Acknowledgments. We thank T. Iwaniec and N. Fusco for helpful discussions on the subject.

References

[1] H. Brezis, *Analyse Fonctionnelle, théorie et applications*, Masson, 1983.
[2] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.
[3] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. of Math. Stud. 105, Princeton, 1983.
[4] M. Giaquinta and G. Modica, *Regularity results for some classes of higher order nonlinear elliptic systems*, J. Reine Angew. Math. 311/312 (1979), 145–169.
[5] E. Giusti, *Metodi diretti del calcolo delle variazioni*, Unione Matematica Italiana, 1994.
[6] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Ann. of Math. 136 (1992), 589–624.
[7] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. 454 (1994), 143–161.
[8] T. Iwaniec, C. Scott and B. Stroffolini, *Nonlinear Hodge theory on manifolds with boundary*, preprint.
[9] D. Giachetti, F. Leonetti and R. Schianchi, *On the regularity of very weak minima*, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 287–296.
[10] —, —, —, *Boundary regularity and uniqueness for very weak $\mathcal{A}$ harmonic functions*, in preparation.
[11] N. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa 17 (1963), 189–206.
[12] N. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions*, Duke Math. J. 42 (1975), 121–136.
[13] G. Moscariello, *Weak minima and quasiminima of variational integrals*, Boll. Un. Mat. Ital., to appear.
[14] —, *On weak minima of certain integral functionals*, preprint.
[15] B. Stroffolini, *On weakly A-harmonic tensors*, Studia Math. 114 (1995), 289–301.

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Facoltà di Ingegneria
Università di Roma "La Sapienza"
Via A. Scarpa 16
00161 Roma, Italy
E-mail: giachett@dmmm.uniroma1.it
schianch@dmmm.uniroma1.it

Received July 10, 1996
Revised version September 23, 1996 (3711)

STUDIA MATHEMATICA 123 (2) (1997)

On a theorem of Gelfand and its local generalizations

by

DRISS DRISSI (Kuwait)

Abstract. In 1941, I. Gelfand proved that if a is a doubly power-bounded element of a Banach algebra A such that $\mathrm{Sp}(a) = \{1\}$, then $a = 1$. In [4], this result has been extended locally to a larger class of operators. In this note, we first give some quantitative local extensions of Gelfand–Hille's results. Secondly, using the Bernstein inequality for multivariable functions, we give short and elementary proofs of two extensions of Gelfand's theorem for m commuting bounded operators, $T_1, \ldots, T_m$, on a Banach space X.

1. Introduction. In 1941, I. Gelfand [14] proved that if T is a bounded linear operator on a complex Banach space X which satisfies $\mathrm{Sp}(T) = \{1\}$ and $\sup_{k\in\mathbb{Z}} \|T^k\| < \infty$, then $T = I$. This result was generalized by E. Hille in 1944 (see [15] or [16, Theorem 4.10.1]), who proved that if $\mathrm{Sp}(T) = \{1\}$ and $\|T^k\| = o(|k|)$ for $k \in \mathbb{Z}$, then $T = I$. In [4, Theorem 3.4], we generalized these results locally to a wider class of operators. A natural question arises: What happens to each of these results if we drop the assumption on the boundedness of the negative powers of the operator T? On the other hand, in 1955 H. F. Bohnenblust and S. Karlin [9] asked the following question: Is 0 the only quasi-nilpotent dissipative element in a Banach algebra? In 1961, G. Lumer and R. S. Phillips [22] gave a negative answer to this question, but nobody noticed that Shilov's negative answer to Gelfand's problem in [28] is also a negative answer to H. F. Bohnenblust and S. Karlin's question. Just looking at the Gelfand problem, the condition $\mathrm{Sp}(T) = \{1\}$ implies, using the F. Riesz and N. Dunford holomorphic functional calculus, that $T = e^S$ with S quasi-nilpotent and the hypothesis $\sup_{n\geq 0} \|T^n\| < \infty$ implies that S is dissipative for an equivalent norm.

