Suppose that \( m \geq n \) and \( \alpha \leq n/(n+1) \). \( \mathfrak{A} \) is generated by the \( m \) generators \( x_j(\theta) = \exp(i\theta_j), \ j = 1, \ldots, m \), and, by considering functions which are constant on the other \( \theta_k \), each \( n \)-element subset of these generators lies in a subalgebra isomorphic with \( \text{lip}_\alpha T^n \), which is \( n \)-dimensionally weakly amenable. Thus if \( T \) is an alternating \( n \)-derivation with values in a symmetric Banach \( \mathfrak{A} \)-module, by restricting to the subalgebra we see that \( T \) is zero for any \( n \)-generators, and so by Corollary 2.6, \( T = 0 \).

Suppose that \( m \geq n \) and \( \alpha > n/(n+1) \). By restricting to \( \{ \theta : \theta \in T^n, \theta_{n+1} = \ldots = \theta_m = 0 \} \) we see that \( \text{lip}_\alpha T^n \) is a quotient of \( \text{lip}_\alpha T^m \) and so by Theorem 3.1(1) the latter is not \( n \)-dimensionally weakly amenable because the former is not.

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Hereditarily finitely decomposable Banach spaces
by
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Abstract. A Banach space is said to be HD\(n\) if the maximal number of subspaces of \( X \) forming a direct sum is finite and equal to \( n \). We study some properties of HD\(n\) spaces, and their links with hereditarily indecomposable spaces; in particular, we show that if \( X \) is complex HD\(n\), then \( \dim(\ell_1(X)/\mathcal{S}(X)) \leq n^2 \), where \( \mathcal{S}(X) \) denotes the space of strictly singular operators on \( X \). It follows that if \( X \) is a real hereditarily indecomposable space, then \( \ell_1(X)/\mathcal{S}(X) \) is a division ring isomorphic either to \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \), the quaternionic division ring.

1. Introduction

General introduction. The problems discussed in this article concern natural questions after the article of W. T. Gowers and B. Maurey ([GM]) solving the unconditional basic sequence problem. In a Banach space \( X \), a sequence \( (e_n)_{n \in \mathbb{N}} \) is said to be a basis if every vector in \( X \) can be written uniquely in the form \( x = \sum_{i=0}^{\infty} \lambda_i e_i \), where the \( \lambda_i \)'s are scalars. It is an unconditional basis if there is a constant \( C \) such that the inequalities

\[
\left\| \sum_{i \in E} \lambda_i e_i \right\| \leq C \left( \sum_{i=0}^{\infty} \left| \lambda_i \right| \right)
\]

hold for all subsets \( E \) of \( \mathbb{N} \) and all coefficients \( \lambda_i \). A sequence is a basic sequence (resp. an unconditional basic sequence) if it is a basis (resp. an unconditional basis) of its closed linear span. Details about these notions can be found in [LT].

A lot of classical spaces have an unconditional basis (spaces \( l_p \), for \( p \geq 1 \) have one) but for example \( L_1 \) does not have one; an example of a Banach space without a basis is even harder to find, but was given by P. Enflo in 1973 ([E]). On the other hand, it is a classical result that every Banach space contains a basic sequence; but the question “Does every Banach space contain an unconditional basic sequence?” has remained unsolved for many years.

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years. In 1991, Gowers and Maurey finally gave a counterexample, that we shall call $X_{GM}$, to this question.

In fact, the space $X_{GM}$ has a stronger property, namely, it is hereditarily indecomposable (or H.I.), which means that no subspace of $X_{GM}$ is decomposable (i.e. the topological direct sum of two infinite-dimensional subspaces). A Banach space $X$ with the H.I. property is easily shown not to contain unconditional basic sequences, but also not to be isomorphic to its square, and, with a little more work, not to be isomorphic to its proper subspaces. But the H.I. property not only provides counterexamples to many important dichotomy theorems, proved by Gowers. According to Gowers' theorem, every Banach space contains a subspace that is either hereditarily indecomposable or spanned by an unconditional basis ([G1], [G2]). Because of this theorem, it is particularly important to know about general properties of hereditarily indecomposable spaces.

Most known results on H.I. spaces are about operators on complex H.I. spaces. In particular, Gowers and Maurey showed in [GM] that if $X$ is H.I. complex, then every operator on $X$ is a strictly singular perturbation of a multiple of the identity (we recall that an operator $S$ is strictly singular if no restriction of $S$ is an into isomorphism). The author generalized their result showing that for every subspace $Y$ of $X$, every operator from $Y$ to $X$ is a strictly singular perturbation of a multiple of the canonical inclusion map from $Y$ to $X$ ([F1], and Theorem 1 below). The fact that the space of operators on a H.I. space is, in a sense, small is a crucial point for proving properties of these spaces.

In this article, we study the property $HD_n$: a space is $HD_n$ if the supremum of the $m$'s such that $X$ contains a direct sum of $m$ infinite-dimensional subspaces is equal to $n$. For $n \geq 2$, these are spaces without an unconditional basic sequence that are not H.I., so they give a good illustration of Gowers' dichotomy theorem. On the other hand, we show that $HD_n$ spaces are spaces with "few" operators, which provides them with properties similar to the properties of H.I. spaces. Finally, our results allow us to solve the real case: in a real H.I. space, the quotient of the space of operators by the space of strictly singular operators is a division ring that is either real, complex or quaternionic.

**Notation and definitions.** In the following, $X$, $Y$, $Z$, $W$ are complex or real Banach spaces. The real field is denoted by $\mathbb{R}$, the complex field by $\mathbb{C}$, and the quaternionic division ring by $\mathbb{H}$. By a space (resp. subspace) we shall always mean an infinite-dimensional closed space (resp. subspace). The set of bounded operators from $Z$ to $X$ is denoted by $L(Z,X)$. Recall that an operator $S$ from $Z$ to $X$ is said to be strictly singular if for every subspace $W$ of $Z$, the restriction $S|_W$ of $S$ to $W$ is not an into isomorphism. This is equivalent to saying that for every $\varepsilon > 0$ and every subspace $W$ of $Z$, there exists a subspace $W'$ of $W$ such that $\|S|_{W'}\| \leq \varepsilon$. The set of strictly singular operators from $Z$ to $X$ is a subspace of $L(Z,X)$ that we shall denote by $S(Z,X)$, and we recall that when $TU$ is defined, it is a strictly singular operator whenever $T$ or $U$ is strictly singular.

