Suppose that $m \geq n$ and $\alpha \leq n/(n+1)$. $\mathfrak{A}$ is generated by the m generators $z_j(\theta) = \exp(i\theta_j)$, $j = 1, \ldots, m$, and, by considering functions which are constant on the other θ_k, each n-element subset of these generators lies in a subalgebra isomorphic with $\mathrm{lip}_\alpha \mathbb{T}^n$, which is n-dimensionally weakly amenable. Thus if T is an alternating n-derivation with values in a symmetric Banach $\mathfrak{A}$-module, by restricting to the subalgebra we see that T is zero for any n-generators, and so by Corollary 2.6, $T = 0$.

Suppose that $m \geq n$ and $\alpha > n/(n+1)$. By restricting to $\{\theta : \theta \in \mathbb{T}^m,\ \theta_{n+1} = \ldots = \theta_m = 0\}$ we see that $\mathrm{lip}_\alpha \mathbb{T}^n$ is a quotient of $\mathrm{lip}_\alpha \mathbb{T}^m$ and so by Theorem 3.1(i) the latter is not n-dimensionally weakly amenable because the former is not.

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Hereditarily finitely decomposable Banach spaces

by

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Abstract. A Banach space is said to be HD_n if the maximal number of subspaces of X forming a direct sum is finite and equal to n. We study some properties of HD_n spaces, and their links with hereditarily indecomposable spaces; in particular, we show that if X is complex HD_n, then $\dim(\mathcal{L}(X)/\mathcal{S}(X)) \leq n^2$, where $\mathcal{S}(X)$ denotes the space of strictly singular operators on X. It follows that if X is a real hereditarily indecomposable space, then $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the quaternionic division ring.

1. Introduction

General introduction. The problems discussed in this article came as natural questions after the article of W. T. Gowers and B. Maurey ([GM]) solving the unconditional basic sequence problem. In a Banach space X, a sequence $(e_n)_{n\in\mathbb{N}}$ is said to be a *basis* if every vector x in X can be written uniquely in the form $x = \sum_{i=0}^{\infty} \lambda_i e_i$, where the λ_i's are scalars. It is an *unconditional basis* if there is a constant C such that the inequalities

$$\Big\|\sum_{i\in E} \lambda_i e_i\Big\| \leq C\Big\|\sum_{i=0}^{\infty} \lambda_i e_i\Big\|$$

hold for all subsets E of $\mathbb{N}$ and all coefficients λ_i. A sequence is a *basic sequence* (resp. an *unconditional basic sequence*) if it is a basis (resp. an unconditional basis) of its closed linear span. Details about these notions can be found in [LT].

A lot of classical spaces have an unconditional basis (spaces l_p, for $p \geq 1$, have one) but for example L_1 does not have one; an example of a Banach space without a basis is even harder to find, but was given by P. Enflo in 1973 ([E]). On the other hand, it is a classical result that every Banach space contains a basic sequence; but the question "Does every Banach space contain an unconditional basic sequence?" has remained unsolved for many

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years. In 1991, Gowers and Maurey finally gave a counterexample, that we shall call X_{GM}, to this question.

In fact, the space X_{GM} has a stronger property, namely, it is *hereditarily indecomposable* (or *H.I.*), which means that no subspace of X_{GM} is decomposable (i.e. the topological direct sum of two infinite-dimensional subspaces). A Banach space X with the H.I. property is easily shown not to contain unconditional basic sequences, but also not to be isomorphic to its square, and, with a little more work, not to be isomorphic to its proper subspaces. But the H.I. property not only provides counterexamples to many questions about general Banach spaces; it also appears in the following important dichotomy theorem, proved by Gowers. According to Gowers' theorem, every Banach space contains a subspace that is either hereditarily indecomposable or spanned by an unconditional basis ([G1], [G2]). Because of this theorem, it is particularly important to know about general properties of hereditarily indecomposable spaces.

Most known results on H.I. spaces are about operators on complex H.I. spaces. In particular, Gowers and Maurey showed in [GM] that if X is H.I. complex, then every operator on X is a strictly singular perturbation of a multiple of the identity (we recall that an operator S is strictly singular if no restriction of S is an into isomorphism). The author generalized their result showing that for every subspace Y of X, every operator from Y to X is a strictly singular perturbation of a multiple of the canonical inclusion map from Y to X ([F1], and Theorem 1 below). The fact that the space of operators on a H.I. space is, in a sense, small is a crucial point for proving properties of these spaces.