In this paper, we will study these cases locally for a general class of operators. We will also give an extension to n commuting operators $T_1, \ldots, T_n$ in a Banach space X. For this we need to introduce some preliminaries on

1991 *Mathematics Subject Classification*: 47B10, 47B15.
Key words and phrases: locally power-bounded operator, local spectrum, local spectral radius.
Research supported by the Kuwait University Research Grant No. SM 138.

local spectral theory; for more details on this subject we refer to [11] or [5].

Let $T \in \mathcal{B}(X)$ and $x \in X$. We define Ω_x to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α of α with u analytic on V_α having values in X such that $(\lambda - T)u(\lambda) = x$ on V_α. This set is open and contains the complement of the spectrum of T. The function u is called a *local resolvent* of T on V_α. By definition the *local spectrum* of T at x, denoted by $\mathrm{Sp}_x(T)$, is the complement of Ω_x, so it is a compact subset of $\mathrm{Sp}(T)$.

In general, this set may be empty even for $x \neq 0$ (take the left shift operator on l^2 with $x = e_1 = (1, 0, \ldots)$). But for $x \neq 0$, the local spectrum of T at x is non-empty if T has the uniqueness property for the local resolvent. That is, $(\lambda - T)v(\lambda) = 0$ implies $v = 0$ for any analytic function v defined on any domain D of $\mathbb{C}$ with values in a Banach space X. It is easy to see that an operator T having spectrum without interior points has this property (for more details see [11]). For operators with this property there is a unique local resolvent which is the analytic extension of $(\lambda - T)^{-1}x$ to Ω_x. Also in this case the *local spectral radius* $r_x(T) = \max\{|z| : z \in \mathrm{Sp}_x(T)\}$ is equal to $\limsup_{k\to\infty} \|T^k x\|^{1/k}$. In general, we only have $r_x(T) \le \limsup_{k\to\infty} \|T^k x\|^{1/k}$.

2. Local properties for a single operator. In [4], using the Levin subordination theory for entire functions of exponential type (see [19] for more details), the following generalization of Gelfand's and Hille's results was proved:

THEOREM 2.1. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $r_x(T) = 0$, *and*

(ii) $\|(I + T)^k x\| = O(|k|^r)$ *as* $|k| \to \infty$, *for some positive integer* r.

Then $T^n x = 0$ *for every* $n \ge r + 1$.

The purpose here is to give a local quantitative extension of Gelfand's theorem.

THEOREM 2.2. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $\|T^n x\| = O(1)$ *as* $n \to \infty$, *and*

(ii) *there exists* $S \in \mathcal{B}(X)$ *such that* $T = e^S$.

Then

$$\|Tx - x\| \le 2\tan(\tau_x/2),$$

where $\tau_x = (2/\sqrt{e})\sqrt{\limsup_{k\to\infty} k\|S^k x\|^{1/k}}$.

Proof. Suppose that $\|T^n x\| = O(1)$ as $n \to \infty$. Let $S \in \mathcal{B}(X)$ be such that $e^S = T$. Let $g(z) = u(e^{z^2 S}x)$, where u is a functional of norm one.

Then g is an entire function of exponential type τ_x, with

$$\begin{aligned}\tau_x &= \limsup_{n\to\infty} \|g^{(n)}(0)\|^{1/n} \\ &= \limsup_{k\to\infty} \|g^{(2k)}(0)\|^{1/(2k)} \le \frac{2}{\sqrt{e}}\sqrt{\limsup_{k\to\infty} k\|S^k x\|^{1/k}}.\end{aligned}$$

So, if $\limsup_{n\to\infty} n\|S^n x\|^{1/n} < \pi^2 e/4$, then by Bernstein's theorem [8, Theorem 11.4.1, p. 214], we obtain

$$\|e^S x - x\| = |g(1) - g(0)| \le 2\tan(\tau_x/2),$$

which gives us $\|Tx - x\| \le 2\tan(\tau_x/2)$. If $\tau_x = 0$, then we have $Tx = x$.

THEOREM 2.3. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $n\|T^n x\|^{1/n} \to 0$, *as* $n \to \infty$, *and*

(ii) $\|(I + T)^k x\| = O(k^r)$ *as* $k \to \infty$, *for some positive integer* r.