Let $Z \subset X$, and $A \in L(Z,X)$. By Proposition 2.c.10 of [LT], if $A$ is of the form $Id+S$, where $S$ is strictly singular, then it is an isomorphism on some finite-dimensional subspace of $Z$. If such an operator $A$ is an isomorphism on $Z$, we say that $A$ is an $Id+S$-isomorphism. Then the inverse isomorphism $A^{-1}$ of $A$ defined on $A(Z)$ satisfies $A^{-1} - Id = (Id - A)A^{-1}$, so it is an $Id+S$-isomorphism on $A(Z)$. Two subspaces $Y$ and $Z$ of a Banach space $X$ are said to be $Id+S$-isomorphic if there is an $Id+S$-isomorphism from $Y$ onto $Z$. Replacing "strictly singular" by "compact" in the above, we also define $Id+K$-isomorphisms. We recall that every compact operator is strictly singular.

Two spaces are said to be totally incomparable if no subspace of the first is isomorphic to a subspace of the second.

Let $R$ be a finite set. Then $|R|$ stands for its cardinality. Given subspaces $(X_i)_{i \in R}$ of $X$, their sum is direct if for every $i$, the projection $p_i$ from $\sum_{i \in R} X_i$ to $X_i$ is well defined and continuous; the sum is then denoted by $\bigoplus_{i \in R} X_i$. For any $Q \subset R$, we will denote by $X_Q$ the sum $\bigoplus_{i \in Q} X_i$. For any operator $T$ on $X_R$, we will denote by $T_Q$ the restriction of $T$ to $X_Q$ (if $Q = \emptyset$, we write $T_Q$ for $T_{\emptyset}$). Given another direct sum $X_Q'$, we say that $X_Q'$ is smaller than $X_R$ if there exists a permutation $\sigma$ on $R$ such that for all $i \in R$, $X_{\sigma(i)}' \subset X_i$ (notice that when two sums are comparable, they are necessarily sums over the same finite set).

Recall that a space that can be written as a direct sum of at least two (infinite-dimensional) subspaces is said to be decomposable and that a space is said to be hereditarily indecomposable (or H.I.) if it has no decomposable subspace.

More generally, we say that a space is $HD_n$ if the maximal number of subspaces of $X$ in a direct sum is finite and equal to $n$ (Definition 2) and we will prove (Corollary 2) that a simple example of a $HD_n$ space is the direct sum of $n$ H.I. spaces. In Section 2, we define and study the convenient notion of quasi-maximality (Definition 1). In Section 3, we show that the sum of a $HD_m$ and of a $HD_n$ space is $HD_{m+n}$ (Proposition 1). This leads to considering a certain type of $HD_n$ spaces called fundamental (Definition 3), which we study in Section 4. In Section 5, we show that consequently, if $X$ is complex $HD_n$, then the dimension of $L(X)/S(X)$ is smaller than $n^2$ (Proposition 4). In Section 6, we study spectral properties of (real and complex)
HD$_n$ spaces. In Section 7, we show our main theorem: if $X$ is a real H.I. space, then $L(X)/S(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (Theorem 2).

2. Quasi-maximality. We first define an important technical notion.

**Definition 1.** Let $X$ be a Banach space. Let $X$ be a subspace of $X$. The space $X$ is said to be quasi-maximal in $X$ if for every subspace $Z$ of $X$, the sum $X + Z$ is not direct.

**Lemma 1.** Let $X$ be a Banach space. Let $Y, Z$ be subspaces of $X$. If for every subspace $W$ of $Z$, the sum $Y + W$ is not direct, then $Y$ and $Z$ have $\text{Id} + K$-isomorphic subspaces.

**Proof.** Let $Y, Z$ satisfy the hypothesis. Passing to a subspace, we may assume that $Z$ has a basis. For every $\varepsilon > 0$, for any subspace $W$ of $Z$ the sum $Y + W$ is not direct, so there exist two unit vectors $y \in Y$ and $w \in W$ such that $||y - w|| \leq \varepsilon$. Using repeatedly this remark, one can build a normalized block basic sequence $(z_i)_{i \in \mathbb{N}}$ in $Z$ and a sequence $(y_i)_{i \in \mathbb{N}}$ in $Y$ such that $\sum_{i \in \mathbb{N}} ||y_i - z_i|| < \infty$. Let $Z'$ be the subspace generated by $(z_i)_{i \in \mathbb{N}}$. The operator $A$ from $Z'$ into $Y$ defined by $A(z_i) = y_i$ for all $i$ is of the form $\text{Id} + K$, where $K$ is compact as a uniform limit of finite rank operators. Passing to a finite-codimensional subspace, we may assume that it is an isomorphism. Then $Z'$ and $A(Z')$ are $\text{Id} + K$-isomorphic subspaces of $Z$ and $Y$.

**Corollary 1.** Let $X$ be a Banach space, and $X$ a subspace of $X$. Then $X$ is quasi-maximal if and only if for every subspace $Z$ of $X$, $X$ and $X$ have $\text{Id} + S$-isomorphic subspaces. The same statement holds if we replace $\text{Id} + S$ by $\text{Id} + K$.

**Proof.** This follows from Lemma 1, from the definition of a strictly singular operator, and from the fact that every compact operator is strictly singular.

We now prove two lemmas about quasi-maximal subspaces $X$ of $X$. They show that, as long as we study operators on $X$ only up to strictly singular operators, it is enough to study operators on some quasi-maximal subspace $X$ of $X$.

**Lemma 2.** Let $X$ be a Banach space and let $X$ be a quasi-maximal subspace of $X$. Let $Z$ be any Banach space, and let $T \in L(X, Z)$. Then $T$ is strictly singular if and only if the restriction $T|_X$ of $T$ to $X$ is strictly singular.

