

**Boundary higher integrability for
the gradient of distributional solutions
of nonlinear systems**

by

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Abstract. We consider distributional solutions to the Dirichlet problem for nonlinear elliptic systems of the type

$$\begin{cases} \operatorname{div} A(x, u, Du) = \operatorname{div} f & \text{in } \Omega, \\ u - u_0 \in W_0^{1,r}(\Omega), \end{cases}$$

with r less than the natural exponent p which appears in the coercivity and growth assumptions for the operator A . We prove that $Du \in W^{1,p}(\Omega)$ if $|r - p|$ is small enough.

1. Introduction. In this paper we consider boundary value problems of the type

$$(1.1) \quad \begin{cases} \operatorname{div} A(x, u, Du) = \operatorname{div} f & \text{in } \Omega, \\ u - u_0 \in W_0^{1,r}(\Omega, \mathbb{R}^N), \end{cases}$$

where Ω is a bounded open set in \mathbb{R}^n with a Lipschitz boundary $\partial\Omega$, $A = A(x, s, \xi) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \rightarrow \mathbb{R}^{nN}$ is a Carathéodory vector-valued function which satisfies

$$(1.2) \quad A(x, s, \xi)\xi \geq a|\xi|^p, \quad p > 1,$$

$$(1.3) \quad |A(x, s, \xi) - A(x, s, \eta)| \leq \begin{cases} b|\xi - \eta|^{p-1} & \text{if } 1 < p \leq 2, \\ b|\xi - \eta|(|\xi| + |\eta|)^{p-2} & \text{if } p > 2, \end{cases}$$

$$(1.4) \quad |A(x, s, 0)| \leq d|s|^{p-1} + |h(x)|$$

where a, b, d are positive constants, $h \in L^{p/(p-1)}$ and $\max\{1, p-1\} < r < p$.

Dirichlet problems with $f \equiv 0$, A not explicitly depending on s and homogeneous with respect to ξ , have been studied in [7], where the authors prove both existence of solutions and higher integrability of the gradient Du . In [9] we removed the homogeneity assumption, we considered operators depending explicitly on s and we proved local higher integrability of Du . We

1991 *Mathematics Subject Classification*: 49N60, 35J60.

The work has been supported by M.U.R.S.T. (60% and 40%).

remark that $f \equiv 0$ in [9], but it is not difficult to extend the result to the case $f \neq 0$ handling the term $f(x)$ as the term $h(x)$ in (1.4).

In this paper we are concerned with global higher integrability of Du , u a solution to (1.1).

The existence of solution of (1.1) under our assumptions is still an open problem.

More precisely, we prove:

THEOREM 1.1. *If $f \in L^{r/(p-1)+\eta}(\Omega, \mathbb{R}^{nN})$, $\eta > 0$, $u_0 \in W^{1,r+\varepsilon}(\Omega, \mathbb{R}^N)$, $\varepsilon > 0$, then under assumptions (1.2)–(1.4) there exists $r_1 = r_1(a, b, d, p, n, N, \partial\Omega)$ with $\max\{1, p-1\} < r_1 < p$ such that if $u \in W^{1,r}(\Omega, \mathbb{R}^N)$ is a distributional solution of (1.1) and $r_1 \leq r < p$, then $Du \in L^{1,r+\delta}(\Omega, \mathbb{R}^{nN})$ for a suitable $\delta = \delta(r, \varepsilon, \eta, a, b, n, \partial\Omega)$.*

COROLLARY 1.1. *If $f \in L^{p/(p-1)}(\Omega, \mathbb{R}^{nN})$ and $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$ then, if $u \in W^{1,r}(\Omega, \mathbb{R}^N)$ is a distributional solution of (1.1) with $r_1 \leq r < p$, r_1 as in Theorem 1.1, then $u \in W^{1,p}(\Omega, \mathbb{R}^N)$.*

Remark 1.1. Let us point out that we do not require $f \in L^{p/(p-1)+\eta}$ and $u_0 \in W^{1,p+\varepsilon}$ in order to have $u \in W^{1,p}$.

COROLLARY 1.2. *If $u_0 \equiv 0$ and $f \equiv 0$, then $u \equiv 0$.*

Classical results on higher integrability of Du for linear problems are given in [11]. The nonlinear case is studied in [12] for $r \geq p$.