In this article, we study the property HD_n: a space is HD_n if the supremum of the m's such that X contains a direct sum of m infinite-dimensional subspaces is equal to n. For $n \geq 2$, these are spaces without an unconditional basic sequence that are not H.I., so they give a good illustration of Gowers' dichotomy theorem. On the other hand, we show that HD_n spaces are spaces with "few" operators, which provides them with properties similar to the properties of H.I. spaces. Finally, our results allow us to solve the real case: in a real H.I. space, the quotient of the space of operators by the space of strictly singular operators is a division ring that is either real, complex or quaternionic.

Notation and definitions. In the following, X, Y, Z, W are complex or real Banach spaces. The real field is denoted by $\mathbb{R}$, the complex field by $\mathbb{C}$, and the quaternionic division ring by $\mathbb{H}$. By a *space* (resp. *subspace*) we shall always mean an infinite-dimensional closed space (resp. subspace). The set of bounded operators from Z to X is denoted by $\mathcal{L}(Z,X)$. Recall that an operator S from Z to X is said to be *strictly singular* if for every subspace W of Z, the restriction $S_{|W}$ of S to W is not an into isomorphism. This is equivalent to saying that for every $\varepsilon > 0$ and every subspace W of Z, there exists a subspace W' of W such that $\|S_{|W'}\| \leq \varepsilon$. The set of strictly singular operators from Z to X is a subspace of $\mathcal{L}(Z,X)$ that we shall denote by $\mathcal{S}(Z,X)$, and we recall that when TU is defined, it is a strictly singular operator whenever T or U is strictly singular.

Let $Z \subset X$, and $A \in \mathcal{L}(Z,X)$. By Proposition 2.c.10 of [LT], if A is of the form $\mathrm{Id}+S$, where S is strictly singular, then it is an isomorphism on some finite-codimensional subspace of Z. If such an operator A is an isomorphism on Z, we say that A is an $\mathrm{Id}+S$-*isomorphism*. Then the inverse isomorphism A^{-1} of A defined on $A(Z)$ satisfies $A^{-1} - \mathrm{Id} = (\mathrm{Id} - A)A^{-1}$, so it is an $\mathrm{Id}+S$-isomorphism on $A(Z)$. Two subspaces Y and Z of a Banach space X are said to be $\mathrm{Id}+S$-*isomorphic* if there is an $\mathrm{Id}+S$-isomorphism from Y onto Z. Replacing "strictly singular" by "compact" in the above, we also define $\mathrm{Id}+K$-*isomorphisms*. We recall that every compact operator is strictly singular.

Two spaces are said to be *totally incomparable* if no subspace of the first is isomorphic to a subspace of the second.

Let R be a finite set. Then $|R|$ stands for its cardinality. Given subspaces $(X_i)_{i\in R}$ of X, their sum is *direct* if for every i, the projection p_i from $\sum_{i\in R} X_i$ to X_i is well defined and continuous; the sum is then denoted by $\bigoplus_{i\in R} X_i$. For any $Q \subset R$, we will denote by X_Q the sum $\bigoplus_{i\in Q} X_i$. For any operator T on X_R, we will denote by $T_{|Q}$ the restriction of T to X_Q (if $Q = \{i\}$, we write $T_{|i}$ for $T_{|\{i\}}$). Given another direct sum X'_R, we say that X'_R is *smaller* than X_R if there exists a permutation σ on R such that for all $i \in R$, $X'_i \subset X_{\sigma(i)}$ (notice that when two sums are comparable, they are necessarily sums over the same finite set).

Recall that a space that can be written as a direct sum of at least two (infinite-dimensional) subspaces is said to be *decomposable* and that a space X is said to be *hereditarily indecomposable* (*or H.I.*) if it has no decomposable subspace.