Then $T^n x = 0$ *for all* $n \ge r + 1$.

Proof. The case $r = 0$ follows from Theorem 2.2, and the rest of the proof follows the same idea as in [3] but locally.

Remark. Let T be a bounded operator on a Banach space X and let $x \in X$ be such that $\|(I + T)^n x\| = o(|n|^r)$. Suppose that $T^r x \neq 0$. It follows from Theorem 2.2 that $\limsup_{n\to\infty} n\|T^n x\|^{1/n} \neq 0$. So the series $\sum_{n=1}^{\infty} \|T^n x\|/\|T^{n-1}x\|$ diverges. Consequently, we obtain a result by Pedersen [26], which is an extension of a result of J. Esterle and F. Zouakia's.

The following two theorems are well known. Esterle's theorem [13, Theorem 9.1] was the first significant generalization of Gelfand's theorem. Subsequently, Y. Katznelson and L. Tzafriri [17] generalized this result to a wider class of operators where condition (i) of Theorem 2.1 is replaced by $\mathrm{Sp}(T) \cap \Gamma \subset \{1\}$, Γ being the unit circle. G. E. Shilov [28] showed that Hille's result, and G. Allan & T. J. Ransford [2] as well as Vũ Quôc Phóng [31] showed that the Esterle–Katznelson–Tzafriri results, are all consequences of Gelfand's theorem. So this brings us to think that any improvement of Gelfand's theorem has significant consequences on Esterle–Katznelson–Tzafriri's results. For other related work see [6], [7], [10], [12], [20], [21], [23] and [30]. Related results in a particular case can also be found in [1].

THEOREM 2.4 [E. Hille]. *Let $T \in \mathcal{B}(X)$ be such that*

(i) $\mathrm{Sp}(T) = \{1\}$, *and*

(ii) $\|T^k\| = O(k^r)$ *as* $|k| \to \infty$ *for some positive integer* r.

Then $(T - I)^n = 0$ *for every* $n \ge r + 1$.

THEOREM 2.5 [J. Esterle]. *Let $T \in \mathcal{B}(X)$ be such that*

(i) $\mathrm{Sp}(T) = \{1\}$, *and*
(ii) $\sup_{n\geq 0} \|T^n\| < \infty$.

Then $\|T^n - T^{n+1}\| \to 0$ *as* $n \to \infty$.

Is it possible to split Hille's theorem in a way similar to Esterle's theorem, at least for the case when $\|T^n\| = o(n)$ as $n \to \infty$? The answer is of course negative, but if we put restrictions on the behaviour of $(T-I)^n$ we obtain from Theorem 2.3 the following local extension.

COROLLARY 2.6. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $n\|T^n x\|^{1/n} \to 0$ *as* $n \to \infty$, *and*
(ii) $\|(I+T)^k x\| = o(k)$ *as* $k \to \infty$.

Then $Tx = 0$.

In [33] J. Zemánek proved the following result.

THEOREM 2.7. *Let $T \in \mathcal{B}(X)$ be invertible. Suppose that*

(i) $\mathrm{Sp}(T) = \{1\}$, *and*
(ii) $\|T^n - T^{n+1}\| = O(n^{r-1})$ *as* $|n| \to \infty$, *for some positive integer* r.

Then $(T-I)^n = 0$ *for every* $n \geq r+1$.

Here we give two local extensions relaxing condition (ii) to a more general class of operators as follows:

THEOREM 2.8. *Let $T \in \mathcal{B}(X)$ be invertible and $x \in X$. Suppose that*

(i) $\mathrm{Sp}_x(T) = \{1\}$,
(ii) $\|T^n x - T^{n+1}x\| = O(n^p)$ *as* $n \to -\infty$, *for some integer* $p \geq 2$, *and*
(iii) $\|T^n x - T^{n+1}x\| = O(n)$ *as* $n \to \infty$.

Then $(T-I)^3 x = 0$.

