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A new Taylor type formula and C^∞ extensions for asymptotically developable functions

by

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Abstract. The paper studies the relation between asymptotically developable functions in several complex variables and their extensions as functions of real variables. A new Taylor type formula with integral remainder in several variables is an essential tool. We prove that strongly asymptotically developable functions defined on polysectors have C^∞ extensions from any subpolysector; the Gevrey case is included.

1. Introduction. The goal of this paper is to prove a new type of the Taylor formula and to obtain a characterization of strongly asymptotically developable functions by means of C^∞ extensions. These functions appear as solutions of integrable connections with irregular singular points (see [Mj]). In dimension one the Poincaré asymptotic expansions were used to study irregular singular points, but Malgrange [M] suggested that the C^∞ extensions would be more convenient. Here we prove that in any finite dimension the two procedures are equivalent.

Two types of asymptotic expansion have been introduced to understand the irregular singular points of integrable connections and Pfaff systems: strong expansions (see [Mj]) and weak ones (see [GS]). Here we prove that the two approximations are essentially different. We give an example of a function which is weakly, but not strongly asymptotically developable.

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2. Notations. Let $\mathbb{K}$ be the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$. Let n be a positive integer > 1. An *interval* in $\mathbb{K}^n$ is a set of the form $Q = I_1 \times \ldots \times I_n$ where the I_j are convex connected open sets in $\mathbb{K}$. Suppose that 0 is a limit point for each I_j and let $O = (0, \ldots, 0) \in \mathbb{K}^n$. We then call O a *vertex* of Q. For instance, the following sets are intervals:

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$(0,\infty)\times\ldots\times(0,\infty)\subset\mathbb{R}^n$ and $S_1\times\ldots\times S_n\subset\mathbb{C}^n$, where S_j are open sectors in $\mathbb{C}$.

For any $x\in\mathbb{K}^n$, γ_x denotes the segment from O to x, that is, $\gamma_x(t)=tx$ for $t\in[0,1]$.

We use the following multiindex notations:

Let $J=\{j_1,\ldots,j_s\}$, $j_1<\ldots<j_s$, be a nonempty subset of $\{1,\ldots,n\}$ and let $\alpha=(\alpha_1,\ldots,\alpha_n)$ be in $\mathbb{N}^n$. Then

$$|\alpha|=\alpha_1+\ldots+\alpha_n,\qquad \alpha_J=(\alpha_{j_1},\ldots,\alpha_{j_s})\in\mathbb{N}^J,$$
$$1_J=(1,\ldots,1)\in\mathbb{N}^J,\qquad 0_J=(0,\ldots,0)\in\mathbb{N}^J,$$

and $\mathcal{Q}_J=I_{j_1}\times\ldots\times I_{j_s}$ for an interval $\mathcal{Q}=I_1\times\ldots\times I_n$ in $\mathbb{K}^n$. Moreover, we write $J^c=\{1,\ldots,n\}\setminus J$ and, if $J=\{s\}$, s^c stands for $\{s\}^c$. We put $(\alpha_J,\alpha_{J^c})=\alpha$, $e_j=(1_j,0_{j^c})$,

$$x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n}\quad\text{ with } x=(x_1,\ldots,x_n)\in\mathbb{K}^n,$$

and for $s=(s_1,\ldots,s_n)\in[0,\infty)^n$,

$$\alpha!^s=\alpha_1!^{s_1}\ldots\alpha_n!^{s_n}.$$

Now, for $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$ in $\mathbb{N}^n$, we say $\alpha\le\beta$ if and only if $\beta_j-\alpha_j$ is a nonnegative integer for each $j=1,\ldots,n$.

Let $\mathcal{Q}$ be an interval in $\mathbb{K}^n$ and $\eta=(\eta_1,\ldots,\eta_n)\in\mathbb{N}^n$. We denote by $\mathcal{C}^\eta_{\mathrm{e}}(\mathcal{Q})$ the set of functions $f:\mathcal{Q}\to\mathbb{K}$ whose partial derivatives $D^\alpha f$, $\alpha\le\eta$, exist in $\mathcal{Q}$ and are continuously extendable to $\overline{\mathcal{Q}}$. For simplicity, we also write $D^\alpha f$ for the extension to $\overline{\mathcal{Q}}$.

Let $\mathcal{F}$ be a family of continuous functions of the form

$$\mathcal{F}=\{f_{\alpha_J}\}_{\emptyset\ne J\subset\{1,\ldots,n\},\,\alpha_J\in\mathbb{N}^J},$$

where $f_{\alpha_J}:\mathcal{Q}_{J^c}\to\mathbb{K}$ and $f_{\alpha_{\{1,\ldots,n\}}}\in\mathbb{K}$. Let $\alpha\in\mathbb{N}^n$ and $x=(x_1,\ldots,x_n)\in\mathcal{Q}$. We put

$$\text{(i)}\qquad \mathrm{App}_\alpha(\mathcal{F},x)=\sum_{\emptyset\ne J\subset\{1,\ldots,n\}}(-1)^{\#J+1}\sum_{0_J\le\beta_J\le\alpha_J-1_J}f_{\beta_J}(x_{J^c})x_J^{\beta_J}.$$

Given f in $\mathcal{C}^\eta_{\mathrm{e}}(\mathcal{Q})$, $j\in\{1,\ldots,n\}$ and $x_j\in I_j$, we define

$$\mathcal{I}_jf:\overline{\mathcal{Q}}\ni x\mapsto\int_{\gamma_{x_j}}f(t,x_{j^c})\,dt\in\mathbb{K},$$

and for $s\le\eta_j$,

$$\mathcal{A}^{(s)}_jf:\overline{\mathcal{Q}}\ni x\mapsto f(0_j,x_{j^c})+\ldots+\frac{1}{s!}\frac{\partial^sf}{\partial x_j^s}(0_j,x_{j^c})x_j^s\in\mathbb{K}.$$