**Proof.** The direct implication is clear. Now suppose that $T|_X$ is strictly singular. Let $Y$ be a subspace of $X$. According to Corollary 1, there is a subspace $Y'$ of $Y$ and an embedding $\alpha = \text{Id}_{Y'} + S$ of $Y'$ into $X$. Then we can write $T|_{V'} = T(\text{Id}_{Y'} + S - S) = T|_X A - T|_X S$. As $T|_X$ and $S$ are strictly singular, so is $T|_{V'}$, in particular, for every $\varepsilon > 0$, there is a unit vector $y$ in $Y'$ such that $||y|| \leq \varepsilon$, so $T$ is strictly singular.

**Lemma 3.** Let $X$ be a Banach space and let $X$ be a quasi-maximal subspace of $X$. Let $Z$ be any Banach space, and let $T \in L(Z, X)$. Then there exist $Z' \subset Z$ and $T' \in L(Z', X)$ with $\text{Im} T' \subset X$ such that $T|_{Z'} = T'$ is strictly singular (or compact).

**Proof.** It is easy to check that for every strictly singular operator $S$ on $Z$, there exists a subspace $W$ such that $S|_W$ is compact; so it is enough to prove the strictly singular part of the assertion. Now let $T \in L(Z, X)$. If $T$ is strictly singular then we can choose $Z' = Z$ and $T' = 0$. If not, passing to a subspace, we may assume that $T$ is an isomorphism into $X$. Applying Corollary 1 to $TZ$ and $X$, we find a subspace $Z'$ of $X$ such that $TZ'$ embeds into $X$ by an isomorphism $A = \text{Id} + S$. Let $T' = AT|_Z$. We have $\text{Im} T' \subset X$; furthermore, $T|_{Z'} = T'$ is strictly singular.

3. HD$_n$ spaces

**Definition 2.** A space $X$ is hereditarily finitely decomposable if the maximal number of (infinite-dimensional) subspaces of $X$ forming a direct sum in $X$ is finite. For $n \geq 1$, $X$ is HD$_n$ if this number is equal to $n$.

**Some easy remarks.** A space is HD$_1$ if and only if it is H.I. More generally, we will see that the direct sum of $n$ H.I. spaces is HD$_n$ (Corollary 2).

A space $X$ is HD$_n$ if and only if it contains direct sums of $n$ subspaces and every such sum is quasi-maximal. In particular, $X$ is H.I. if and only if every subspace of $X$ is quasi-maximal.

For every HD$_n$ space $X$, every subspace $Y$ of $X$ is HD$_m$ for some $m \leq n$.

**Proposition 1.** Let $X$ be HD$_m$ and $Y$ be HD$_n$. Then $X \oplus Y$ is HD$_{m+n}$.

**Proof.** It is based on a type of Gaussian elimination method. Let $p$ be the projection from $X \oplus Y$ onto $X$, and $q$ be the projection from $X \oplus Y$ onto $Y$. By definition, there exist $m$ subspaces of $X$ forming a direct sum in $X$, and $n$ subspaces of $Y$ forming a direct sum in $Y$. The sum of all these subspaces is a direct sum of $m + n$ subspaces of $X \oplus Y$. Now let $K = \{1, \ldots, k\}$ and let $Z_K$ form a direct sum in $X \oplus Y$. It is enough to prove that $k \leq m + n$.

Passing to subspaces of the $Z_i$'s, we may assume that each $p_i$ and each $q_i$ is either strictly singular or an isomorphism. Let $I_X$ be the set of $i$ such that $p_i$ is an isomorphism. For $i \in I_X$, let $X_i = p_i(Z_i)$.

Let $\mathcal{R}$ be the set of subsets $R$ of $I_X$ such that $p$ is an isomorphism on $Z_R$ or on some smaller sum (in the sense of the definition given in the
introduction). Let \( r = \max \{|R| : R \in \mathcal{R}\} \). As \( X \) is HD, we have \( r \leq m \). We may assume that \( r \) is attained for \( R = [1, r] \) and, passing to a smaller sum for the \( Z_i \)'s, that \( p_{ij} \) is an isomorphism. Let \( p_{ij}^{-1} \) be the inverse isomorphism of \( p_{ij} \). Let \( i \in I_X \setminus R \). By the definition of \( r, p \) is not an isomorphism on any sum smaller than \( Z_R \oplus Z_i \). This implies, in particular, that no subspace of \( X_i \) forms a direct sum with \( X_R \); indeed, assume \( X_i' \subset X_i \) and \( X_R \) form a direct sum, and let \( Z_i' = p^{-1}(X_i') \); then on \( Z_R \oplus Z_i' \), \( p \) is an isomorphism. By Lemma 1, it follows that some subspace of \( X_i \) is \( Id + S \)-isomorphic to a subspace of \( Z_R \). So, passing to a subspace of \( Z_i \), we may assume that \( X_i \) embeds into \( Z_R \) by an operator of the form \( Id_S + S \). This yields an embedding \( b_i = p_{ij}(Id_S + S)p_{ij} \) of \( Z_i \) into \( Z_R \). For \( i \) not in \( R \) and not in \( I_X \), we let \( b_i = 0 \). Let \( b \) be the operator from \( Z_{K \setminus R} \) into \( Z_R \) whose matrix is the \( (r, k - r) \)-matrix \( \begin{bmatrix} b_{1,1} & \cdots & b_{1,k} \\
 & \ddots & \iddots \\
 & \cdots & b_{k-1,k} \end{bmatrix} \). We now consider another decomposition \( Z'_{K \setminus R} = Z_1' + \cdots + Z_k' \) of \( Z_{K \setminus R} \), defined by \( Z_i' = Z_i \) for \( 1 \leq i \leq r \) and \( Z_i' = \{ (b_{ji}z_i, z_i) : z_i \in Z_i \} \) for \( r + 1 \leq i \leq k \).

We show that the restriction of \( p \) to \( Z'_{K \setminus R} \) is strictly singular. Let \( i \in K \setminus R \). Then either \( i \in I_X \) and then the restriction of \( p \) to \( Z'_{i} \) is defined by

\[
p(b_{ji}z_i, z_i) = - (Id + S)p_{ij}z_i + p_{ij}z_i = - S(p_{ij}z_i),
\]

so it is strictly singular; or \( i \notin I_X \) and then the restriction of \( p \) to \( Z'_{i} \) is defined by

\[
p(-b_{ji}z_i, z_i) = p_{ij}z_i
\]

(where \( p_{ij} \) is strictly singular), so it is strictly singular.