Proof. By a local version of Theorem 2.7, we have $(T-I)^{p+2}x = 0$. Suppose that $(T-I)^{p+2}x = 0$ for some $p \geq 2$. Let $y = (T-I)^p x$. Then $(T-I)^2 y = 0$ which implies

$$\frac{M_n(T)y}{n} = \left(\frac{n-1}{2n}\right)(Ty - y) + \frac{y}{n} \to \frac{1}{2}(Ty-y) \quad (\text{as } n \to \infty)$$

where $M_n(T) = (I + T + \ldots + T^{n-1})/n$. On the other hand, from (iii), we get

$$\frac{M_n(T)y}{n} = (T-I)^{p-1}\left(\frac{T^n - I}{n^2}\right)(T-I)x \to 0 \quad (\text{as } n \to \infty).$$

Hence $(T-I)y = 0$, which implies $(T-I)^{p+1}x = 0$. Therefore, by induction, we get $(T-I)^3 x = 0$.

THEOREM 2.9. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $n\|T^n x\|^{1/n} \to 0$ *as* $n \to \infty$, *and*
(ii) $\|T^n x - T^{n+1}x\| = O(n^{r-1})$ *as* $n \to \infty$, *for some positive integer* r.

Then $(T-I)^n x = 0$ *for every* $n \geq r+1$.

Proof. Condition (ii) implies $\|T^n y\| = O(n^r)$, with $y = (T-I)x$. Applying Theorem 2.3 we obtain $(T-I)^r y = 0$, hence the result.

Here we give an elementary proof of the following local extension to Theorem 6 [33]:

THEOREM 2.10. *Let $T \in \mathcal{B}(X)$ and let $x \in X$ be such that*

$$\mathrm{Sp}_x(T) = \{1\} \quad \textit{and} \quad \limsup_{n\to\infty} \|T^n x - T^{n+1}x\|^{1/n} < 1.$$

Then $Tx = x$.

Proof. The condition $\limsup_{n\to\infty} \|T^n x - T^{n+1}x\|^{1/n} < 1$ is equivalent to $r_{(T-I)x}(T) < 1$. So the local spectrum of T at $(T-I)x$ is strictly included in the unit disk. On the other hand, we have

$$\mathrm{Sp}_{(T-I)x}(T) \subset \mathrm{Sp}_x(T) = \{1\}.$$

So $\mathrm{Sp}_{(T-I)x}(T) = \emptyset$. Since T has the property of uniqueness of the local resolvent, we conclude that $(T-I)x = 0$.

In [25], M. Mbekhta and J. Zemánek proved the following generalization of Gelfand's theorem.

THEOREM 2.11. *Let $T \in \mathcal{B}(X)$ be such that*

(i) $\mathrm{Sp}(T) = \{1\}$, *and*
(ii) $M_n(T)$ *and* $M_n(T^{-1})$ *are bounded.*

Then $T = I$.

A natural question was raised in [32]: It is interesting to give a characterization of the asymptotic behaviour of order n for the mean $M_n(T)$ for the operator T as n tends to infinity.

Using ideas from the proof of Theorem 2.1, we give here a partial answer to this question and thereby give a local extension to the above result.

THEOREM 2.12. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $n\|(T-I)^n x\|^{1/n} \to 0$ *as* $n \to \infty$, *and*
(ii) $\|M_n(T)x\| = o(n^r)$ *as* $n \to \infty$, *for some positive integer* r.

Then $(T-I)^n x = 0$ *for every* $n \geq r+1$. *However, if* $r = 1$, *then we obtain* $Tx = x$.

Proof. From condition (ii) and the relation

$$\frac{I-T^n}{n}x=(I-T)M_n(T)x,$$

it follows that $\|T^n x\|=o(n^{r+1})$. By Theorem 2.3, we get the result.

For the second assertion, we have $\|T^n x\|=o(n)$ as $n\to\infty$. By applying Theorem 2.2, we get $(T-I)x=0$.

The following result is an extension of [25, Theorem 2]. It also improves [18, Corollary 7].

THEOREM 2.13. *Let $T\in\mathcal{B}(X)$ and $x\in X$. Suppose that*

(i) $\mathrm{Sp}_x(T)=\{1\}$, *and*
(ii) $\|(I-T)^k M_n(T)x\|\to 0$ *as* $n\to\infty$, *for some positive integer* k.