In our case, $r < p$, one cannot use test functions proportional to u , since Du does not have the right summability properties. This difficulty was first overcome in [7] by using Hodge decomposition (see Lemma 2.4 below), which is in fact the main tool in the present proofs.

In this framework, $r < p$, other results can be found, for example, in [8] and [13]–[15].

2. Notations and preliminaries. Ω is an open bounded subset of \mathbb{R}^n , $n \geq 2$; $\partial\Omega$ is the boundary of Ω ; U and V are neighbourhoods of some point $x^0 \in \partial\Omega$. If $z \in \mathbb{R}^n$ we put

$$\begin{aligned} Q_R(z) &= \{x \in \mathbb{R}^n : |x_i - z_i| < R, i = 1, \dots, n\}, \\ Q_R^+(z) &= \{x \in Q_R(z) : x_n > 0\}, \quad Q_R^-(z) = \{x \in Q_R(z) : x_n < 0\}, \\ \Gamma_R(z) &= \{x \in Q_R(z) : x_n = 0\}. \end{aligned}$$

We denote by Q, Q^+, Q^-, Γ respectively $Q_1(0), Q_1^+(0), Q_1^-(0), \Gamma_1(0)$. For every set ω we denote by $\bar{\omega}$ its closure, and by $|\omega|$ its Lebesgue measure.

In the following we shall use some lemmas which we state below.

LEMMA 2.1. *Let $f : [R, 2R] \rightarrow [0, \infty)$ be a bounded function satisfying*

$$f(\varrho) \leq \theta f(\sigma) + \frac{A}{(\sigma - \varrho)^r} + B$$

for some constants $A, B \geq 0, r \geq 1, 0 < \theta < 1$ and for every ϱ, σ such that $0 < R \leq \varrho < \sigma \leq 2R$; then

$$f(R) \leq c(\theta, r) \left(\frac{A}{R^r} + B \right)$$

where

$$c(\theta, r) = \frac{2^{1+r}}{1-\theta} \left[\left(\frac{2}{1+\theta} \right)^{1/r} - 1 \right]^{-r}$$

is increasing with respect to r .

For the proof see [3].

LEMMA 2.2 (Gehring’s lemma). *If $U \in L^r(Q)$ and $G \in L^s(Q)$, $1 < r < s$, are nonnegative functions such that*

$$\int_{Q_R} U^r dx \leq c \left[\left(\int_{Q_{2R}} U dx \right)^r + \int_{Q_{2R}} G^r dx \right], \quad c > 1,$$

for every pair of concentric cubes $Q_R \subset Q_{2R} \Subset Q$, then there exists $\varepsilon > 0$ such that $U \in L_{loc}^{r+\varepsilon}(Q)$.

For the proof see [3]–[5].

LEMMA 2.3 (Sobolev–Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary $\partial\Omega$. If $u \in W^{1,p}(\Omega)$, $p < n$ and $u \equiv 0$ in a set $A \subset \Omega$ with a positive measure, then*

$$\left(\int_{\Omega} |u|^{p^*} dx \right)^{1/p^*} \leq c \left(\frac{|\Omega|}{|A|} \right)^{1/p^*} \left(\int_{\Omega} |Du|^p dx \right)^{1/p}$$

with a positive constant only depending on n and p .

For the proof see [5].

LEMMA 2.4 (Hodge decomposition). *Let $\omega \subset \mathbb{R}^n$ be a regular domain (for the definition see [7]), $w \in W_0^{1,r}(\omega, \mathbb{R}^N)$, $r > 1$, and let $-1 < \varepsilon < r-1$. Then there exist $\phi : \omega \rightarrow \mathbb{R}^N$ and $H : \omega \rightarrow \mathbb{R}^{nN}$ such that $H \in L^{r/(1+\varepsilon)}(\omega, \mathbb{R}^{nN})$, $\text{div } H = 0$, $\phi \in W_0^{1,r/(1+\varepsilon)}(\omega, \mathbb{R}^N)$ and*

$$(2.1) \quad |Dw|^\varepsilon Dw = D\phi + H,$$

$$(2.2) \quad \|H\|_{L^{r/(1+\varepsilon)}(\omega)} \leq c_\omega(r, n, N) |\varepsilon| \cdot \|Dw\|_{L^r(\omega)}^{1+\varepsilon},$$

$$(2.3) \quad \|D\phi\|_{L^{r/(1+\varepsilon)}(\omega)} \leq (1 + c_\omega(r, n, N) |\varepsilon|) \|Dw\|_{L^r(\omega)}^{1+\varepsilon}.$$

For the proof see [6], [7].