Next, we introduce the operators

$$\mathcal{I}^q_j=\mathcal{I}_j\circ\overset{q}{\ldots}\circ\mathcal{I}_j\qquad\text{for } q\in\mathbb{N}\setminus0.$$

For each nonempty subset $J=\{j_1,\ldots,j_s\}$ of $\{1,\ldots,n\}$, $j_1<\ldots<j_s$, we put

$$\mathcal{I}^{\alpha_J}_J=\mathcal{I}^{\alpha_{j_1}}_{j_1}\circ\ldots\circ\mathcal{I}^{\alpha_{j_s}}_{j_s},\qquad \mathcal{A}^{\alpha_J}_J=\mathcal{A}^{(\alpha_{j_1})}_{j_1}\circ\ldots\circ\mathcal{A}^{(\alpha_{j_s})}_{j_s},$$
$$\mathcal{E}^{\alpha_J}=\mathcal{I}^{\alpha_J}_J\circ D^{(\alpha_J,0_J)},$$

that is,

$$(\mathcal{E}^{\alpha_J}f)(x)$$
$$=\int_{\gamma_{x_{j_1}}}dt_{j_1,1}\int_{\gamma_{t_{j_1,1}}}dt_{j_1,2}\ldots\int_{\gamma_{t_{j_1,\alpha_{j_1}-1}}}dt_{j_1,\alpha_{j_1}}\ldots\int_{\gamma_{x_{j_k}}}dt_{j_k,1}\int_{\gamma_{t_{j_k,1}}}dt_{j_k,2}\ldots$$
$$\ldots\int_{\gamma_{t_{j_k,\alpha_{j_k}-1}}}\frac{\partial^{|\alpha_J|}f}{\partial t^{\alpha_{j_1}}_{j_1,\alpha_{j_1}}\ldots\partial t^{\alpha_{j_k}}_{j_k,\alpha_{j_k}}}(t_{j_1,\alpha_{j_1}},\ldots,t_{j_k,\alpha_{j_k}},x_{J^c})\,dt_{j_k,\alpha_{j_k}}.$$

Finally, $\mathcal{E}^\alpha=\mathcal{E}^{\alpha_J}$ for $J=\{1,\ldots,n\}$.

3. The formula

THEOREM 3.1. *Let $\mathcal{Q}$ be an interval in $\mathbb{K}^n$ and $O=(0,\ldots,0)\in\mathbb{K}^n$ a vertex of $\mathcal{Q}$. Let $\eta\in\mathbb{N}^n\setminus0$. For all $\alpha\in\mathbb{N}^n\setminus0$ and $f\in\mathcal{C}^\eta_{\mathrm{e}}(\mathcal{Q})$, we have equality*

$$(\star)\qquad f(x)-\mathrm{App}_\alpha(f,x)=(\mathcal{E}^\alpha f)(x)\quad\textit{for every } x\in\overline{\mathcal{Q}},$$

where $\mathrm{App}_\alpha(f,x)=\mathrm{App}_\alpha(\mathcal{F},x)$ for the family $\mathcal{F}=\{f_{\alpha_I}\}_{\emptyset\ne I\subset\{1,\ldots,n\},\,\alpha_I\in\mathbb{N}^I}$, with

$$f_{\alpha_I}(x_{I^c})=\frac{1}{\alpha_I!}D^{(\alpha_I,0_{I^c})}f(0_I,x_{I^c}).$$

Remark 3.1. Observe that the remainder is of order $|x^\alpha|$:

$$|(\mathcal{E}^\alpha f)(x)|\le M_\alpha\frac{|x^\alpha|}{\alpha!},$$

where $M_\alpha=\sup\{|D^\alpha f(x)|:x\in\overline{\mathcal{Q}}\}$.

4. Preliminary results

Remark 4.1. Let $j\in\{1,\ldots,n\}$. If $g\in\mathcal{C}^\eta_{\mathrm{e}}(\mathcal{Q})$ does not depend on x_j, then for every $0<p\le\eta_j$ we have

$$(\mathcal{I}^p_jg)(x)=\frac{1}{p!}g(x)x_j^p\qquad\text{for } x\in\overline{\mathcal{Q}}.$$

PROPOSITION 4.2. *Let $\mathcal{Q}=I_1\times\ldots\times I_n$ be an interval in $\mathbb{K}^n$. For all f in $\mathcal{C}^\eta_{\mathrm{e}}(\mathcal{Q})$, we have:*

(1) *$\partial f/\partial x_j$ is in $\mathcal{C}^{\eta-e_j}_{\mathrm{e}}(\mathcal{Q})$.*
(2) *$\mathcal{I}_jf$ belongs to $\mathcal{C}^{\eta+e_j}_{\mathrm{e}}(\mathcal{Q})$.*

(3) *If* $i \neq j$, *then* $\mathcal{I}_i(\mathcal{I}_j f) = \mathcal{I}_j(\mathcal{I}_i f)$.
(4) *If* $i \neq j$, *then* $\mathcal{I}_i(\partial f/\partial x_j) = (\partial/\partial x_j)(\mathcal{I}_i f)$.