Thus the restriction of \( p \) to \( Z'_{K \setminus R} \) is strictly singular, and as \( q = Id - p \), it follows that the restriction of \( q \) to some finite-codimensional subspace of \( Z_{K \setminus R} \) is an isomorphism: up to a finite-dimensional perturbation, \( Z_{K \setminus R} \) embeds as a direct sum into \( Y \). As \( Y \) is HD, it follows that \( k - r \leq n \), so \( k \leq m + n \).

**Corollary 2.** Let \( n \in \mathbb{N} \). For \( i = 1, \ldots, n \), let \( X_i \) be a H.I. space. Then the space \( \bigoplus_{i=1}^{n} X_i \) is HD.

**Remarks.** A hereditarily finitely decomposable space does not contain any unconditional basic sequence. Indeed, if \( X \) contains an unconditional basic sequence \( (e_k)_{k \leq n} \), then for every \( n \geq 1 \), \( X \) contains the direct sum \( \bigoplus_{i=1}^{n} E_i \), where \( E_i \) is the space generated by \( (e_{n+i})_{i \leq n} \).

Corollary 2 provides us with the first examples of HD \( n \) spaces for \( n \geq 2 \); by the previous remark, notice that these are spaces without an unconditional basic sequence that are not H.I. However, direct sums of n H.I. spaces are the only examples of HD \( n \) spaces: a HD \( n \) space not of this form was built in [F2], for a different purpose.

Finally, notice a property that was already known for H.I. spaces: if \( X \) is finitely hereditarily indecomposable, then for every \( k \neq l \), \( X^k \) and \( X^l \) are not isomorphic. Indeed, by Proposition 1, if \( X \) is HD, then \( X^k \) is HD, while \( X^l \) is HD.

**4. Fundamental HD \( n \) spaces.** In this section, we study a particular type of HD \( n \) spaces, called fundamental: fundamental HD \( n \) spaces are direct sums of \( n \) H.I. spaces, with some "normalizing" condition. We will see that given a HD \( n \) space \( X \), it is enough to know about the operators on any of its fundamental HD \( n \) subspaces to know about the operators on \( X \).

**Definition.** A Banach space \( X \) is a fundamental HD \( n \) space if it is the direct sum of \( n \) H.I. spaces any two of which are either isomorphic or totally incomparable.

Let \( X \) be a fundamental HD \( n \) space of the form \( X = \bigoplus_{i=1}^{n} X_i \). For \( i, j \) in \( \{1, \ldots, n\} \), we say that \( i \simeq j \) if \( X_i \) and \( X_j \) are isomorphic. For every \( i \), we denote by \( p_i \) the projection from \( X \) onto \( X_i \). For every \( i \simeq j \), we denote by \( \alpha_{ij} \) a fixed isomorphism from \( X_i \) onto \( X_j \), by \( p_{ij} \) the map \( \alpha_{ij}p_i \) from \( X \) onto \( X_j \). We denote by \( \alpha(X) \) the set of equivalence classes of indices for the relation \( \simeq \). We also want to define a set \( s(X) \), so that does not depend on the choice of the labelling: ordering the classes by decreasing cardinality, we let \( s(X) \) be the finite sequence of these cardinalities. We also define a positive number \( n(X) \) by \( n(X)^2 = \sum_{i \leq n} |s_i|^2 \). Writing these numbers as functions of \( X \) is an abuse of notation, because a priori, they depend on the choice of the decomposition of \( X \) as \( \bigoplus_{i=1}^{n} X_i \); we will allow this notation because we will see (Proposition 2) that \( s(X) \) and \( n(X) \) are in fact uniquely determined by \( X \). Until then, we think of \( s \) and \( n \) as functions of \( X \) under a particular decomposition.

We will see that \( s(X) \) and \( n(X) \) characterize in a way the space of operators on \( X \). Notice also that \( n(X) \leq n \).

**Remark.** Let \( X \) be fundamental. A sum smaller than \( X \) can always be chosen fundamental, by passing to further subspaces (if two spaces are not totally incomparable, then, passing to subspaces, we may assume that they are isomorphic; repeating this procedure, we end up with a fundamental sum). We will always choose such smaller sums, without necessarily saying it.

It is clear that both \( s(X) \) and \( n(X) \) are preserved when we apply an isomorphism to \( X \). Notice also that neither \( s(X) \) nor \( n(X) \) changes when we pass to a smaller sum; again this is an abuse of notation, but we can allow it here because comparing two fundamental sums is by definition comparing two given decompositions of the sums (see the definition in the introduction). The proof is the following. Let \( X \) be a fundamental sum, and \( Y \) a smaller sum, and to simplify the notation, assume that for all \( i \), \( Y_i \subset X_i \). Then if \( X_i \) and \( X_j \) are totally incomparable, then so are the subspaces \( Y_i \) and \( Y_j \).
and \( Y_j \); if \( X_i \) and \( X_j \) are isomorphic, then by the H.L. property (apply Corollary 1, for example), \( Y_i \) and \( Y_j \) are not totally incomparable, so they must be isomorphic.

**Lemma 4.** Every HD\(_n\) space contains a fundamental HD\(_n\) space.

**Proof.** Let \( X \) be HD\(_n\). Then \( X \) contains a direct sum of \( n \) subspaces. Each of these subspaces must be H.L., otherwise \( X \) would contain a direct sum of \( n+1 \) subspaces. By the previous remark, passing to appropriate subspaces, we may assume that this sum is a fundamental HD\(_n\) space.

**Remark.** If \( X \) is HD\(_n\), then every fundamental HD\(_n\) subspace \( X \) of \( X \) is quasi-maximal in \( X \). This is an important remark with respect to Lemmas 2 and 3.

We now study the relative position of fundamental subspaces of a HD\(_n\) space \( X \). We will see that the fundamental HD\(_n\) subspaces of \( X \) form a filter for the inclusion up to isomorphism.