More generally, we say that a space is HD_n if the maximal number of subspaces of X in a direct sum is finite and equal to n (Definition 2) and we will prove (Corollary 2) that a simple example of a HD_n space is the direct sum of n H.I. spaces. In Section 2, we define and study the convenient notion of *quasi-maximality* (Definition 1). In Section 3, we show that the sum of a HD_m and of a HD_n space is HD_{m+n} (Proposition 1). This leads to considering a certain type of HD_n spaces called *fundamental* (Definition 3), which we study in Section 4. In Section 5, we show that consequently, if X is complex HD_n, then the dimension of $\mathcal{L}(X)/\mathcal{S}(X)$ is smaller than n^2 (Proposition 4). In Section 6, we study spectral properties of (real and complex)

HD_n spaces. In Section 7, we show our main theorem: if X is a real H.I. space, then $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (Theorem 2).

2. Quasi-maximality. We first define an important technical notion.

DEFINITION 1. Let X be a Banach space. Let $\mathbf{X}$ be a subspace of X. The space $\mathbf{X}$ is said to be *quasi-maximal* in X if for every subspace Z of X, the sum $\mathbf{X}+Z$ is not direct.

LEMMA 1. *Let X be a Banach space. Let Y, Z be subspaces of X. If for every subspace W of Z, the sum $Y+W$ is not direct, then Y and Z have* $\mathrm{Id}+K$*-isomorphic subspaces.*

Proof. Let Y, Z satisfy the hypothesis. Passing to a subspace, we may assume that Z has a basis. For every $\varepsilon>0$, for any subspace W of Z the sum $Y+W$ is not direct, so there exist two unit vectors $y\in Y$ and $w\in W$ such that $\|y-w\|\le\varepsilon$. Using repeatedly this remark, one can build a normalized block basic sequence $(z_i)_{i\in\mathbb{N}}$ in Z and a sequence $(y_i)_{i\in\mathbb{N}}$ in Y such that $\sum_{i\in\mathbb{N}}\|y_i-z_i\|<\infty$. Let Z' be the subspace generated by $(z_i)_{i\in\mathbb{N}}$. The operator A from Z' into Y defined by $A(z_i)=y_i$ for all i is of the form $\mathrm{Id}+K$, where K is compact as a uniform limit of finite rank operators. Passing to a finite-codimensional subspace, we may assume that it is an isomorphism. Then Z' and $A(Z')$ are $\mathrm{Id}+K$-isomorphic subspaces of Z and Y.

COROLLARY 1. *Let X be a Banach space, and $\mathbf{X}$ a subspace of X. Then $\mathbf{X}$ is quasi-maximal if and only if for every subspace Z of X, Z and $\mathbf{X}$ have* $\mathrm{Id}+S$*-isomorphic subspaces. The same statement holds if we replace* $\mathrm{Id}+S$ *by* $\mathrm{Id}+K$.

Proof. This follows from Lemma 1, from the definition of a strictly singular operator, and from the fact that every compact operator is strictly singular.

We now prove two lemmas about quasi-maximal subspaces $\mathbf{X}$ of X. They show that, as long as we study operators on X only up to strictly singular operators, it is enough to study operators on some quasi-maximal subspace $\mathbf{X}$ of X.

LEMMA 2. *Let X be a Banach space and let $\mathbf{X}$ be a quasi-maximal subspace of X. Let Z be any Banach space, and let $T\in\mathcal{L}(X,Z)$. Then T is strictly singular if and only if the restriction $T_{|\mathbf{X}}$ of T to $\mathbf{X}$ is strictly singular.*

Proof. The direct implication is clear. Now suppose that $T_{|\mathbf{X}}$ is strictly singular. Let Y be a subspace of X. According to Corollary 1, there is a subspace Y' of Y and an embedding $A=\mathrm{Id}_{|Y'}+S$ of Y' into $\mathbf{X}$. Then we can write $T_{|Y'}=T(\mathrm{Id}_{|Y'}+S-S)=T_{|\mathbf{X}}A-TS$. As $T_{|\mathbf{X}}$ and S are strictly singular, so is $T_{|Y'}$; in particular, for every $\varepsilon>0$, there is a unit vector y in Y' such that $\|Ty\|\le\varepsilon$; so T is strictly singular.

LEMMA 3. *Let X be a Banach space and let $\mathbf{X}$ be a quasi-maximal subspace of X. Let Z be any Banach space, and let $T\in\mathcal{L}(Z,X)$. Then there exist $Z'\subset Z$ and $T'\in\mathcal{L}(Z',X)$ with $\mathrm{Im}\,T'\subset\mathbf{X}$ such that $T_{|Z'}-T'$ is strictly singular* (*or compact*).