Remark 4.3. Let $j \in \{1,\ldots,n\}$, $\eta \in \mathbb{N}^n$ and $q \in \mathbb{N}$, $q \le \eta_j - 1$. If $f \in \mathcal{C}^\eta_e(\mathcal{Q})$ then

$$\mathcal{I}_j^{q+1} \frac{\partial^{q+1} f}{\partial x_j^{q+1}} = f - \mathcal{A}_j^{(q)} f,$$

by induction on q and Remark 4.1, as

$$\mathcal{I}_j \frac{\partial^{p+1} f}{\partial x_j^{p+1}} = \frac{\partial^p f}{\partial x_j^p} - \frac{\partial^p f}{\partial x_j^p}(0_j, \cdot).$$

This is nothing but the classical Taylor formula (with respect to x_j) with the integral remainder.

PROPOSITION 4.4. *Let* $i, j \in \{1,\ldots,n\}$, $i \neq j$, *and let* p *be a positive integer. For all* f *in* $\mathcal{C}^\eta_e(\mathcal{Q})$, *we have the following two equalities*:

(1) $\mathcal{A}_j^{(p)}(\mathcal{I}_i f) = \mathcal{I}_i(\mathcal{A}_j^{(p)} f)$.
(2) $\mathcal{A}_j^{(p)}(\partial f/\partial x_i) = (\partial/\partial x_i)(\mathcal{A}_j^{(p)} f)$.

Proof. The proposition is an easy consequence of 4.2.

5. The proof of the formula. It is sufficient to establish the formula with α_J in place of α, where $J = \{1,\ldots,p\}$, for every $p \le n$. We proceed by induction. If $p = 1$, then our statement is Remark 4.3. So assume the formula to be valid for $J_0 = J \setminus p$ in place of J.

Let $\mathrm{App}_\alpha f$ denote the function $x \mapsto \mathrm{App}_\alpha(f,x)$. By (i) of Section 2 we can write

$$\mathrm{App}_{(\alpha_J, 0_{J^c})}(f,x) = \mathrm{App}_{(\alpha_{J_0}, 0_{J_0^c})}(f,x) + (\mathcal{A}_p^{(\alpha_p - 1)} f)(x) + S(x)$$

(we put $\mathcal{A}_p^{(-1)} = 0$), where

$$S(x) = \sum_{\substack{\emptyset \neq L \subset J \\ \{p\} \subsetneq L}} (-1)^{\# L + 1} \sum_{0_L \le \mu_L \le \alpha_L - 1_L} f_{\mu_L}(x_{L^c}) x_L^{\mu_L}.$$

Observe that $S = -\mathcal{A}_p^{(\alpha_p - 1)}(\mathrm{App}_{(\alpha_{J_0}, 0_{J_0^c})} f)$ since

$$\frac{1}{\nu!} \frac{\partial^\nu f_{\beta_K}}{\partial x_p^\nu}(x_{K^c \setminus p}, 0_p) = f_{(\nu, \beta_K)}(x_{K^c \setminus p})$$

for $\nu = 0, \ldots, \alpha_p - 1$ and $\emptyset \neq K \subset J \setminus p$. So, using the induction hypothesis

(ii) $$\mathcal{E}^{\alpha_{J_0}} f = f - \mathrm{App}_{(\alpha_{J_0}, 0_{J_0^c})} f$$

we get, by 4.2–4.4, the equalities

$$\begin{aligned}
\mathcal{E}^{\alpha_J} f &= \mathcal{E}^{\alpha_{J_0}} f - \mathcal{E}^{\alpha_{J_0}}(\mathcal{A}_p^{(\alpha_p - 1)} f) \\
&= f - \mathrm{App}_{(\alpha_{J_0}, 0_{J_0^c})} f - \mathcal{A}_p^{(\alpha_p - 1)}(\mathcal{E}^{\alpha_{J_0}} f) \\
&= f - \mathrm{App}_{(\alpha_{J_0}, 0_{J_0^c})} f - \mathcal{A}_p^{(\alpha_p - 1)} f - S = f - \mathrm{App}_{(\alpha_J, 0_{J^c})} f,
\end{aligned}$$

as desired.

6. Asymptotic expansions in several complex variables. In the literature we can find two different notions of asymptotic behavior of a holomorphic function f defined on a polysector $V = V_1 \times \ldots \times V_n$ near the vertex $O = (0,\ldots,0) \in \mathbb{C}^n$ of V. From now on, each V_i will be supposed to be open and to have an amplitude $< 2\pi$. In 1979 Gérard and Sibuya [GS] introduced the following idea:

Given a formal power series $\widehat{f} = \sum a_\alpha z^\alpha$ with complex coefficients, we will call $\widehat{f}$ the *weak asymptotic expansion* of f as $z \to O$ in V if for every $t \in \mathbb{N}$ there exists a constant $C_t > 0$ such that

$$\Big| f(z) - \sum_{|\alpha| < t} f_\alpha z^\alpha \Big| \le C_t |z|^t, \quad \forall z = (z_1, \ldots, z_n) \in V,$$

where $|z| = \max\{|z_j| : 1 \le j \le n\}$. Later, in 1984, Majima [Mj] defined the following beautiful concept:

Let $\mathcal{F} = \{f_{\alpha_J}\}_{\emptyset \neq J \subset \{1,\ldots,n\},\, \alpha_J \in \mathbb{N}^J}$ be a family such that $f_{\alpha_J} \in \mathcal{O}(V_{J^c})$ (recall that $\mathcal{O}(U)$ is the algebra of holomorphic functions on the open set U). Suppose that for each proper subpolysector ([1]) $W \neq \emptyset$ of V and for $N \in \mathbb{N}^n$, there exists $K_{W,N} > 0$ such that

$$|f(z) - \mathrm{App}_N(\mathcal{F}, z)| \le K_{W,N} |z^N| \quad \text{for all } z \in W.$$

Then we say that $\mathcal{F}$ is the *total asymptotic expansion* of f (in V) and we call f a *strongly asymptotically developable function*. We put $\mathrm{TA}(f) = \mathcal{F}$. The formal power series

$$\mathrm{FA}(f) = \sum_{\alpha_{\{1,\ldots,n\}} \in \mathbb{N}^n} f_{\alpha_{\{1,\ldots,n\}}} z^{\alpha_{\{1,\ldots,n\}}}$$

is called the *formal asymptotic series* of f.