**Notation.** Let \( X = \bigoplus_{i=1}^{n} X_i \) be a fundamental HD\(_n\) space and let \( Y = \bigoplus_{i=1}^{n} Y_i \) be a fundamental HD\(_m\) subspace of \( X \). We denote by \( I_{Y,X} \) the injection of \( Y \) into \( X \), written as an \((n,m)\)-matrix with coefficients in \( L(Y_i, X_i) \) for \( 1 \leq j \leq m \) and \( 1 \leq i \leq n \). For \( 1 \leq i \leq n \), we recall that \( p_i \) denotes the projection on \( X_i \), and according to the notation defined in the introduction, \( p_{ij} \) is the coefficient in the \( i \)th line and \( j \)th column of the matrix of \( I_{Y,X} \).

**Lemma 5.** Let \( X = \bigoplus_{i=1}^{n} X_i \) be a fundamental HD\(_n\) space. Let \( Y = \bigoplus_{i=1}^{m} Y_i \) be a fundamental HD\(_m\) subspace of \( X \). Then there exists a sum \( Y' \) smaller than \( Y \), an isomorphism \( L \) from \( Y' \) into \( Y \), and a permutation matrix \( E \) on \( X \) such that \( EI_{Y,X} = \begin{pmatrix} D + S \\ V \end{pmatrix} \),

where \( D \) stands for the block-diagonal \((m,m)\)-matrix of an isomorphism, \( S \) for the \((m,m)\)-matrix of a strictly singular operator, and \( V \) for some \((n-m,m)\)-matrix.

**Proof.** We use a type of Gaussian elimination method. We shall say that the matrix of an operator from \( Y' \) to \( X \) is of Gaussian form on the first \( k \) lines if it is of the form \( \begin{pmatrix} D + S \\ V \end{pmatrix} \), where \( D \) is the block-diagonal \((k,k)\)-matrix of an isomorphism, \( S \) (resp. \( S' \)) the \((k,k)\)-matrix (resp. \((k,m-k)\)-matrix) of a strictly singular operator, and \( V \) (resp. \( V' \)) some \((n-k,k)\)-matrix (resp. \((n-k,m-k)\)-matrix).

Assume \( A_{k-1} \) is a matrix of Gaussian form on the first \( k-1 \) lines and the matrix of an isomorphism. Clearly, it is enough to show that, for some further restriction of \( Y' \), we can find an automorphism \( B \) on \( Y' \) such that, up to a permutation on the \( X_i \)'s, \( A_k = A_{k-1}B \) is of Gaussian form on the first \( k \) lines (it is then clear that \( A_k \) is the matrix of an isomorphism).

Let \( N = \{1, \ldots, n\} \) and \( M = \{1, \ldots, m\} \). As \( A_{k-1} \) is an isomorphism, the restriction of \( A_{k-1} \) to \( Y' \) is not strictly singular, so there exists \( i \) such that the restriction \( p_{ij} \) of \( p_i \) to \( Y' \) is not strictly singular, and by our hypothesis on the form of \( A_{k-1} \), we have \( i \geq k \). Up to a permutation on the \( X_i \)’s, we may assume that \( i = k \). Passing to a smaller sum, we may assume that for every \( j \), the restriction \( p_{kj} \) of \( p_k \) to \( Y' \) is either strictly singular or an isomorphism. Let \( J \) be the set of \( j \) such that \( p_{kj} \) is an isomorphism. In particular, \( k \in J \). For every \( j \in J \), let \( H_j = \text{Im}(Y'_j) \); \( H_j \) and \( H_k \) are infinite-dimensional subspaces of the H.L. space \( X_k \), so by Corollary 1, \( H_j \) and \( H_k \) have \( 1d + S \)-isomorphic subspaces. Passing to a smaller sum, we may assume that \( H_j = \text{Ind}(H_j) \) (in particular, we choose \( s_k = 0 \)).

For \( j \in J \), let \( b_j \) be the operator \( p^{-1}_j \text{Ind}(s_j) p_{kj} \) from \( Y'_j \) to \( Y'_k \); let \( b_j = 0 \)). Let \( B \) be the automorphism on \( Y_m' \) with matrix

\[
\begin{pmatrix}
I_{d_{k-1}} & 0 & 0 \\
-b((k-1),k-1) & I_d & -b((k+1),m) \\
0 & 0 & I_{m-n}
\end{pmatrix}
\]

Then let \( A_k = A_{k-1}B \). As \( A_{k-1} \) is of the form

\[
\begin{pmatrix}
\text{Diag}(p_i)^{k-1} + S & S_1 & S_2 & S_3 \\
0 & V_1 & \text{Ind}(p_{ik}) & V_2 & \text{Ind}(p_{ik}) & V_3
\end{pmatrix}
\]

it follows that \( A_k \) is of the form

\[
\begin{pmatrix}
\text{Diag}(p_i)^{k-1} + S' & S_2 & S_3 \\
0 & V_1 & \text{Ind}(p_{ik}) & V_2 & \text{Ind}(p_{ik}) & V_3
\end{pmatrix}
\]

All the coefficients of the \( k \)th line except \( p_{ik} \) are now strictly singular. Indeed, let \( j \not= k \); then either \( j \not\in J \) and then \( (p_k - p_{kj})_{ij} \) is equal to \( p_{ij} \) strictly singular; or \( j \in J \) and then

\[
(p_k - p_{kj})_{ij} = p_{kj} - (\text{Ind}(s_j)p_{kj} = -s_j p_{kj}),
\]

so that it is also strictly singular.

So \( A_k \) is of Gaussian form on the first \( k \) lines, and is the matrix of an isomorphism. Repeating this procedure until \( k = m \), we finally get the result.

**Corollary 3.** Let \( X = \bigoplus_{i=1}^{n} X_i \) be a fundamental HD\(_n\) space. Let \( Y = \bigoplus_{i=1}^{m} Y_i \) be a fundamental HD\(_m\) subspace of \( X \). Then there exist a sum \( Y' \) smaller than \( Y \), a subset \( M' \) of \( \{1, \ldots, n\} \) of cardinality \( m \) and an isomorphism \( A \) on \( Y' \) such that \( AY' \) is smaller than \( X_{M'} \).
Proof. Let $Y', D, L$ be as in Lemma 5. Then $Z = DLX'$ is smaller than $X_{M'}$ for some $M'$ of cardinal $m$.