3. Proofs

Proof of Theorem 1.1. The result will be achieved in several steps.

Step 1: Reduction to a problem in Q^+ . First let us remark that we only have to prove the regularity near the boundary $\partial\Omega$ because of the local higher integrability result obtained in [9].

Since $\bar{\Omega}$ is compact, $\partial\Omega$ can be covered by a finite number of neighbourhoods V of its points; it will then be enough to prove the higher integrability of Du in $V \cap \Omega$. Since $\partial\Omega$ is Lipschitz, one can find G which is Lipschitz together with its inverse such that

$$(3.1) \quad G(V) = Q, \quad G(V \cap \Omega) = Q^+, \quad G(V \setminus \bar{\Omega}) = Q^-, \quad G(V \cap \partial\Omega) = \Gamma.$$

By standard arguments we can reduce the problem to proving higher integrability in Q^+ of $D\bar{u}$ for $\bar{u} = u \circ G^{-1}$ which satisfies

$$(3.2) \quad \int_{Q^+} \mathcal{A}(x, \bar{u}, D\bar{u}) D\phi \, dx = \int_{Q^+} \bar{f} D\phi \, dx \quad \forall \phi \in W^{1,r/(r-p+1)}(Q^+)$$

where \mathcal{A} is a Carathéodory vector-valued function which satisfies assumptions of type (1.2), ..., (1.4) with different constants (see [10], Th. 3.2.5, and Lemma 3.2.8 of [2]) and $\bar{f} = f \circ G^{-1}$.

In the following, to simplify the notations, we shall denote \bar{u} , \bar{f} and $\bar{u}_0 = u_0 \circ G^{-1}$ by u , f and u_0 respectively.

Step 2: Higher integrability of Du in Q^+ . By the assumption on u_0 it will be enough to prove higher integrability of $D(u - u_0)$ in Q^+ . To this end we consider $u - u_0$ and its natural extension by zero in Q^- . It belongs to $W^{1,r}(Q)$ and we still denote it by $u - u_0$.

We prove a reverse Hölder inequality for $D(u - u_0)$ which implies the statement of Theorem 1.1, by Lemma 2.2. More precisely, for every $y_0 \in Q$ and for every $Q_R(y_0) \subset Q_{2R}(y_0) \Subset Q$ we prove

$$(3.3) \quad \int_{Q_R(y_0)} |D(u - u_0)|^r \, dx \leq c \left[\left(\int_{Q_{2R}(y_0)} |D(u - u_0)|^{nr/(n+r)} \right)^{(n+r)/n} dx + \int_{Q_{2R}(y_0)} F \, dx \right]$$

where

$$(3.4) \quad F = \begin{cases} |Du_0|^r + |u_0|^r + |h|^{r/(p-1)} + |f|^{r/(p-1)} & \text{if } x \in Q^+, \\ 0 & \text{if } x \in Q^-. \end{cases}$$

Let us consider three different situations.

If $Q_{2R}(y_0) \subset Q^+$ inequality (3.3) has been proved in [9], since it is related to local higher integrability.

If $Q_R(y_0) \subset Q^-$ inequality (3.3) is obvious since its left hand side is identically equal to zero.

Now we prove (3.3) if $Q_R(y_0) \cap Q^+ \neq \emptyset$. By Lemmas 3.1 and 3.2 below, we need only prove it for every $y_0 \in \Gamma$ and this will be done in the next step 3.

Step 3: Proof of (3.3) for $y_0 \in \Gamma$ and $Q_R(y_0) \cap Q^+ \neq \emptyset$. In this step the center y_0 of all the cubes will be omitted.

Let $R \leq \varrho < \sigma \leq 2R$ and $\mu \in C_0^1(Q_\sigma)$ be such that $\mu = 1$ in Q_ϱ and $|D\mu| \leq c/(\sigma - \varrho)$. We consider $w = \mu(u - u_0) \in W_0^{1,r}(Q_\sigma^+)$ and apply Lemma 2.4 to w , $\varepsilon = r - p$ and $\omega = Q_\sigma^+$.

Let us point out that balls and cubes are regular domains for which the constant c_ω in (2.2) and (2.3) does not depend on the dimension nor on the center of the domain itself.