Set

$$\mathbf{A}_{\mathrm{st}}(V) = \{f \in \mathcal{O}(\mathcal{V}) : f \text{ is strongly asymptotically developable in } V\},$$
$$\mathbf{A}_{\mathrm{wk}}(V) = \{f \in \mathcal{O}(\mathcal{V}) : f \text{ has a weak asymptotic expansion in } V\}.$$

([1]) A *proper subpolysector* of $V = V_1 \times \ldots \times V_n$ is a subset of V of the form $W = W_1 \times \ldots \times W_n$ where W_j is a sector of $\mathbb{C}$ such that $\overline{W}_i \subset V_i \cup 0$.

PROPOSITION 6.1. *Let f be a strongly asymptotically developable function in a polysector V. Let $\mathcal{F} = \{f_{\alpha_J}\}_{J,\alpha_J}$ be the total asymptotic expansion of f in V. Then f_{α_J} is strongly asymptotically developable in V_{J^c}, for all $J \subset \{1,\ldots,n\}$, $J \neq \emptyset$, and $\alpha_J \in \mathbb{N}^J$. Moreover,*

$$(\#) \qquad D^{\alpha_{J^c}} f_{\alpha_J}(w_{J^c}) = \frac{1}{\alpha_J!} \lim_{\substack{(z_J,z_{J^c})\to(0_J,w_{J^c})\\ (z_J,z_{J^c})\in W_J\times W_{J^c}}} D^{(\alpha_J,\alpha_{J^c})} f(z_J, z_{J^c}),$$

where $W \neq \emptyset$ is a proper subpolysector of V, $w_{J^c} \in W_{J^c}$, $\alpha_J \in \mathbb{N}^J$.

Proof. Let $\mathrm{TA}(f) = \{f_{\beta_K}\}_{K,\beta_K}$. Set

$$\mathcal{F}_{\alpha_J} = \{f_{\alpha_{I\cup J}}(z_{(I\cup J)^c})\}_{\emptyset\neq I\subset J^c,\,\alpha_I\in\mathbb{N}^I}.$$

We claim that $\mathrm{TA}(f_{\alpha_J}) = \mathcal{F}_{\alpha_J}$. We proceed by induction on $\#J$ and $|\alpha_J|$. For all $N_{J^c} \in \mathbb{N}^{J^c}$ we have

$$\begin{aligned}(\mathrm{a}) \quad & \mathrm{App}_{(\alpha_J+1_J,N_{J^c})}(\mathcal{F},z)\\ &= (-1)^{\#J+1} \sum_{0_J\le\beta_J\le\alpha_J} z_J^{\beta_J}[f_{\beta_J}(z_{J^c}) - \mathrm{App}_{N_{J^c}}(\mathcal{F}_{\beta_J}, z_{J^c})]\\ &\quad + \mathrm{App}_{(0_J,N_{J^c})}(\mathcal{F},z)\\ &\quad + \sum_{\emptyset\neq K\subsetneq J} (-1)^{\#K} \sum_{0_K\le\beta_K\le\alpha_K} z_K^{\beta_K}[\mathrm{App}_{(0_K,N_{J^c})}(\mathcal{F}_{\beta_K}, z_{K^c}) - f_{\beta_K}(z_{K^c})].\end{aligned}$$

Now fix $w = (w_1,\ldots,w_n) \in W$. Suppose $J = \{p\}$. If $\alpha_p = 0$, then, by (a),

$$|f_{0_J}(z_{J^c}) - \mathrm{App}_{N_{J^c}}(\mathcal{F}_{0_J}, z_{J^c})| \le |z_{J^c}^{N_{J^c}}|(K_{W,(1_J,N_{J^c})}|w_J| + K_{W,(0_J,N_{J^c})}),$$

where $K_{W,(1_J,N_{J^c})}$ and $K_{W,(0_J,N_{J^c})}$ are given by the hypothesis. Define $C_{W_{J^c},0_J,N_{J^c}} = K_{W,(1_J,N_{J^c})}|w_J| + K_{W,(0_J,N_{J^c})}$. If $|\alpha_J| > 0$, then, by (a) and the induction hypothesis,

$$|f_{\alpha_J}(z_{J^c}) - \mathrm{App}_{N_{J^c}}(\mathcal{F}_{\alpha_J}, z_{J^c})| \le C_{W_{J^c},\alpha_J,N_{J^c}}|z_{J^c}^{N_{J^c}}|,$$

where $C_{W_{J^c},\alpha_J,N_{J^c}}$ is defined inductively by

$$\begin{aligned}C_{W_{J^c},\alpha_J,N_{J^c}} &= K_{W,(\alpha_J+1_J,N_{J^c})}|w_J| + K_{W,(0_J,N_{J^c})}\frac{1}{|w_J^{\alpha_J}|}\\ &\quad + \sum_{0_J\le\beta_J\le\alpha_J-1_J} |w_J^{\beta_J-\alpha_J}|C_{W_{J^c},\beta_J,N_{J^c}}.\end{aligned}$$

