamma})} \le \sum_{n=-\infty}^{\infty} \|\phi_\lambda(\cdot + 2\pi n)\|_{L^1(\Gamma)} \\ &= \|\phi_\lambda\|_{L^1(p\Gamma)} \le \sqrt{1+N^2}\{\mathrm{dist}(\lambda, \mathbf{S}^0_\omega)\}^{-1}, \end{aligned}$$

where we have used the bounds of $\|\phi_\lambda\|_{L^1(p\Gamma)}$ obtained in [McQ1].

The above estimate implies that $D_{\widetilde{\Gamma},p}$ is a *type-ω* operator ([Mc]) that allows us to define $b(D_{\widetilde{\Gamma},p})$ via spectral integrals first for those H^∞-functions b with good decay properties at both 0 and ∞:

$$b(D_{\widetilde{\Gamma},p}) = \frac{1}{2\pi i}\int_\delta b(\eta)(D_{\widetilde{\Gamma},p} - \eta I)^{-1}\,d\eta,$$

where δ is a path consisting of four rays: $\{s\exp(-i\theta) : s \text{ goes from } \infty \text{ to } 0\}$ $\cup\, \{s\exp(i\theta) : s \text{ goes from } 0 \text{ to } \infty\} \cup \{s\exp(i(\pi-\theta)) : s \text{ goes from } \infty \text{ to } 0\}$ $\cup\, \{s\exp(i(\pi+\theta)) : s \text{ goes from } 0 \text{ to } \infty\}$, where $\arctan N < \delta < \omega$.

It is not difficult to show, using the above estimate, that each $b(D_{\widetilde{\Gamma},p})$ is a bounded operator, and $b(D_{\widetilde{\Gamma},p}) = \widetilde{M}_b = T_{(\widetilde{\Phi},0)}$. Taking limits of sequences of Calderón–Zygmund operators (see [Mc] or [CM2]), one can then extend the definition of $b(D_{\widetilde{\Gamma},p})$ to all the functions in $H^\infty(\mathbf{S}^0_\omega)$, and prove

$$b(D_{\widetilde{\Gamma},p}) = \widetilde{M}_b = T_{(\widetilde{\Phi},\widetilde{\Phi}_1)},$$

with the properties

$$\begin{gathered} \|b(D_{\widetilde{\Gamma},p})\| \le C_\omega\|b\|_\infty, \\ (b_1b_2)(D_{\widetilde{\Gamma},p}) = b_1(D_{\widetilde{\Gamma},p})b_2(D_{\widetilde{\Gamma},p}), \\ (\alpha_1b_1 + \alpha_2b_2)(D_{\widetilde{\Gamma},p}) = \alpha_1b_1(D_{\widetilde{\Gamma},p}) + \alpha_2b_2(D_{\widetilde{\Gamma},p}) \end{gathered}$$

whenever $b_1, b_2 \in H^\infty(\mathbf{S}^0_\omega)$ and α_1, α_2 are complex numbers.

5. Fourier multipliers on star-shaped Lipschitz curves. In this section we shall not restrict ourselves to the H^∞-multipliers. We wish to point out that all the results and methods of the Fourier multiplier theory for the infinite Lipschitz graph case developed in [McQ3] can be adapted to the present case. The major changes are: the class $\mathcal{A}(\widetilde{\Gamma})$ is good enough for our purpose, and whenever we deal with a kernel on Γ we refer to its corresponding kernel on $p\Gamma$ via the Poisson summation formula. We shall state two results without proofs. Both can be proved using the corresponding Schur lemma in the present case (see [McQ3]).

For $b=(b_n)_{n=-\infty}^{\infty}\in l^\infty$, define

$$\|b\|_{M_p(\widetilde{\Gamma})}=\sup\Big\{\Big\|\sum b_n\widehat{\widetilde{F}}(n)z^n\Big\|_{L^p(\widetilde{\Gamma})}:\|\widetilde{F}\|_{L^p(\widetilde{\Gamma})}\le 1\Big\},$$

and

$$M_p(\widetilde{\Gamma})=\{b:\|b\|_{M_p(\widetilde{\Gamma})}<\infty\}.$$

Functions b in $M_p(\widetilde{\Gamma})$ are called *$L^p(\widetilde{\Gamma})$-Fourier multipliers*.

THEOREM 7. *Let $\widetilde{\Phi}$ be a holomorphic function defined in a simply connected open neighbourhood of the difference set $\widetilde{\Gamma}-\widetilde{\Gamma}=\{z-\zeta:z,\zeta\in\widetilde{\Gamma}\}$ satisfying $|\widetilde{\Phi}(r\exp(i\theta))|\le\psi(\exp(i\theta))$, where $\int_{-\pi}^{\pi}\psi(\exp(i\theta))\,d\theta<\infty$. Then $b=(\widehat{\widetilde{\Phi}}(n))_{n=-\infty}^{\infty}\in M_p(\widetilde{\Gamma})$, $1<p<\infty$, and the associated convolution operator $T_{\widetilde{\Phi}}$ is given by*

$$T_{\widetilde{\Phi}}\widetilde{F}(z)=\int_{\widetilde{\Gamma}}\widetilde{\Phi}(z\eta^{-1})\widetilde{F}(\eta)\,\frac{d\eta}{\eta},\qquad \widetilde{F}\in\mathcal{A}(\widetilde{\Gamma}).$$