A classical reflection argument allows us to consider regular all rectangles with integer ratio between the two different dimensions, therefore for $\omega = Q_\sigma^+(y_0)$ the constant c_ω in (2.2) and (2.3) is independent of σ and y_0 .

Let us also remark that this constant c_ω is independent of r when r belongs to a suitable compact set (see [9], p. 290).

We consider $p > 2$. Analogous calculations hold true for $1 < p \leq 2$.

We insert $D\phi$ given by (2.1) in (3.2). We get

$$\begin{aligned} & \int_{Q_\sigma^+} \mathcal{A}(x, u, Dw) |Dw|^{r-p} Dw \, dx \\ &= \int_{Q_\sigma^+} \mathcal{A}(x, u, Dw) H \, dx \\ &+ \int_{Q_\sigma^+} [\mathcal{A}(x, u, Dw) - \mathcal{A}(x, u, Du)] D\phi \, dx + \int_{Q_\sigma^+} f D\phi \, dx. \end{aligned}$$

By coercivity assumption (1.2) and growth conditions (1.3) and (1.4) we have

$$(3.5) \quad \begin{aligned} a \int_{Q_\sigma^+} |Dw|^r \, dx &\leq \int_{Q_\sigma^+} [b|Dw|^{p-1} + d|u|^{p-1} + h(x)] |H| \, dx \\ &+ b \int_{Q_\sigma^+} |Du - Dw| (|Du| + |Dw|)^{p-2} |D\phi| \, dx \\ &+ \int_{Q_\sigma^+} |f| \cdot |D\phi| \, dx \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

For simplicity we give estimates for I, II and III in the appendix below.

Collecting them we finally obtain

$$(3.6) \quad a \int_{Q_\varepsilon^+} |Dw|^r dx \leq [c|r-p| + \varepsilon] \int_{Q_\varepsilon^+} |Dw|^r dx \\ + c(\varepsilon) \int_{Q_\varepsilon^+ \setminus Q_{2\varepsilon}^+} |Du - Du_0|^r dx \\ + c(\varepsilon) \left[\int_{Q_\varepsilon^+} |Du_0|^r dx + \frac{1}{(\sigma - \varrho)^r} \int_{Q_\varepsilon^+} |u - u_0|^r dx + \int_{Q_\varepsilon^+} |u_0|^r dx \right. \\ \left. + \int_{Q_\varepsilon^+} |u - u_0|^r dx + \int_{Q_\varepsilon^+} |h|^{r/(p-1)} dx + \int_{Q_\varepsilon^+} |f|^{r/(p-1)} dx \right].$$

Now we choose r_1 close enough to p and ε small enough in such a way that $c|r-p| + \varepsilon \leq a/2$ for $r_1 \leq r < p$.

Moreover, for $R < 1$, from (3.6) we have

$$\int_{Q_\varepsilon^+} |Du - Du_0|^r dx \leq c(\varepsilon) \int_{Q_\varepsilon^+ \setminus Q_{2\varepsilon}^+} |Du - Du_0|^r dx + \frac{c(\varepsilon)}{(\sigma - \varrho)^r} \int_{Q_{2\varepsilon}^+} |u - u_0|^r dx \\ + c(\varepsilon) \left[\int_{Q_{2\varepsilon}^+} |Du_0|^r dx + \int_{Q_{2\varepsilon}^+} |u_0|^r dx + \int_{Q_{2\varepsilon}^+} (|h|^{r/(p-1)} + |f|^{r/(p-1)}) dx \right].$$

Adding $c(\varepsilon) \int_{Q_\varepsilon^+} |D(u - u_0)|^r dx$ to both sides we get

$$\int_{Q_\varepsilon^+} |D(u - u_0)|^r dx \leq \frac{c(\varepsilon)}{1 + c(\varepsilon)} \int_{Q_\varepsilon^+} |D(u - u_0)|^r dx + \frac{c(\varepsilon)}{(\sigma - \varrho)^r} \int_{Q_{2\varepsilon}^+} |u - u_0|^r dx \\ + c(\varepsilon) \int_{Q_{2\varepsilon}^+} (|Du_0|^r + |u_0|^r + |h|^{r/(p-1)} + |f|^{r/(p-1)}) dx.$$