So our claim holds if $\#J = 1$. Suppose it is true for every subset L of $\{1,\ldots,n\}$ such that $\#L \le k-1$. Let $J \subset \{1,\ldots,n\}$ have cardinality k. If $\alpha_J = 0_J$, then (a) implies

$$|f_{0_J}(z_{J^c}) - \mathrm{App}_{N_{J^c}}(\mathcal{F}_{0_J}, z_{J^c})| \le C_{W_{J^c},0_J,N_{J^c}}|z_{J^c}^{N_{J^c}}|$$

with

$$\begin{aligned}C_{W_{J^c},0_J,N_{J^c}} &= K_{W,(1_J,N_{J^c})}|w_J| + K_{W,(0_J,N_{J^c})}\\ &\quad + \sum_{\emptyset\neq K\subsetneq J} C_{W_{K^c},0_K,(0_{K^c\cap J},N_{J^c})},\end{aligned}$$

where $K_{W,(1_J,N_{J^c})}$ and $K_{W,(0_J,N_{J^c})}$ are given by the hypothesis, and $C_{W_{K^c},0_K,(0_{K^c\cap J},N_{J^c})}$ were constructed by induction. If $|\alpha_J| > 0$, then

$$\begin{aligned}&|f_{\alpha_J}(z_{J^c}) - \mathrm{App}_{N_{J^c}}(\mathcal{F}_{\alpha_J}, z_{J^c})|\\ &\le |z_{J^c}^{N_{J^c}}|\Big[K_{W,(\alpha_J+1_J,N_{J^c})}|w_J| + K_{W,(0_J,N_{J^c})}\frac{1}{|w_J^{\alpha_J}|}\\ &\quad + \sum_{\emptyset\neq K\subsetneq J}\ \sum_{0_K\le\beta_K\le\alpha_K} \frac{|w_K^{\beta_K}|}{|w_J^{\alpha_J}|}C_{W,\beta_K,(0_K,N_{J^c})} + \sum_{\substack{0_J\le\beta_J\le\alpha_J\\ \beta_J\neq\alpha_J}} \frac{|w_J^{\beta_J}|}{|w_J^{\alpha_J}|}C_{W,\beta_J,N_{J^c}}\Big],\end{aligned}$$

where $K_{W,(\alpha_J+1_J,N_{J^c})}$ and $K_{W,(0_J,N_{J^c})}$ are given by the hypothesis, and $C_{W_{K^c},\beta_K,(0_{K^c\cap J},N_{J^c})}$ and $C_{W_{J^c},\beta_J,N_{J^c}}$ were constructed by induction. So, we get our claim.

The proof of (#) goes as follows. For $n = 1$ it is a classical result (see Wasow [Wa]). It is now enough to prove the two equalities

$$(\mathrm{b}) \qquad \lim_{\substack{(z_J,z_{J^c})\to(0_J,w_{J^c})\\ (z_J,z_{J^c})\in W_J\times W_{J^c}}} D^{(\alpha_J,\alpha_{J^c})}(f(z) - \mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},z)) = 0$$

and

$$(\mathrm{c}) \qquad \lim_{\substack{(z_J,z_{J^c})\to(0_J,w_{J^c})\\ (z_J,z_{J^c})\in W_J\times W_{J^c}}} D^{(\alpha_J,\alpha_{J^c})}\mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},z) = \alpha_J!D^{\alpha_{J^c}}f_{\alpha_J}(w_{J^c}).$$

Proof of (b). Let $\lambda > 0$ be such that for all $z = (z_1,\ldots,z_n) \in W$ we have

$$\gamma_j = \{z_j + \lambda|z_j|e^{i\theta} : \theta\in[0,2\pi]\} \subset V_j, \qquad j \in J.$$

The Cauchy formula for the function $f - \mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},\cdot)$ integrated on $\gamma_1\times\ldots\times\gamma_n$ implies

$$\begin{aligned}&|D^{(\alpha_J,\alpha_{J^c})}(f(z) - \mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},z))|\\ &\qquad\le \frac{\alpha_J!\alpha_{J^c}!K_{W',(\alpha_J+1_J,0_{J^c})}(1+\lambda)^{|\alpha_J|+\#J}}{\lambda^{|\alpha_J|+|\alpha_{J^c}|}|z_{J^c}^{\alpha_{J^c}}|}|z_J^{1_J}|,\end{aligned}$$

where W is a proper subpolysector of V containing W. Hence we get (b).

Proof of (c). An easy computation gives the formula

$$(\mathrm{d}) \qquad D^{(\alpha_J,\alpha_{J^c})}\mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},z) = \sum_{\emptyset\neq L\subset J} (-1)^{\#L+1}\alpha_L!D^{\alpha_{L^c}}f_{\alpha_L}(z_{L^c}).$$

The cardinality of L^c is $< n$; so, by induction on n,

$$\lim_{\substack{(z_J,z_{J^c})\to(0_J,w_{J^c})\\(z_J,z_{J^c})\in W_J\times W_{J^c}}} D^{\alpha_{L^c}} f_{\alpha_L}(z_{L^c}) = \alpha_{J\setminus L}! D^{\alpha_{J^c}} f_{\alpha_J}(w_{J^c}).$$

Then, by (d), we obtain

$$\lim_{\substack{(z_J,z_{J^c})\to(0_J,w_{J^c})\\(z_J,z_{J^c})\in W_J\times W_{J^c}}} D^{(\alpha_J,\alpha_{J^c})}\mathrm{App}_{(\alpha_J+1_J,0_{J^c})}(\mathcal{F},z) = \alpha_J! D^{\alpha_{J^c}} f_{\alpha_J}(w_{J^c}),$$

as

$$\sum_{\emptyset\neq L\subset J}(-1)^{\#L+1} = \sum_{i=1}^{k}\binom{k}{i}(-1)^{i+1} = 1 \quad (\text{with } k=\#J).$$

COROLLARY 6.2. *If $f\in \mathbf{A}_{\mathrm{st}}(V)$, then $D^\alpha f\in\mathbf{A}_{\mathrm{st}}(V)$ for all $\alpha\in\mathbb{N}^n$.*

COROLLARY 6.3. *The total asymptotic expansion $\mathcal{F}$ of a strongly asymptotically developable function f is unique in a given polysector V.*

7. The main theorem

THEOREM 7.1. *Let V be a nonempty polysector in $\mathbb{C}^n$ and f a holomorphic function on V. Then the following two statements are equivalent:*

(1) f is strongly asymptotically developable in V.