PROPOSITION 2. Let $X$ be a $HD_n$ space. Let $X$ and $X'$ be fundamental $HD_n$ subspaces of $X$. Then there exists a fundamental $HD_n$ subspace $Y$ of $X$ (resp. $Y'$ of $X'$) such that $Y$ and $Y'$ are isomorphic.

Proof. Write $X = \bigoplus_{i=1}^{n} X_i$, $X' = \bigoplus_{i=1}^{n} X'_i$. We first find a sum $Y$ smaller than $X$ that $Id + S$-embeds in $X'$. As $X'$ is fundamental in $X$, it is quasi-maximal: applying Corollary 1 for every $i$ in $\{1, \ldots, n\}$, we find a subspace $Y_i$ of $X_i$ that embeds into $X'$ by an $Id + S$-isomorphism. The sum $Y = \bigoplus_{i=1}^{n} Y_i$ is smaller than $X$ and embeds into $X'$ by an $Id + S$-operator $B$, which we may assume to be an isomorphism by passing to finite-codimensional subspaces. Let $Z = B(Y)$. As $Y \subset X$, we may apply Corollary 3: without loss of generality, taking a new restriction of the $Y_i$, and up to relabelling, we may assume that there exists an isomorphism $L$ such that for every $i$, $L(Z_i) \subset X_i$. The sum $Y' = L(Z) = LB(Y)$ is smaller than $X$ and isomorphic to $Y$.

DEFINITION. It follows from the filter structure on the set of fundamental subspaces of $X$ (Proposition 2), and from the fact that both $s(X)$ and $n(X)$ are preserved when taking a smaller sum or an isomorphism (Remark after Definition 3) that for $X$ a $HD_n$ space and $X$ a fundamental subspace of $X$, neither $s(X)$ nor $n(X)$ depends on the choice of the decomposition of $X$ as a direct sum, or even on the choice of $X$. We define $s(X)$ (resp. $n(X)$) to be this common value.

We are now able to study the space of operators on a fundamental complex $HD_n$ space, and then on a general complex $HD_n$ space.

5. Operators on a complex $HD_n$ space. We first recall a theorem from [F1].

THEOREM 1. Let $X$ be a complex $H.I.$ space and $Y \subset X$. Then every operator from $Y$ to $X$ is of the form $\lambda Y \times S$, where $\lambda$ is complex, $Y \times S$ is the canonical inclusion map from $Y$ to $X$, and $S$ is strictly singular.

Let $X = \bigoplus_{i=1}^{n} X_i$ be a fundamental complex $HD_n$ space. Recall that $\alpha_{ij}$ denotes $\alpha_{ij}p_i$, where $p_i$ is the projection on $X_i$, and $\alpha_{ij}$ is a given isomorphism from $X_i$ onto $X_j$. We denote by $\sum_{i,j}$ the sum $\sum_{C} \sum_{i,j \in C}$ where $C$ runs over the classes of indices for $\alpha$.

PROPOSITION 3. Let $X = \bigoplus_{i=1}^{n} X_i$ be a fundamental complex $HD_n$ space. Let $Y \subset X$. Then every operator from $Y$ to $X$ has the form $\sum_{i \leq j} \lambda_{ij}p_{ij} + S$, where $\lambda_{ij}$ is a constant and $S$ is strictly singular.

Proof. Let $X$ be the $HD_n$ space with $m \leq n$. Let $Y = \bigoplus_{i=1}^{m} Y_i$ be a fundamental $HD_m$ subspace of $Y$. According to Lemma 5, we may, numbering the $X_i$, assume that the matrix $I_{Y_i}^X$ of the injection of $Y_i$ into $X$ is of the form $(D \times S)$, where $D$ is the $(m, m)$-block-diagonal matrix of an isomorphism, $S$ is the $(m, m')$-matrix of a strictly singular operator, and $V$ some $(m, n)$-matrix. We may also assume (passing to a smaller sum) that for all $i$ in $\{1, \ldots, n\}$, $p_{ij}$ is an isomorphism. For all such $i$, let $H_i = p_i(Y_i)$. Then $p_{ij}$ is the inverse operator of $p_{ij}$ (defined on $H_i$).

Let now $T$ be any operator from $Y$ to $X$. For every $j \approx i$, $H_i$ is isomorphic by $\alpha_{ij}$ to a subspace of $X_j$, so by Theorem 1, every operator from $H_i$ to $X_j$ is of the form $\lambda_{ij} + S$; while for every $j \not\approx i$, $H_i$ and $X_j$ are totally incomparable, so every operator from $H_i$ to $X_j$ is strictly singular. As the operator $T_{ij}^{-1}$ is from $H_i$ to $X$, it follows that $T_{ij}^{-1}$ is of the form $\sum_{i \leq j} \lambda_{ij} \alpha_{ij} + S_i$, where $S_i$ is strictly singular. It follows that for every $i \leq m$, $T_{ij}$ is of the form $(\lambda_{ij} \alpha_{ij})^{-1} + S_i$. If we let $\lambda_{ij} = 0$ for $i \leq m + 1$, then

$$T_{ij} = \sum_{i \leq j} \lambda_{ij}p_{ij} + S.$$ 

So the restriction to $Y$ of the operator $S$ defined by $S = T - \sum_{i \leq j} \lambda_{ij}p_{ij}$ is strictly singular. By Lemma 2, $S$ is also strictly singular on $Y$.

PROPOSITION 4. Let $X$ be a complex $HD_n$ space. Let $Y \subset X$ and let $m \leq n$ be such that $Y$ is $HD_m$. Then $\dim(L(Y, X)/S(Y, X)) \leq n(Y)n(X) \leq mn$.

Proof. Let $Y = \bigoplus_{i=1}^{m} Y_i$ be a fundamental $HD_m$ subspace of $Y$. Let $X = \bigoplus_{i=1}^{n} X_i$ be a fundamental $HD_n$ subspace of $X$ containing $Y$. Without loss of generality, we may also assume that the injection from $Y$ to $X$ is of the form given by Lemma 5.