Let $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ be two curves of the type under consideration. Define

$$M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_2)=\{b\in l^\infty:\|b\|_{M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_2)}<\infty\},$$

where

$$\|b\|_{M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_2)}=\sup\Big\{\frac{\|\sum b_n\widehat{\widetilde{F}}(n)z^n\|_{L^p(\widetilde{\Gamma}_2)}}{\|\widetilde{F}\|_{L^p(\widetilde{\Gamma}_1)}}:\widetilde{F}\in\mathcal{A}(\widetilde{\Gamma}_1)\cap\mathcal{A}(\widetilde{\Gamma}_2)\Big\}.$$

If $\widetilde{\Gamma}_3$ is a third such curve, and $b_1\in M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_2)$, $b_2\in M_p(\widetilde{\Gamma}_2,\widetilde{\Gamma}_3)$, then $b_2b_1\in M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_3)$, and

$$\|b_2b_1\|_{M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_3)}\le\|b_2\|_{M_p(\widetilde{\Gamma}_2,\widetilde{\Gamma}_3)}\|b_1\|_{M_p(\widetilde{\Gamma}_1,\widetilde{\Gamma}_2)}.$$

THEOREM 8. *Let $b\in l^\infty$ and $f_\beta(n)=b(n)\exp(2\beta|n|)$. If $f_\beta\in M_p(\mathbb{T})$ for some $\beta>M=\max A(x)$, where $\mathbb{T}$ is the unit circle and $1<p<\infty$, then $b\in M_p(\widetilde{\Gamma})$ and*

$$\|b\|_{M_p(\widetilde{\Gamma})}\le(2\pi\beta)^2(\beta^2-M^2)^{-1}(1+N^2)^{1/2}\|f_\beta\|_{M_p(\mathbb{T})}.$$

The following example shows that we cannot expect the relation $\|b\|_{M_2(\widetilde{\Gamma})}\le C_{\widetilde{\Gamma}}\|b\|_\infty$ to hold in general ([McQ3]), although it does hold when the curve Γ is flat.

Take $\Gamma(x)=x+iA(x)$ to be a piece of Lipschitz curve defined on $[-\pi,\pi]$ with $A(-\pi)=A(\pi)$, $A(0)>0$ and $m=\min A(x)<0$. For any integer S let b_S be the sequence in l^∞ defined by $b_S(n)=1$ if $n\le S$ and $b_S(n)=0$ otherwise. Using $F(z)=\frac{1}{1-\exp(iz)}$ as a test function one can show that for any $\varepsilon>0$,

$$\|b_S\|_{M_2(\widetilde{\Gamma})}\ge C_\varepsilon\exp(-S(m+\varepsilon)).$$

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Received May 26, 1994
Revised version November 8, 1996 (3278)

STUDIA MATHEMATICA 123 (3) (1997)

A Phragmén–Lindelöf type quasi-analyticity principle

by

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Abstract. Quasi-analyticity theorems of Phragmén–Lindelöf type for holomorphic functions of exponential type on a half plane are stated and proved. Spaces of Laplace distributions (ultradistributions) on $\mathbb{R}$ are studied and their boundary value representation is given. A generalization of the Painlevé theorem is proved.

1. Introduction and statement of the main results. The well-known Phragmén–Lindelöf theorem consists of two parts. The first one ([H]), called the maximum principle, says that a function holomorphic and of exponential type on a sector S of opening less than π is bounded if it is bounded on the boundary of S. The second one ([T]), called the quasi-analyticity principle, says that a holomorphic function F on a sector S vanishes if the opening of S is greater than π and F is exponentially decreasing in S.

In the present paper we study the quasi-analyticity principle in the critical case of a half plane Π. To ensure vanishing of F in that case we assume that F is of exponential type in Π and decreases exponentially along the boundary of Π. More precisely, we have

THEOREM 1 (Quasi-analyticity principle, continuous version). *Let* $F \in \mathcal{O}(\{\operatorname{Re} z > 0\}) \cap C^0(\{\operatorname{Re} z \ge 0\})$ *be of exponential type, i.e.*

(1) $$|F(z)| \le Ce^{c|z|} \quad \textit{for } \operatorname{Re} z \ge 0 \textit{ with some } C < \infty \textit{ and } c < \infty.$$

If

(2) $$|F(\pm ir)| \le Ce^{c^{\pm}r} \quad \textit{for } r \ge 0$$

with some $c^{\pm} \in \mathbb{R}$ *such that* $c^+ + c^- < 0$ *then* $F \equiv 0$.

The elementary proof of Theorem 1 is based on the Laplace integral representation of holomorphic functions of exponential type.

1991 *Mathematics Subject Classification*: 30D15, 44A15, 46F12, 46F20.

Key words and phrases: quasi-analyticity, Laplace distributions, Laplace ultradistributions, boundary values.

Partially supported by KBN grant No 2 PO3A 006 08.