Now we apply Lemma 2.1 with

$$\theta = \frac{c(\varepsilon)}{1 + c(\varepsilon)}, \quad A = c(\varepsilon) \int_{Q_{2\varepsilon}^+} |u - u_0|^r dx,$$

$$B = c(\varepsilon) \int_{Q_{2\varepsilon}^+} (|Du_0|^r + |u_0|^r + |h|^{r/(p-1)} + |f|^{r/(p-1)}) dx.$$

Finally, we have

$$\int_{Q_\varepsilon^+} |D(u - u_0)|^r dx \leq c(\theta, r) \left[\frac{1}{R^r} \int_{Q_{2\varepsilon}^+} |u - u_0|^r dx + B \right].$$

Extending all the integrands in the previous inequality by zero in $Q_{2\varepsilon}^-$,

applying Lemma 2.3 and dividing by R^n , we get

$$\int_{Q_R} |D(u - u_0)|^r dx \leq c \left[\left(\int_{Q_{2R}} |D(u - u_0)|^{nr/(n+r)} dx \right)^{(n+r)/n} dx \right. \\ \left. + \int_{Q_{2R}} (|Du_0|^r + |u_0|^r + |h|^{r/(p-1)} + |f|^{r/(p-1)}) dx \right],$$

which implies (3.3) with F given by (3.4) and concludes the proof of Theorem 1.1.

Let us now prove the following lemmas:

LEMMA 3.1. *If $y_0 \in Q^-$, then (3.3) holds true.*

LEMMA 3.2. *If $y_0 \in Q^+$, then (3.3) holds true.*

In the proofs of the two lemmas we set

$$g = |D(u - u_0)|^r, \quad s = \frac{n}{n+r}$$

and we confine ourselves to the case $n = 2$ for simplicity.

In the following estimates the constant c may change from line to line.

Proof of Lemma 3.1. $Q_R^+(y_0)$ is a rectangle with dimensions $2R$ and $\varepsilon = R - \text{dist}(y_0, \Gamma)$. First suppose $\varepsilon > R/2$. There exist two squares $Q_\varepsilon(x_i)$ with $x_i \in \Gamma$ for $i = 1, 2$ such that $Q_{2\varepsilon}(x_i) \subseteq Q_{2R}(y_0)$ and $\bigcup_{i=1}^2 Q_\varepsilon(x_i) \supset Q_R^+(y_0)$. Then

$$\int_{Q_R(y_0)} g dx \leq \sum_{i=1}^2 \int_{Q_\varepsilon(x_i)} g dx.$$

By (3.3) which has been proved for cubes centered on Γ , we get

$$(3.7) \quad \int_{Q_R(y_0)} g dx \leq \frac{1}{4R^2} \sum_{i=1}^2 \int_{Q_\varepsilon(x_i)} g dx \\ \leq c \left(\frac{\varepsilon}{R} \right)^2 \sum_{i=1}^2 \left[\left(\int_{Q_{2\varepsilon}(x_i)} g^s dx \right)^{1/s} + \int_{Q_{2\varepsilon}(x_i)} F dx \right] \\ \leq c \left[\left(\frac{\varepsilon}{R} \right)^{2-2/s} \left(\int_{Q_{2R}(y_0)} g^s dx \right)^{1/s} + \int_{Q_{2R}(y_0)} F dx \right] \\ \leq c \left[\left(\int_{Q_{2R}(y_0)} g^s dx \right)^{1/s} + \int_{Q_{2R}(y_0)} F dx \right]$$

since $\varepsilon \geq R/2$ and $2 - 2/s < 0$.

Now we suppose $0 < \varepsilon \leq R/2$. Then there exist two squares $Q_{R/2}(x_i)$, $x_i \in \Gamma$ for $i = 1, 2$, such that $Q_R(x_i) \subseteq Q_{2R}(y_0)$ and $\bigcup_{i=1}^2 \bar{Q}_{R/2}(x_i) \supseteq$

$Q_R^+(y_0)$. By applying again (3.3) to $Q_{R/2}(x_i)$ we have

$$(3.8) \quad \int_{Q_R(y_0)} g \, dx \leq c \sum_{i=1}^2 \int_{Q_{R/2}(x_i)} g \, dx \\ \leq c \left[\left(\int_{Q_R(x_i)} g^s \, dx \right)^{1/s} + \int_{Q_R(x_i)} F \, dx \right] \\ \leq c \left[\left(\int_{Q_{2R}(y_0)} g^s \, dx \right)^{1/s} + \int_{Q_{2R}(y_0)} F \, dx \right].$$

By (3.7) and (3.8) the assertion of Lemma 3.1 follows.