Then $(I-T)^k x=0$.

Proof. *Case* $k=2$. Let $T\in\mathcal{B}(X)$ and $x\in X$. It is easy to see that

$$\frac{I-T^n}{n}(I-T)^{k-1}x=(I-T)^k M_n(T)x. \tag{1}$$

So, from condition (ii) with $k=2$, we get

$$\|T^{n+1}x-T^n x\|=o(|n|)\quad \text{as } n\to\infty. \tag{2}$$

Let $y=(T-I)x$. Then (2) implies $\|T^n y\|=o(|n|)$ as $n\to\infty$. On the other hand,

$$\mathrm{Sp}_y(T)=\mathrm{Sp}_{(T-I)x}(T)\subset \mathrm{Sp}_x(T)=\{1\}.$$

Hence, from Theorem 2.1, we then get $(T-I)y=0$, which implies that $(T-I)^2x=0$.

Case $k=3$. Suppose that $\|(I-T)^3M_n(T)x\|\to 0$. Then from

$$\frac{I-T^n}{n}(I-T)^2x=(I-T)^3M_n(T)x,$$

we have $\|\frac{T^n(I-T)^2x}{n}\|\to 0$ as $n\to\infty$. But

$$-T^n(T-I)^2x=T^n(T-I)x-T^{n+1}(T-I)x.$$

So, if we put $y=(I-T)x$, we obtain

$$\|T^{n+1}y-T^n y\|=o(|n|)\quad \text{as } n\to\infty.$$

It follows from the case $k=2$ that $(I-T)^2y=0$, which implies $(I-T)^3x=0$.

Finally, suppose that $\|(I-T)^kM_n(T)x\|\to 0$ as $n\to\infty$, and let $y=(T-I)^{k-2}x$. Then from (1) we have

$$\|T^{n+1}y-T^n y\|=o(|n|)\quad \text{as } n\to\infty,$$

and $\mathrm{Sp}_y(T)=\{1\}$. By applying Theorem 2.1, we obtain

$$(T-I)^k x=(T-I)^2y=0.$$

3. Local extensions for n commuting operators. Let $T=(T_1,\ldots,T_n)\in\mathcal{B}(X)^n$ be a commuting multi-operator and let $x\in X$. Then T is said to be *locally power-bounded* at $x\in X$ if

$$\sup_{j_1,\ldots,j_n\ge 0}\|T^{j_1}\ldots T^{j_n}x\|<\infty.$$

T being locally doubly power-bounded is defined similarly. We can easily see that if each T_i is locally power-bounded at x, then $T=(T_1,\ldots,T_n)$ is locally power-bounded at x, but the inverse implication is not true. For related work see [24].

Using the following generalization of Bernstein's theorem on multivariable functions, we give elementary proofs of two local extensions of Gelfand's theorem for n commuting operators $T_1,\ldots,T_n$ in a Banach space X.

THEOREM 3.1. *Let $f:\mathbb{R}^m\to\mathbb{C}$ be real-analytic. Suppose that*

(i) $|f|\le M$ *on* $\mathbb{R}^m$, *and*
(ii) *for every* $k=1,\ldots,m$,

$$\limsup_{n\to\infty}\left|\frac{\partial^n f}{\partial t_k^n}(t_1,\ldots,t_m)\right|^{1/n}\le\tau_k<\infty\quad \text{on } \mathbb{R}^m.$$

Then for every $n=1,2,\ldots,$ and all non-negative integers $j_1,\ldots,j_m$ such that $j_1+\ldots+j_m=n$, we have

$$\left|\frac{\partial^n f}{\partial t_1^{j_1}\ldots\partial t_m^{j_m}}(t_1,\ldots,t_m)\right|\le M\tau_1^{j_1}\ldots\tau_m^{j_m}\quad \text{on } \mathbb{R}^m.$$

THEOREM 3.2. *Let $T=(T_1,\ldots,T_n)\in\mathcal{B}(X)^n$ be a commuting multi-operator and let $x\in X$. Suppose that*