Proof. It is easy to check that for every strictly singular operator S on Z, there exists a subspace W such that $S_{|W}$ is compact; so it is enough to prove the strictly singular part of the assertion. Now let $T\in\mathcal{L}(Z,X)$. If T is strictly singular then we can choose $Z'=Z$ and $T'=0$. If not, passing to a subspace, we may assume that T is an isomorphism into X. Applying Corollary 1 to TZ and $\mathbf{X}$, we find a subspace Z' of Z such that TZ' embeds into $\mathbf{X}$ by an isomorphism $A=\mathrm{Id}+S$. Let $T'=AT_{|Z'}$. We have $\mathrm{Im}\,T'\subset\mathbf{X}$; furthermore, $T_{|Z'}-T'=ST_{|Z'}$ is strictly singular.

3. HD_n spaces

DEFINITION 2. A space X is *hereditarily finitely decomposable* if the maximal number of (infinite-dimensional) subspaces of X forming a direct sum in X is finite. For $n\ge1$, X is HD_n if this number is equal to n.

Some easy remarks. A space is HD_1 if and only if it is H.I. More generally, we will see that the direct sum of n H.I. spaces is HD_n (Corollary 2).

A space X is HD_n if and only if it contains direct sums of n subspaces and every such sum is quasi-maximal. In particular, X is H.I. if and only if every subspace of X is quasi-maximal.

For every HD_n space X, every subspace Y of X is HD_m for some $m\le n$.

PROPOSITION 1. *Let X be HD_m and Y be HD_n. Then $X\oplus Y$ is HD_{m+n}.*

Proof. It is based on a type of Gaussian elimination method. Let p be the projection from $X\oplus Y$ onto X, and q be the projection from $X\oplus Y$ onto Y. By definition, there exist m subspaces of X forming a direct sum in X, and n subspaces of Y forming a direct sum in Y. The sum of all these subspaces is a direct sum of $m+n$ subspaces of $X\oplus Y$. Now let $K=\{1,\ldots,k\}$ and let Z_K form a direct sum in $X\oplus Y$. It is enough to prove that $k\le m+n$.

Passing to subspaces of the Z_i's, we may assume that each $p_{|i}$ and each $q_{|i}$ is either strictly singular or an isomorphism. Let I_X be the set of i such that $p_{|i}$ is an isomorphism. For i in I_X, let $X_i=p(Z_i)$.

Let $\mathcal{R}$ be the set of subsets R of I_X such that p is an isomorphism on Z_R or on some smaller sum (in the sense of the definition given in the introduction). Let $r = \max\{|R| : R \in \mathcal{R}\}$. As X is HD_m, we have $r \le m$. We may assume that r is attained for $R = [1, r]$ and, passing to a smaller sum for the Z_i's, that $p_{|R}$ is an isomorphism. Let $p_{|R}^{-1}$ be the inverse isomorphism of $p_{|R}$. Let i be in $I_X \setminus R$. By the definition of r, p is not an isomorphism on any sum smaller than $Z_R \oplus Z_i$. This implies, in particular, that no subspace of X_i forms a direct sum with X_R: indeed, assume $X_i' \subset X_i$ and X_R form a direct sum, and let $Z_i' = p^{-1}(X_i')$; then on $Z_R \oplus Z_i'$, p is an isomorphism. By Lemma 1, it follows that some subspace of X_i is $\mathrm{Id} + S$-isomorphic to a subspace of X_R. So, passing to a subspace of Z_i, we may assume that X_i embeds into X_R, by an operator of the form $\mathrm{Id}_{|i} + S_i$. This yields an embedding $b_i = p_{|R}^{-1}(\mathrm{Id}_{|i} + S_i)p_{|i}$ of Z_i into Z_R. For i not in R and not in I_X, we let $b_i = 0$. Let b be the operator from $Z_{K\setminus R}$ into Z_R whose matrix is the $(r, k-r)$-matrix $[b_{r+1}, \ldots, b_k]$. We now consider another decomposition $Z_K' = Z_1' \oplus \ldots \oplus Z_k'$ of Z_K, defined by $Z_i' = Z_i$ for $1 \le i \le r$ and $Z_i' = \{(-b_i z_i, z_i) : z_i \in Z_i\}$ for $r+1 \le i \le k$.