Proof of Lemma 3.2. Since $y_0 \in Q^+$, $Q_R^+(y_0)$ is a rectangle with dimensions $2R$ and $R+d$ with $d = \text{dist}(y_0, \Gamma) > 0$.

If $0 < d \leq R$ we can cover $Q_R^+(y_0)$ with a finite number n of squares $Q_r(x_i)$, n independent of d , such that $Q_{2r}(x_i) \subset Q_{2R}(y_0)$ and

- (i) if $x_i \in \Gamma$ then $r = R/2$,
- (ii) if $x_i \in Q_R^+(y_0)$ then $r = R/4$ and $Q_{R/2}(x_i) \subset Q_R^+(y_0)$.

Let now $R < d < \frac{3}{2}R$. Then we can cover $Q_R^+(y_0)$ with a finite number n of squares $Q_{R/4}(x_i)$, n independent of d , such that $Q_{R/2}(x_i) \subset Q_{2R}(y_0)$ and

- (i) $x_i \in \Gamma$ for $i = 1, 2, 3, 4$,
- (ii) $x_i \in Q_R^+(y_0)$ for $i > 4$ and $Q_{R/2}(x_i) \subset Q_R^+(y_0)$.

Finally, if $d > \frac{3}{2}R$ we can cover $Q_R^+(y_0)$ with a finite number n of squares $Q_{R/4}(x_i)$, n independent on d , $x_i \in Q_R^+(y_0)$ such that $Q_{R/2}(x_i) \subset Q_{2R}^+(y_0)$.

Our estimate is achieved by arguing as in the previous Lemma 3.1 taking into account that in both cases (i) the estimate (3.3) has been proved and in the other cases we are reduced to the local estimate proved in [9].

Proof of Corollary 1.1. Set $E = \{s \in [r_1, p] : Du \in L^s(\Omega, \mathbb{R}^{nN})\}$. Then $E \neq \emptyset$ since $r_1 \in E$.

Following the outline of the proof of Theorem 1.2 of [9] we can prove that E is both open and closed in $[r_1, p]$; hence $E = [r_1, p]$.

Proof of Corollary 1.2. By Corollary 1.1, $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Since $u \in W_0^{1,r_1}(\Omega, \mathbb{R}^N)$, it follows that $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ (see [1], p. 275).

We can use u as test function in the weak form of the equation and, by coercivity, we conclude $u \equiv 0$.

4. Appendix. In the following c will denote a constant independent of r, σ, y_0 , while ε and $c(\varepsilon)$ respectively small and large constants coming from Young's inequality which will be used several times. Let us point out that all

these constants may change from line to line. Moreover, when not explicitly mentioned, the integrations will be carried out on Q_σ^+ .

Now we proceed to estimate I, II, III in (3.5).

By using Hölder's and Young's inequalities and Lemma 2.4 we have

$$I = b \int |Dw|^{p-1} |H| \, dx + d \int |u|^{p-1} |H| \, dx + \int |h(x)| \cdot |H| \, dx \\ \leq c|r-p| \left[\int |Dw|^r \, dx + \int |u-u_0|^r \, dx + \int |u_0|^r \, dx + \int |h|^{r/(p-1)} \, dx \right].$$

In order to estimate II we recall that $Du - Dw = (1-\eta)Du + \eta Du_0 - (u-u_0)D\eta$, therefore

$$II \leq c \int |Du - Dw| (|Du - Dw|^{p-2} + |Dw|^{p-2}) |D\phi| \, dx \\ = c \left[\int |Du - Dw|^{p-1} |D\phi| \, dx + \int |Du - Dw| \cdot |Dw|^{p-2} |D\phi| \, dx \right] \\ \leq c \left[\int (|(1-\eta)Du|^{p-1} + |\eta Du_0|^{p-1} + |u-u_0|^{p-1} |D\eta|^{p-1}) |D\phi| \, dx \right. \\ \left. + \int (|(1-\eta)Du| + |\eta Du_0| + |u-u_0| \cdot |D\eta|) |Dw|^{p-2} |D\phi| \, dx \right].$$