(2) For every proper subpolysector $W\neq\emptyset$ of V, the restriction of f to W, f_W, admits a $\mathcal{C}^\infty$ extension to $\mathbb{R}^{2n}$ as a function of real variables.

Remark 7.2. If we replace (2) by

(2)′ f admits a $\mathcal{C}^\infty$ extension to some neighbourhood of $\overline{V}\subset\mathbb{R}^{2n}$,

then the assertion of the theorem is false (take $f(z)=e^{-1/z}$ in $V=\{\mathrm{Re}\, z>0\}$).

8. The proof of Theorem 7.1.

Let us start with the proof (1)⇒(2).

LEMMA 8.1. *Let $W\neq\emptyset$ be a polysector in $\mathbb{C}^n=\mathbb{R}^{2n}$ and let F be a $\mathcal{C}^\infty$ function defined on W. If for all $\alpha\in\mathbb{N}^n$, $D^\alpha F$ has a continuous extension to $\overline{W}$, then F has a $\mathcal{C}^\infty$ extension to $\mathbb{R}^{2n}$.*

Proof. Let F_m be a $\mathcal{C}^m$ extension of f to $\mathbb{R}^{2n}$ (the extension exists because W is a convex domain; see [W2]). Let J^mF_m be the m-jet of the restriction of F_m to $\overline{W}$. So, the Whitney map $(J^mF_m)_{m\in\mathbb{N}}$ on $\overline{W}$ has a $\mathcal{C}^\infty$ extension, by the Whitney extension theorem (see [W1]).

PROPOSITION 8.2. *Assume (1) of 7.1. Then the partial derivatives*

$$\frac{\partial^{|\alpha|+|\beta|}\mathrm{Re}\, f}{\partial x^\alpha\partial y^\beta},\quad \frac{\partial^{|\alpha|+|\beta|}\mathrm{Im}\, f}{\partial x^\alpha\partial y^\beta},\quad \alpha,\beta\in\mathbb{N}^n,$$

each have a continuous extension to $\overline{W}$.

Proof. This is an easy consequence of (#) in 6.1.

In order to prove (2)⇒(1) we will use our formula (⋆) of Taylor type.

Let W be a nonempty proper subpolysector of V. By (2) we have $f_W\in\mathcal{C}^\infty_{\mathrm{e}}(W)$. Set

$$f_{W,\alpha_J}(z_{J^c}) = \frac{1}{\alpha_J!}\frac{\partial^{|\alpha_J|}f_W}{\partial z_J^{\alpha_J}}(0_J,z_{J^c}) \quad \text{in } W_{J^c}$$

for $\emptyset\neq J\subset\{1,\dots,n\}$ and $\alpha_J\in\mathbb{N}^J$. This function is holomorphic. Observe that the right-hand side does not depend on W (when $W_{J^c}\ni z_{J^c}$). So, all these functions define a function f_{α_J} that is holomorphic in V_{J^c}. Set $\mathcal{F}=\{f_{\alpha_J}\}$.

In order to evaluate $f-\mathrm{App}_N(\mathcal{F},z)$ let us proceed as follows:

Given a proper subpolysector $W\neq\emptyset$ of V, let $N=(N_1,\dots,N_n)\in\mathbb{N}^n$ and put $C_{W,N}=\max\{|D^Nf(z)| : z\in\overline{W}\}$. By (⋆) we get

$$|f(z)-\mathrm{App}_N(\mathcal{F},z)| = |\mathcal{E}_Nf(z)|\le C_{W,N}|z^N|$$

for all $z\in W$, as desired.

9. Applications.

Let $V\neq\emptyset$ be a polysector in $\mathbb{C}^n$. Set

$$\mathbf{U}_{\mathrm{st}}(V)=\{f\in\mathbf{A}_{\mathrm{st}}(V) : 1/f\in\mathbf{A}_{\mathrm{st}}(V)\}.$$

The following proposition completes the result of Haraoka [Ha] on the inverse of a strongly asymptotically developable function.

PROPOSITION 9.1. *Let $f\in\mathbf{A}_{\mathrm{st}}(V)$. Let $\mathrm{TA}(f)=\{f_{\alpha_J}\}$. Then the following statements are equivalent:*

(1) $f\in\mathbf{U}_{\mathrm{st}}(V)$.

(2) For all $z\in V$ and for all $J\subset\{1,\dots,n\}$, $J\neq\emptyset$, we have $f(z)\neq0$ and $f_{0_J}(z_{J^c})\neq0$.

Proof. Suppose (1). For $h=1/f$, we have

$$1=\lim_{\substack{z_J\to0_J\\z_J\in W_J}} f(z)h(z) = f_{0_J}(z_{J^c})h_{0_J}(z_{J^c})$$

and we get (2), since V is connected.