Let $T \in L(Y, X)$. As $X$ is quasi-maximal in $X$, for $1 \leq i \leq m$, we may apply Lemma 3 with $Z = Y_i$; we obtain $Y_i' \subset Y_i$, and $T_{ij}$ a strictly singular perturbation of $T_{ij}'$ with $\text{Im}T_{ij}' \subset X_i$. Let $Y' = \bigoplus_{i=1}^{m} Y_i'$. Passing to subspaces, we may assume that $Y'$ is fundamental. The unique operator $T'$ on $Y'$ such that for all $i$, $T'_{ij} = T_{ij}'$ takes its values in $X$ and is a strictly singular perturbation of $T_{ij}'$. The form of $T'$ is then the one given by Proposition 3, and $T_{ij}'$ is of the same form. As $Y'$ is quasi-maximal in $Y$, Lemma 2 implies that $T'$ is of the same form on the whole of $Y$. Now because of the form of $Y'$, the elements of any equivalence class in $\alpha(Y)$ embed into the elements of one and only one equivalence class in $\alpha(X)$. This defines a quotient map $e$ from $\alpha(Y)$ into $\alpha(X)$. It follows that

$$\dim(L(Y, X)/S(Y, X)) = \sum_{\alpha \in \alpha(Y)} |\alpha| \cdot |e(\alpha)|,$$
so by the Cauchy–Schwarz inequality,

$$\dim(L(Y, X)/S(Y, X)) \leq n(Y)n(X).$$

Now for $Z \subset X$ and $U \in L(Z, X)$, let $\tilde{U}$ be the class of $U$ modulo\nstrictly singular operators. We define a map $\delta : L(Y, X)/S(Y, X) \rightarrow L(Y, X)/S(Y, X)$ by $\delta(\tilde{U}) = T_{Y}Y$. It is clear that $\delta$ is a linear mapping and,\nby Lemma 2, it is an injection. It follows that

$$\dim(L(Y, X)/S(Y, X)) \leq n(Y)n(X) \leq mn.$$

**Corollary 4.** Let $X$ be a complex $HD_n$ space. Then

$$\dim(L(X)/S(X)) \leq n(X)^2 \leq n^2.$$

**Remarks.** These inequalities are sharp because, by Proposition 3, if $X$ is\nfor example a $HD_n$ space of the form $\bigoplus_{i=1}^n Y$, we have $\dim(L(X)/S(X)) = n(X)^2 = n^2$.

However, the inequalities can be strict. It is clear that $n(X)^2$ can be\nstrictly less than $n^2$, as soon as we know the existence of totally\nincomparable H.I. spaces. One can see examples of totally\nincomparable H.I. spaces in [F2], but the simplest examples are probably a\nclass of different versions of $X_{GM}$ with different values for the function $f$ (see [GM] for an\nexplicit definition of $X_{GM}$).

Also, $\dim(L(X)/S(X))$ is not necessarily equal to $n(X)^2$: take a H.I.\nspace $Y$, a subspace $Z$ of $Y$ of infinite codimension, and let $X$ be the\n$HD_2$ space $Y \oplus Z$; then $Z \oplus Y$ is fundamental in $X$, so $n(X)^2 = 4$, while\n$\dim(L(Y)/S(Y)) = 3$ because $L(Y, Z) = S(Y, Z)$.

It is an open problem whether each value between 1 and $n^2$ can actually\nbe obtained for $\dim(L(X)/S(X))$; we do not even have an example of a non-H.I. space for which this dimension is 1.

6. Spectral theory in $HD_n$ spaces

**Definitions.** Let $X$ be a Banach space, $T$ be an operator on $X$. Let\$S(T)$ denote the essential spectrum of $T$: by definition, it is the spectrum of the\nclass of $T$ in the Calkin algebra $L(X)/K(X)$. Let $\delta S(T)$ be the boundary of $S(T)$. We say that the operator $T$ is \textit{infinitely singular} if for every\nfinitely-codimensional subspace $X'$ of $X$, the restriction of $T$ to $X'$ is not an isomorphism. By Proposition 2.2.4 of [LT], this is equivalent to saying that for every $\varepsilon > 0$, there exists a subspace $Y$ of $X$ such that $\|T|_Y\| \leq \varepsilon$. A scalar $\lambda$ is \textit{infinitely singular} for $T$ if $T - \lambda I$ is infinitely singular. Let $I(T)$ be the set of infinitely singular values for $T$.

We also recall that $T$ is Fredholm if its image is closed, and its kernel and cokernel are finite-dimensional. Fredholm operators are exactly the operators that are invertible modulo compact (resp. strictly singular) operators. We

say that $T$ is semi-Fredholm if its image is closed, and its kernel or cokernel is finite-dimensional. The \textit{generalized Fredholm index}, defined by $i(T) = \dim(\text{Ker}T) - \dim(\text{Coker}T)$, is defined and continuous over the set of semi-Fredholm operators (see [LT] about Fredholm operators).

We first prove a well-known lemma in spectral theory.

**Lemma 6.** Let $X$ be a Banach space, and $T$ be an operator on $X$. Then $\delta S(T) \subset I(T)$.

**Proof.** By definition, $S(T)$ is the set of $\lambda$ such that $T - \lambda I$ is not invertible and $\delta S(T)$ is the set of $\lambda$ such that $T - \lambda I$ is not Fredholm. Now let $\lambda$ be in $\delta S(T)$. As $S(T)$ is closed, $T - \lambda I$ is not Fredholm. Neither is it semi-Fredholm of infinite index, for otherwise, by continuity, $T - \lambda I$ would be semi-Fredholm of infinite index for $\lambda'$ in a neighborhood of $\lambda$, contradicting the fact that $\lambda$ is in the boundary of $S(T)$. It follows that either $T - \lambda I = \infty$, or $(T - \lambda I)(X)$ is not closed; in both cases, $T - \lambda I$ is infinitely singular.

**Proposition 5.** Let $X$ be a complex $HD_n$ space, and $T$ be an operator on $X$. Then the cardinality of $S(T)$ satisfies $|S(T)| \leq n$.

**Proof.** It suffices to prove that $|I(T)| \leq n$; indeed, it then follows from\nLemma 6 that $|\delta S(T)| \leq n$, so that $S(T)$ is a set of at most $n$ isolated points.