Taking into account the properties of the function η and (2.1) we have

$$II \leq c \left[\int_{Q_\sigma^+ \setminus Q_\sigma^+} |Du - Du_0|^{p-1} |D\phi| \, dx + \int |Du_0|^{p-1} |D\phi| \, dx \right. \\ \left. + \int_{Q_\sigma^+ \setminus Q_\sigma^+} |Du - Du_0| \cdot |Dw|^{r-1} \, dx \right. \\ \left. + \int_{Q_\sigma^+ \setminus Q_\sigma^+} |Du - Du_0| \cdot |Dw|^{p-2} |H| \, dx + \int |Du_0| \cdot |Dw|^{p-2} |H| \, dx \right. \\ \left. + \int |Du_0| \cdot |Dw|^{r-1} \, dx + \frac{1}{(\sigma-\varrho)^{p-1}} \int |u-u_0|^{p-1} |D\phi| \, dx \right. \\ \left. + \frac{1}{\sigma-\varrho} \left(\int |u-u_0| \cdot |Dw|^{r-1} \, dx + \int |u-u_0| \cdot |Dw|^{p-2} |H| \, dx \right) \right].$$

All the terms in the previous inequality can be estimated by using Hölder's and Young's inequalities and Lemma 2.4. We get

$$II \leq c|r-p| \int |Dw|^r \, dx \\ + c(\varepsilon) \left[\int_{Q_\sigma^+ \setminus Q_\sigma^+} |Du - Du_0|^r \, dx + \int |Du_0|^r \, dx + \frac{1}{(\sigma-\varrho)^r} \int |u-u_0|^r \, dx \right].$$

Finally, by the same arguments we get

$$III \leq \int |f| \cdot |D\phi| \, dx \leq c(\varepsilon) \int |f|^{r/(p-1)} \, dx + \varepsilon \int |D\phi|^{r/(r-p+1)} \, dx \\ \leq c(\varepsilon) \int |f|^{r/(p-1)} \, dx + \varepsilon \int |Dw|^r \, dx.$$

Acknowledgments. We thank T. Iwaniec and N. Fusco for helpful discussions on the subject.

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Received July 10, 1996
 Revised version September 23, 1996

(3711)

On a theorem of Gelfand and its local generalizations

by

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Abstract. In 1941, I. Gelfand proved that if a is a doubly power-bounded element of a Banach algebra A such that $\text{Sp}(a) = \{1\}$, then $a = 1$. In [4], this result has been extended locally to a larger class of operators. In this note, we first give some quantitative local extensions of Gelfand–Hille’s results. Secondly, using the Bernstein inequality for multivariable functions, we give short and elementary proofs of two extensions of Gelfand’s theorem for m commuting bounded operators, T_1, \dots, T_m , on a Banach space X .

1. Introduction. In 1941, I. Gelfand [14] proved that if T is a bounded linear operator on a complex Banach space X which satisfies $\text{Sp}(T) = \{1\}$ and $\sup_{k \in \mathbb{Z}} \|T^k\| < \infty$, then $T = I$. This result was generalized by E. Hille in 1944 (see [15] or [16, Theorem 4.10.1]), who proved that if $\text{Sp}(T) = \{1\}$ and $\|T^k\| = o(|k|)$ for $k \in \mathbb{Z}$, then $T = I$. In [4, Theorem 3.4], we generalized these results locally to a wider class of operators. A natural question arises: What happens to each of these results if we drop the assumption on the boundedness of the negative powers of the operator T ? On the other hand, in 1955 H. F. Bohnenblust and S. Karlin [9] asked the following question: Is 0 the only quasi-nilpotent dissipative element in a Banach algebra? In 1961, G. Lumer and R. S. Phillips [22] gave a negative answer to this question, but nobody noticed that Shilov’s negative answer to Gelfand’s problem in [28] is also a negative answer to H. F. Bohnenblust and S. Karlin’s question. Just looking at the Gelfand problem, the condition $\text{Sp}(T) = \{1\}$ implies, using the F. Riesz and N. Dunford holomorphic functional calculus, that $T = e^S$ with S quasi-nilpotent and the hypothesis $\sup_{n \geq 0} \|T^n\| < \infty$ implies that S is dissipative for an equivalent norm.

In this paper, we will study these cases locally for a general class of operators. We will also give an extension to n commuting operators T_1, \dots, T_n in a Banach space X . For this we need to introduce some preliminaries on

1991 *Mathematics Subject Classification*: 47B10, 47B15.

Key words and phrases: locally power-bounded operator, local spectrum, local spectral radius.

Research supported by the Kuwait University Research Grant No. SM 138.