We show that the restriction of p to $Z_{K\setminus R}'$ is strictly singular. Let $i \in K \setminus R$. Then either $i \in I_X$ and then the restriction of p to Z_i' is defined by

$$p(-b_i z_i, z_i) = -(\mathrm{Id} + S_i)pz_i + pz_i = -S_i p z_i,$$

so it is strictly singular; or $i \notin I_X$ and then the restriction of p to Z_i' is defined by

$$p(-b_i z_i, z_i) = p_{|i} z_i$$

(where $p_{|i}$ is strictly singular), so it is strictly singular.

Thus the restriction of p to $Z_{K\setminus R}'$ is strictly singular, and as $q = \mathrm{Id} - p$, it follows that the restriction of q to some finite-codimensional subspace of $Z_{K\setminus R}'$ is an isomorphism: up to a finite-dimensional perturbation, $Z_{K\setminus R}'$ embeds as a direct sum into Y. As Y is HD_n, it follows that $k - r \le n$, so $k \le m + n$.

Corollary 2. *Let $n \in \mathbb{N}$. For $i = 1, \ldots, n$, let X_i be a H.I. space. Then the space $\bigoplus_{i=1}^n X_i$ is HD_n.*

Remarks. A hereditarily finitely decomposable space does not contain any unconditional basic sequence. Indeed, if X contains an unconditional basic sequence $(e_k)_{k\in\mathbb{N}}$, then for every $n \ge 1$, X contains the direct sum $\bigoplus_{j=1}^n E_j^n$, where E_j^n is the space generated by $(e_{ni+j})_{i\in\mathbb{N}}$.

Corollary 2 provides us with the first examples of HD_n spaces for $n \ge 2$; by the previous remark, notice that these are spaces without an unconditional basic sequence that are not H.I. However, direct sums of n H.I. spaces are not the only examples of HD_n spaces: a HD_n space not of this form was built in [F2], for a different purpose.

Finally, notice a property that was already known for H.I. spaces: if X is finitely hereditarily indecomposable, then for every $k \ne l$, X^k and X^l are not isomorphic. Indeed, by Proposition 1, if X is HD_n, then X^k is HD_{nk}, while X^l is HD_{nl}.

4. Fundamental HD_n spaces. In this section, we study a particular type of HD_n space, called fundamental: fundamental HD_n spaces are direct sums of n H.I. spaces, with some "normalizing" condition. We will see that given a HD_n space X, it is enough to know about the operators on any of its fundamental HD_n subspaces to know about the operators on X.

Definition 3. A Banach space $\mathbf{X}$ is a *fundamental* HD_n space if it is the direct sum of n H.I. spaces any two of which are either isomorphic or totally incomparable.

Let $\mathbf{X}$ be a fundamental HD_n space of the form $\mathbf{X} = \bigoplus_{i=1}^n X_i$. For i, j in $\{1, \ldots, n\}$, we say that $i \simeq j$ if X_i and X_j are isomorphic. For every i, we denote by p_i the projection from $\mathbf{X}$ onto X_i. For every $i \simeq j$, we denote by α_{ij} a fixed isomorphism from X_i onto X_j, by p_{ij} the map $\alpha_{ij} p_i$ from $\mathbf{X}$ onto X_j. We denote by $\alpha(\mathbf{X})$ the set of equivalence classes of indices for the relation $\simeq$. We also want to define a set $s(\mathbf{X})$ that does not depend on the choice of the labelling: ordering the classes by decreasing cardinality, we let $s(\mathbf{X})$ be the finite sequence of these cardinalities. We also define a positive number $n(\mathbf{X})$ by $n(\mathbf{X})^2 = \sum_{s\in s(\mathbf{X})} |s|^2$. Writing these numbers as functions of $\mathbf{X}$ is an abuse of notation, because a priori, they depend on the choice of the decomposition of $\mathbf{X}$ as $\bigoplus_{i=1}^n X_i$; we will allow this notation because we will see (Proposition 2) that $s(\mathbf{X})$ and $n(\mathbf{X})$ are in fact uniquely determined by $\mathbf{X}$. Until then, we think of s and n as functions of $\mathbf{X}$ under a particular decomposition.