To deduce (1) from (2), take a nonempty proper subpolysector W of V. Choose a proper subpolysector W' of V such that W is a proper subpolysector of W'. By Theorem 7.1 we obtain a $\mathcal{C}^\infty$ extension F of the restriction $f_{W'}$ of f to W'. So, by 6.1,

$$f_{0_J}(z_{J^c}) = F(0_J,z_{J^c}),\qquad z_{J^c}\in W'_{J^c},$$

therefore,

$$F(0_J,z_{J^c})\neq0,\qquad z_{J^c}\in W'_{J^c}.$$

Given $z \in \overline{W}$, we have an open neighbourhood Ω_z in $\mathbb{R}^{2n}$ such that $F(u) \neq 0$ for $u \in \Omega_z$. So $\Lambda_W = \bigcup_z \Omega_z$ is an open neighbourhood of $\overline{W}$ where F never vanishes. Hence the mapping

$$\Lambda_W \ni u \mapsto 1/F(u)$$

is a $\mathcal{C}^\infty$ extension of $1/f$ to an open set Λ_W containing W, and, in consequence, we get (1), by 7.1.

PROPOSITION 9.2. *Let V be a nonempty polysector in $\mathbb{C}^n$. If $f \in \mathbf{A}_{\mathrm{st}}(V)$, then $D^\alpha f \in \mathbf{A}_{\mathrm{wk}}(W)$ for every nonempty proper subpolysector W of V and $\alpha \in \mathbb{N}^n$.*

Proof. It is enough to verify this when $\alpha = 0$, by 6.2. Let $W \neq \emptyset$ be a proper subpolysector of V, and let F be a $\mathcal{C}^\infty$ extension of f_W, the restriction of f to W. For each $z \in W$ consider the function

$$\varphi : [0,1] \ni t \mapsto F(tz) \in \mathbb{C}.$$

The usual Taylor formula in one variable gives the equality

$$f(z) = \varphi(t) + \ldots + \frac{\varphi^{(m)}(t)}{m!}(1-t)^m + \frac{1}{m!}\int_t^1 (1-u)^m \varphi^{(m+1)}(u)\,du.$$

Since

$$\frac{d^k\varphi}{dt^k}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha D^\alpha F(tz),$$

we obtain

$$\lim_{t\to 0} \frac{d^k\varphi}{dt^k}(t) = \sum_{|\alpha|=k} z^\alpha f_\alpha \quad \text{with } f_\alpha = \frac{k!}{\alpha!} D^\alpha F(0).$$

So, we have

$$\Big| f(z) - \sum_{|\beta|=0}^{m} \frac{f_\beta}{\beta!} z^\beta \Big| \le C \sum_{|\alpha|=m+1} |z^\alpha|,$$

with a constant $C > 0$, and that gives our statement.

Remark 9.3. The converse is also true. The first proof of this fact was given by J. A. Hernández in [He]; here we present another proof based on Theorem 7.1.

By hypothesis all $D^\alpha f$ are bounded, which implies that all the derivatives of f are Lipschitzian in W. Hence, we can extend them to $\overline{W}$. Then Lemma 8.1 and Theorem 7.1 imply the statement.

9.4. *Construction of a function which has a weak asymptotic expansion and is not strongly asymptotically developable*

LEMMA 9.4.1. *Let V be a nonempty polysector in $\mathbb{C}^n$. For $f \in \mathbf{A}_{\mathrm{wk}}(V)$ such that $f(z) \neq 0$ in V, denote by f_0 the limit of f at $0 \in \mathbb{C}^n$. If $|f(z) - f_0| < |f_0|$ in $\overline{V}$, then $1/f$ is in $\mathbf{A}_{\mathrm{wk}}(V)$.*

Proof. Set $\alpha(z) = f(z) - f_0$. Our assumption implies $\alpha(V) \subset B(0, |f_0|) = \{z \in \mathbb{C}^n : |z| < |f_0|\}$. Let

$$\gamma : B(0, |f_0|) \ni z \mapsto \frac{1}{f_0 + z} \in \mathbb{C}.$$

So, $1/f = \gamma \circ \alpha \in \mathbf{A}_{\mathrm{wk}}(V)$ (by [GS]).

Now, consider a positive integer $p > 2$. Let $V \neq \emptyset$ be a proper polysector in $\mathbb{C}^p$. Let $\varphi \in \mathbf{A}_{\mathrm{st}}(V)$ be such that $\mathrm{TA}(\varphi) = \{0\}$ and $\varphi \neq 0$. Choose a proper subpolysector $S \neq \emptyset$ of V. We have $\max\{|\varphi(z)| : z \in \overline{S}\} = |\varphi(A)|$ for some $A = (a_J, 0_{J^c})$, $J \subset \{1, \ldots, n\}$, $J \neq \emptyset$.

Consider $V' = \{w \in \mathbb{C} : 0 < |w| < 1/2, \arg w \in (-\pi/4, \pi/4)\}$ and for $(z,w) \in \overline{W}$ the function

$$f(z,w) = \varphi(A) - \varphi(z)(1-w)$$

defined on $W = S \times V'$. It is easy to see that f is in $\mathbf{A}_{\mathrm{st}}(W)$ and $\mathbf{A}_{\mathrm{wk}}(W)$. Moreover,

$$f_0 = \varphi(A) \neq 0 \quad \text{and} \quad |f(z,w) - f_0| \le |\varphi(z)| \cdot |1-w| < |f_0|.$$

Thus, $f(z,w) \neq 0$ in W. So, by Lemma 9.4.1, we have $1/f \in \mathbf{A}_{\mathrm{wk}}(W)$. But we have, by 6.1,

$$f_{0_{J^c}, 0_{p+1}}(a_J) = 0$$

and so it follows that $1/f \notin \mathbf{A}_{\mathrm{st}}(W)$, by 9.1.