(i) *T is locally doubly power-bounded at x, and*
(ii) *there exists j_0 such that $\|S_{j_0}^n x\|^{1/n}\to 0$ as $n\to\infty$, where $e^{S_{j_0}}=T_{j_0}$.*

Then for every $n\ge 1$, and all non-negative integers $j_1,\ldots,j_n$ with $n=j_1+\ldots+j_n$, we have

$$(T_1-I)^{j_1}\ldots(T_n-I)^{j_n}x=0.$$

Proof. As $T_1,\ldots,T_n$ are commuting operators, their local joint spectrum is non-empty. In addition, if $\mathcal{M}$ is the set of characters on the Banach algebra generated by $T_1,\ldots,T_n$, then $\mathrm{Sp}(T_1,\ldots,T_n)=\{(\chi(T_1),\ldots,\chi(T_n)):\chi\in\mathcal{M}\}$.

Consider the following function:

$$f(t_1,\ldots,t_n)=u(e^{it_1S_1}\ldots e^{it_nS_n}x),$$

where $u\in X^*$, $\|u\|=1$, and $e^{iS_i}=T_i$ for $i=1,\ldots,n$. Condition (i) implies that f is bounded on $\mathbb{R}^n$. On the other hand, we have

$$\lim_{n\to\infty}\left|\frac{\partial^n f}{\partial t_i^n}\right|^{1/n} \le \limsup_{n\to\infty}|S_i^n x|^{1/n} = r_x(S_i) \quad \text{for } i=1,\ldots,n.$$

Since T_i are commuting we deduce from condition (ii) and Theorem 3.1 that for every $n\ge 1$, and all non-negative integers $j_1,\ldots,j_n$ with $n=j_1+\ldots+j_n$, we have

$$(T_1-I)^{j_1}\ldots(T_n-I)^{j_n}x=0.$$

THEOREM 3.3. *Let $T=(T_1,\ldots,T_n)\in\mathcal{B}(X)^n$ be a commuting multi-operator and let $x\in X$. Suppose that*

(i) *T is locally power-bounded at x, and*
(ii) *there exists a j_0 such that $n\|S_{j_0}^n x\|^{1/n}\to 0$ as $n\to\infty$, where $e^{S_{j_0}}=T_{j_0}$.*

Then for every $n\ge 1$, and all non-negative integers $j_1,\ldots,j_n$ with $n=j_1+\ldots+j_n$, we have

$$(T_1-I)^{j_1}\ldots(T_n-I)^{j_n}x=0 \quad \textit{for every } n\ge 1.$$

Proof. We use the same idea as in the proof of Theorem 3.2, but with a new function

$$f(t_1,\ldots,t_n)=u(e^{t_1^2S_1}\ldots e^{t_n^2S_n}x),$$

where $u\in X^*$, $\|u\|=1$, and $e^{S_i}=T_i$ for $i=1,\ldots,n$. Condition (i) implies that f is bounded on $\mathbb{R}^n$. On the other hand, we have

$$\lim_{n\to\infty}\left|\frac{\partial^n f}{\partial t_i^n}\right|^{1/n} \le \frac{2}{\sqrt{e}}\sqrt{\limsup_{n\to\infty} n|S_i^n x|^{1/n}} = \tau_x(S_i) \quad \text{for } i=1,\ldots,n.$$

Since T_i are commuting, the result follows from Theorem 3.1 and condition (ii).

Acknowledgements. I am grateful to J. Zemánek for the warm hospitality and stimulating discussions on these questions during my stay at the Banach Center in Warsaw.

References

[1] G. R. Allan, *Sums of idempotents and a lemma of N. J. Kalton*, Studia Math. 121 (1996), 185–192.

[2] G. R. Allan and T. J. Ransford, *Power-dominated elements in a Banach algebra*, ibid. 94 (1989), 63–70.

[3] A. Atzmon, *Operators which are annihilated by analytic functions and invariant subspaces*, Acta Math. 144 (1980), 27–63.

[4] B. Aupetit and D. Drissi, *Some spectral inequalities involving generalized scalar operators*, Studia Math. 109 (1994), 51–66.