We will see that $s(\mathbf{X})$ and $n(\mathbf{X})$ characterize in a way the space of operators on $\mathbf{X}$. Notice also that $n(\mathbf{X}) \le n$.

Remark. Let $\mathbf{X}$ be fundamental. A sum smaller than $\mathbf{X}$ can always be chosen fundamental, by passing to further subspaces (if two spaces are not totally incomparable, then, passing to subspaces, we may assume that they are isomorphic; repeating this procedure, we end up with a fundamental sum). We will always choose such smaller sums, without necessarily saying it.

It is clear that both $s(\mathbf{X})$ and $n(\mathbf{X})$ are preserved when we apply an isomorphism to $\mathbf{X}$. Notice also that neither $s(\mathbf{X})$ nor $n(\mathbf{X})$ changes when we pass to a smaller sum; again this is an abuse of notation, but we can allow it here because comparing two fundamental sums is by definition comparing two given decompositions of the sums (see the definition in the introduction). The proof is the following. Let $\mathbf{X}$ be a fundamental sum, and $\mathbf{Y}$ a smaller sum, and to simplify the notation, assume that for all i, $Y_i \subset X_i$. Then if X_i and X_j are totally incomparable, then so are the subspaces Y_i

and Y_j; if X_i and X_j are isomorphic, then by the H.I. property (apply Corollary 1, for example), Y_i and Y_j are not totally incomparable, so they must be isomorphic.

LEMMA 4. *Every* HD_n *space contains a fundamental* HD_n *space.*

Proof. Let X be HD_n. Then X contains a direct sum of n subspaces. Each of these subspaces must be H.I., otherwise X would contain a direct sum of $n+1$ subspaces. By the previous remark, passing to appropriate subspaces, we may assume that this sum is a fundamental HD_n space.

Remark. If X is HD_n, then every fundamental HD_n subspace $\mathbf{X}$ of X is quasi-maximal in X. This is an important remark with respect to Lemmas 2 and 3.

We now study the relative position of fundamental subspaces of a HD_n space X. We will see that the fundamental HD_n subspaces of X form a filter for the inclusion up to isomorphism.

NOTATION. Let $\mathbf{X} = \bigoplus_{i=1}^n X_i$ be a fundamental HD_n space and let $\mathbf{Y} = \bigoplus_{i=1}^m Y_i$ be a fundamental HD_m subspace of $\mathbf{X}$. We denote by $I_{\mathbf{Y},\mathbf{X}}$ the injection of $\mathbf{Y}$ into $\mathbf{X}$, written as an (n,m)-matrix with coefficients in $\mathcal{L}(Y_j, X_i)$, for $1 \le j \le m$ and $1 \le i \le n$. For $1 \le i \le n$, we recall that p_i denotes the projection on X_i, and according to the notation defined in the introduction, $p_{i|j}$ is the coefficient on the ith line and jth column of the matrix of $I_{\mathbf{Y},\mathbf{X}}$.

LEMMA 5. *Let* $\mathbf{X} = \bigoplus_{i=1}^n X_i$ *be a fundamental* HD_n *space. Let* $\mathbf{Y} = \bigoplus_{i=1}^m Y_i$ *be a fundamental* HD_m *subspace of* $\mathbf{X}$. *Then there exist a sum* $\mathbf{Y}'$ *smaller than* $\mathbf{Y}$, *an isomorphism* L *from* $\mathbf{Y}'$ *into* $\mathbf{Y}$, *and a permutation matrix* E *on* $\mathbf{X}$ *such that*

$$EI_{\mathbf{Y},\mathbf{X}}L = \begin{pmatrix} D+S \\ V \end{pmatrix},$$

where D *stands for the block-diagonal* (m,m)*-matrix of an isomorphism,* S *for the* (m,m)*-matrix of a strictly singular operator, and* V *for some* $(n-m,m)$*-matrix.*

Proof. We use a type of Gaussian elimination method. We shall say that the matrix of an operator from Y' to X is *of Gaussian form on the first* k *lines* if it is of the form $\left(\begin{smallmatrix} D+S & S' \\ V & V' \end{smallmatrix}\right)$, where D is the block-diagonal (k,k)-matrix of an isomorphism, S (resp. S') the (k,k)-matrix (resp. $(k,m-k)$-matrix) of a strictly singular operator, and V (resp. V') some $(n-k,k)$-matrix (resp. $(n-k,m-k)$-matrix).