For every $\lambda$ in $I(T)$ and every $\varepsilon > 0$, there exists a space $X_{\lambda}(\varepsilon)$ on which $T - \lambda I$ is of norm at most $\varepsilon$. For every $\lambda$, using a normalized basic sequence $(y_n)_{n \in \mathbb{N}}$ of vectors such that $\|T(y_n) - \lambda y_n\| \leq 2^{-n}$, we may assume that the map $\varepsilon \mapsto X_{\lambda}(\varepsilon)$ is increasing. We now prove that if $N$ is a finite subset of $I(T)$, then for some $\varepsilon > 0$, the spaces $(X_{\lambda}(\varepsilon))_{\lambda \in N}$ form a direct sum; then from the fact that $X$ is $HD_n$ it follows clearly that $|\delta S(T)| \leq n$.

Assume that the property is false, and let $N$ be a finite subset of $I(T)$ of minimum cardinality among those contradicting the property. It is clear that $|N| \geq 2$. Let $c = \min_{\lambda \neq \lambda'} \|y_\lambda - y_{\lambda'}\|$ and $C = \max_{\lambda \in N} |\lambda|$. Let $\varepsilon > 0$. There are vectors $y_\lambda$ with $y_\lambda \in X_{\lambda}(\varepsilon)$, $\max_{\lambda \in N} \|y_\lambda\| = 1$ and $\|\sum_{\lambda \in N} y_\lambda\| \leq \varepsilon$. Applying $T$ to this inequality, we get

$$\|\sum_{\lambda \in N} \lambda y_\lambda\| \leq (\|T\| + |N|)\varepsilon.$$ 

Let $\lambda_2$ in $N$ be such that $\max_{\lambda \neq \lambda_2} \|y_\lambda\| = 1$ (such a number exists because $|N| \geq 2$). We have

$$\|\sum_{\lambda \neq \lambda_2} (\lambda - \lambda_2) y_\lambda\| \leq (|\lambda_2| + \|T\| + |N|)\varepsilon,$$

while

$$\max_{\lambda \neq \lambda_2} \|y_\lambda - y_{\lambda_2}\| \geq c.$$
Since $\varepsilon \to Y_\varepsilon(\varepsilon)$ is increasing for every $\lambda$, and $\lambda_\varepsilon$ takes a finite number of values, we may assume that $\lambda_\varepsilon$ is constant equal to some $\lambda_0$, so that we get vectors $y_\lambda' \in Y_\varepsilon(\varepsilon)$, $\|\sum_{\lambda \neq \lambda_0} y_\lambda'\| \leq (C + |T| + |N|)\varepsilon$, and $\max_{\lambda \neq \lambda_0} \|y_\lambda'\| \geq \varepsilon$.

Again because $\varepsilon \to Y_\lambda(\varepsilon)$ is increasing, the sum of the $(Y_\lambda(\varepsilon))$ for $\lambda \neq \lambda_0$ is not direct, and this holds for any $\varepsilon > 0$, a contradiction with the minimality of $N$.

Remark. Notice that the inequality in Proposition 5 is sharp; for example, it is an equality in a fundamental HD$_0$ space, for an operator of the form $\sum_{i=1}^n \lambda_i p_i$, with $\lambda_i \neq \lambda_j$ if $i \neq j$.

Lemma 7. Let $X$ be a finitely hereditarily decomposable space. Then every semi-Fredholm operator on $X$ is Fredholm with index 0.

Proof. Let $T$ be a semi-Fredholm operator on $X$. First assume that $X$ is complex. If $T$ is semi-Fredholm with infinite index, then so is $T - \lambda I$ for $\lambda$ small enough, a contradiction with Lemma 6. So $T$ is Fredholm. As $S(T)$ is finite, $C_\lambda S(T)$ is connected, so $\text{ind}(T - \lambda I)$ is constantly equal to $\text{ind}(T)$ on $C_\lambda S(T)$. Furthermore, when $\lambda$ tends to infinity, $T - \lambda I = -\lambda (I - T/\lambda)$ is close to $-\lambda I$, so it is Fredholm with index 0, hence $\text{ind}(T) = 0$.

Assume now that $X$ is real and consider its complexification $X_C$ (that is, $X_C = X \times X$ with the involution $(x, y) = (-y, x)$). Then $X_C$ is a real HD$_{2n}$ space, and as every complex subspace of $X_C$ is also a real subspace, $X_C$ is a complex HD$_m$ space for some $m \leq 2n$. Let $i_T$ be the generalized Fredholm index of $T$. It is easy to check that $T \oplus T$, the complexification of $T$ on $X_C$, is also semi-Fredholm with index $i_T$ (the dimensions in the definition of the index of $T \oplus T$ are over $\mathbb{C}$). By the first paragraph of this proof, and as $X_C$ is finitely hereditarily decomposable, $i_T$ is equal to 0.

Corollary 5. Let $X$ be a finitely hereditarily decomposable space. Then $X$ is not isomorphic to any proper subspace.

Proof. Every isomorphism from $X$ into $X$ is semi-Fredholm; but by Lemma 7, it is Fredholm with index 0, so it must be onto isomorphism.

7. Operators on a real H.I. space.

Theorem 2. Let $X$ be a real H.I. space. Then for all $Y \subset X$,

$$\dim(L(Y, X)/S(Y, X)) \leq 4.$$ 

Furthermore, $L(X)/S(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Proof. The first part of the proof uses a result of [F1]. It was shown in that article that all the spaces $E_Y = L(Y, X)/S(Y, X)$ embed in a limit space $E$ which is a Banach algebra and a division ring. By Gelfand's theorem (whose proof can be found in [R]), the space $E$ is of dimension at most 4 (and must be isomorphic either to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). It follows that $\dim E_Y \leq 4$.

The statement relative to the case $Y = X$ can be proved directly as follows. Let $T$ be a non-strictly singular operator on $X$. Then by Lemma 2, no restriction of $T$ to a subspace of $X$ is strictly singular; by Proposition 2.2 of [LT], the restriction of $T$ to some finite-codimensional subspace of $X$ is an isomorphic isomorphism, and so $T$ is semi-Fredholm. Finally, by Lemma 7, $T$ is Fredholm. This means that every operator on $X$ is either strictly singular, or Fredholm. As Fredholm operators are invertible modulo strictly singular operators, this is equivalent to saying that $L(X)/S(X)$ is a division ring, and so by Gelfand’s theorem it is either real, complex or quaternionic.

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References


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