10. The Gevrey case. Let us recall some definitions for the Gevrey case (following [Ha]).

Let V be a polysector in $\mathbb{C}^n$ and f a holomorphic function in V. Let $s = (s_1, \ldots, s_n) \in [0,\infty)^n$. We will call f *s-Gevrey strongly asymptotically developable* as $z \to 0$ if $f \in \mathbf{A}_{\mathrm{st}}(V)$ and

(G) for each proper subpolysector $W \neq \emptyset$ of V and each $N \in \mathbb{N}^n$ there exists $\varrho > 0$ such that

$$|f(z) - \mathrm{App}_N(\mathcal{F}, z)| \le K_W N!^s \varrho^{|N|} |z^N| \quad \text{for all } z \in W,$$

where $\mathrm{App}_N(\mathcal{F}, z)$ is defined as in Section 2. Set

$$\mathbf{A}^s_{\mathrm{st}}(V) = \{f \in \mathcal{O}(V) : f \text{ is } s\text{-Gevrey strongly asymptotically developable in } V\}.$$

Let V be a nonempty proper polysector of $\mathbb{C}^n$. A holomorphic function f on V is *s-Gevrey* if for every proper subpolysector $W \neq \emptyset$ of V there exist $C_W > 0$ and $\varrho_W > 0$ such that, for all $\alpha \in \mathbb{N}^n$,

$$|D^\alpha f(z)| \le C_W \alpha!^{s+1_{\{1,\dots,n\}}} \varrho_W^{|\alpha|}, \quad \forall z \in W.$$

The algebra of s-Gevrey holomorphic functions on V will be denoted by $\mathbf{O}_s(V)$. Haraoka states the equality ([Ha])

$$(\square) \qquad \mathbf{A}^s_{\text{st}}(V) = \mathbf{A}_{\text{st}}(V) \cap \mathbf{O}_s(V).$$

For the convenience of the reader we include a proof of this fact.

Given f in $\mathbf{A}^s_{\text{st}}(V)$ and W, we choose a proper subpolysector $W' = W'_1 \times \ldots \times W'_n$ of V such that W is a proper subpolysector of W'. Let $\lambda > 0$ be such that for all $z = (z_1, \ldots, z_n) \in W'$ we have

$$\gamma_j = \{z_j + \lambda |z_j| e^{i\theta} : \theta \in [0, 2\pi]\} \subset W'_j, \quad 1 \le j \le n.$$

The Cauchy formula for the function $f - \mathrm{App}_\alpha(\mathcal{F}, \cdot)$ integrated on $\gamma_1 \times \ldots \times \gamma_n$ gives

$$|D^\alpha f(z)| \le K_{W'} \alpha!^{s+1_{\{1,\dots,n\}}} \varrho^{|\alpha|} \left(\frac{1+\lambda}{\lambda}\right)^{|\alpha|} \quad \text{for some } K_{W'} > 0,\ \varrho > 0,$$

because $D^\alpha f = D^\alpha (f - \mathrm{App}_\alpha(\mathcal{F}, \cdot))$. Therefore, $f \in \mathbf{O}_s(V)$.

Conversely, let $f \in \mathbf{A}_{\text{st}}(V) \cap \mathbf{O}_s(V)$. Then for every proper subpolysector $W \neq \emptyset$ there exist $K_W > 0$ and $\varrho_W > 0$ such that, for all $\alpha \in \mathbb{N}^n$,

$$|D^\alpha f(z)| \le K_W \alpha!^{s+1_{\{1,\dots,n\}}} \varrho_W^{|\alpha|}, \quad \forall z \in W.$$

The integral formula $(\star)$ from 3.1 implies

$$|f(z) - \mathrm{App}_\alpha(\mathcal{F}, z)| \le K_W \alpha!^s \varrho_W^\alpha |z^\alpha|, \quad \forall z \in W,$$

since $f \in \mathcal{C}^\infty_{\mathrm{e}}(W)$; so $f \in \mathbf{A}^s_{\text{st}}(V)$.

Remark 10.1. If f is s-Gevrey strongly asymptotically developable, then $\mathrm{FA}(f)$ is a formal power series of Gevrey order s (see [Ha]). This series approaches f with an exponentially flat error (see [Z]).

THEOREM 10.2. *Let $V \neq \emptyset$ be a polysector in $\mathbb{C}^n$. Let $f \in \mathcal{O}(V)$ and $s \in [0, \infty)^n$. Then the following statements are equivalent:*

(1) *f is a s-Gevrey strongly asymptotically developable.*

(2) *For every proper subpolysector $W \neq \emptyset$ of V, the restriction of f to W, f_W, has a C^∞ extension to $\mathbb{R}^{2n}$ as a function of real variables such that, for some $K > 0$ and $\varrho > 0$ we have, for all $\alpha \in \mathbb{N}^n$,*

$$\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)\right| \le K \alpha!^{s+1_{\{1,\dots,n\}}} \varrho^{|\alpha|},$$

where $z \in \overline{W} \subset \mathbb{R}^{2n}$.

Proof. This is an easy consequence of $(\square)$ and Theorem 7.1.

Remark 10.3. The extension in 10.2(2) can be chosen of Gevrey class s, that is, the extension F of f_W can be chosen in such a way that

$$|D^\alpha F| \le K \alpha!^{s+1_{\{1,\dots,n\}}} \varrho^{|\alpha|}, \quad z \in \mathbb{R}^{2n}.$$

The proof goes as the proof of 7.1, with Whitney's theorems replaced by their Gevrey versions (see [K], [B], [BBMT]).

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