[5] B. Aupetit and D. Drissi, *Local spectrum theory and subharmonicity*, Proc. Edinburgh Math. Soc. 39 (1996), 571–579.

[6] C. J. K. Batty, *Asymptotic behaviour of semigroups of operators*, in: Banach Center Publ. 30, Inst. Math., Polish Acad. Sci., 1994, 35–52.

[7] S. J. Bernau and C. B. Huijsmans, *On the positivity of the unit element in a normed lattice ordered algebra*, Studia Math. 97 (1990), 143–149.

[8] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[9] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere of Banach algebras*, Ann. of Math. 62 (1955), 217–229.

[10] A. Brunel et R. Émilion, *Sur les opérateurs positifs à moyennes bornées*, C. R. Acad. Sci. Paris Sér. I Math. 298 (1984),103–106.

[11] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Gordon and Breach, 1968.

[12] R. Émilion, *Mean-bounded operators and mean ergodic theorems*, J. Funct. Anal. 61 (1985), 1–14.

[13] J. Esterle, *Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras*, in: Radical Banach Algebras and Automatic Continuity (Long Beach, Calif., 1981), Lecture Notes in Math. 975, Springer, 1983, 66–162.

[14] I. Gelfand, *Zur Theorie der Charaktere der Abelschen topologischen Gruppen*, Mat. Sb. 9 (1941), 49–50.

[15] E. Hille, *On the theory of characters of groups and semi-groups in normed vector rings*, Proc. Nat. Acad. Sci. U.S.A. 30 (1944), 58–60.

[16] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, 1957.

[17] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. 68 (1986), 313–328.

[18] K. B. Laursen and M. Mbekhta, *Operators with finite chain length*, Proc. Amer. Math. Soc. 123 (1995), 3443–3448.

[19] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., Providence, 1964.

[20] G. Lumer, *Spectral operators, hermitian operators, and bounded groups*, Acta Sci. Math. (Szeged) 25 (1964), 75–85.

[21] —, *Bounded groups and a theorem of Gelfand*, Rev. Un. Mat. Argentina 25 (1971), 239–245.

[22] G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math. 11 (1961), 679–698.

[23] Yu. Lyubich and J. Zemánek, *Precompactness in the uniform ergodic theory*, Studia Math. 112 (1994), 89–97.

[24] M. Mbekhta and F.-H. Vasilescu, *Uniformly ergodic multioperators*, Trans. Amer. Math. Soc. 347 (1995), 1847–1854.

[25] M. Mbekhta and J. Zemánek, *Sur le théorème ergodique uniforme et le spectre*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 1155–1158.

[26] T. V. Pedersen, *Norms of powers in Banach algebras*, Bull. London Math. Soc. 27 (1995), 305–316.

[27] T. Pytlik, *Analytic semigroups in Banach algebras and a theorem of Hille*, Colloq. Math. 51 (1987), 287–294.

[28] G. E. Shilov, *On a theorem of I. M. Gel'fand and its generalizations*, Dokl. Akad. Nauk SSSR 72 (1950), 641–644 (in Russian).

[29] A. M. Sinclair, *The norm of a hermitian element in a Banach Algebra*, Proc. Amer. Math. Soc. 28 (1971), 446–450.
[30] J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tôhoku Math. J. 20 (1968), 417–424.
[31] Vũ Quôc Phóng, *A short proof of the Y. Katznelson's and L. Tzafriri's theorem*, Proc. Amer. Math. Soc. 115 (1992), 1023–1024.
[32] J. Zemánek, *Sur les itérations des opérateurs*, Publ. Math. Univ. Pierre et Marie Curie, Séminaire d'Initiation à l'Analyse, 1994.
[33] —, *On the Gelfand–Hille theorems*, in: Banach Center Publ. 30, Inst. Math., Polish Acad. Sci., 1994, 369–385.

Department of Mathematics and Computer Science
Faculty of Science, Kuwait University
P.O. Box 5969 Safat 13060, Kuwait
E-mail: drissi@math-1.sci.kuniv.edu.kw

Received February 20, 1996
Revised version September 24, 1996 (3621)