Assume A_{k-1} is a matrix of Gaussian form on the first $k-1$ lines and the matrix of an isomorphism. Clearly, it is enough to show that, for some further restriction of $\mathbf{Y}'$, we can find an automorphism B on $\mathbf{Y}'$ such that, up to a permutation on the X_i's, $A_k = A_{k-1}B$ is of Gaussian form on the first k lines (it is then clear that A_k is the matrix of an isomorphism).

Let $N = \{1,\ldots,n\}$, $M = \{1,\ldots,m\}$. As A_{k-1} is an isomorphism, the restriction of A_{k-1} to Y'_k is not strictly singular, so there exists i such that the restriction $p_{i|k}$ of p_i to Y'_k is not strictly singular, and by our hypothesis on the form of A_{k-1}, we have $i \ge k$. Up to a permutation on the X_i, we may assume that $i = k$. Passing to a smaller sum, we may assume that for every j, the restriction $p_{k|j}$ of p_k to Y'_j is either strictly singular or an isomorphism. Let J be the set of j such that $p_{k|j}$ is an isomorphism. In particular, $k \in J$. For every j in J, let $H_j = p_k(Y'_j)$; H_j and H_k are infinite-dimensional subspaces of the H.I. space X_k, so by Corollary 1, H_j and H_k have $\mathrm{Id}+S$-isomorphic subspaces. Passing to a smaller sum, we may assume that $H_k = (\mathrm{Id}+s_j)(H_j)$ (in particular, we choose $s_k = 0$).

For $j \in J$, let b_j be the operator $p_{k|k}^{-1}(\mathrm{Id}_{|j} + s_j)p_{k|j}$ from Y'_j to Y'_k; for $j \notin J$, let $b_j = 0$. Let b be the operator from Y'_M to Y'_k with matrix $[b_1,\ldots,b_m]$. Let B be the automorphism on Y'_M with matrix

$$\begin{pmatrix} \mathrm{Id}_{k-1} & 0 & 0 \\ -b_{|[1,k-1]} & \mathrm{Id} & -b_{|[k+1,m]} \\ 0 & 0 & \mathrm{Id}_{m-k} \end{pmatrix}.$$

Then let $A_k = A_{k-1}B$. As A_{k-1} is of the form

$$\begin{pmatrix} \mathrm{Diag}(p_i)_1^{k-1} + S_1 & S_2 & S_3 \\ p_{k|[1,k-1]} & p_{k|k} & p_{k|[k+1,m]} \\ V_1 & V_2 & V_3 \end{pmatrix},$$

it follows that A_k is of the form

$$\begin{pmatrix} \mathrm{Diag}(p_i)_1^{k-1} + S'_1 & S_2 & S'_3 \\ (p_k - p_k b)_{|[1,k-1]} & p_{k|k} & (p_k - p_k b)_{k|[k+1,m]} \\ V'_1 & V_2 & V'_3 \end{pmatrix}.$$

All the coefficients of the kth line except $p_{k|k}$ are now strictly singular. Indeed, let $j \ne k$; then either $j \notin J$ and then $(p_k - p_k b)_{|j}$ is equal to $p_{|j}$, strictly singular; or $j \in J$ and then

$$(p_k - p_k b)_{|j} = p_{k|j} - (\mathrm{Id}_{|j} + s_j)p_{k|j} = -s_j p_{k|j},$$

so that it is also strictly singular.

So A_k is of Gaussian form on the first k lines, and is the matrix of an isomorphism. Repeating this procedure until $k = m$, we finally get the result.

COROLLARY 3. *Let* $\mathbf{X} = \bigoplus_{i=1}^n X_i$ *be a fundamental* HD_n *space. Let* $\mathbf{Y} = \bigoplus_{i=1}^m Y_i$ *be a fundamental* HD_m *subspace of* $\mathbf{X}$. *Then there exist a sum* $\mathbf{Y}'$ *smaller than* $\mathbf{Y}$, *a subset* M' *of* $\{1,\ldots,n\}$ *of cardinality* m *and an isomorphism* A *on* $\mathbf{Y}'$ *such that* $A\mathbf{Y}'$ *is smaller than* $X_{M'}$.

Proof. Let $\mathbf{Y}'$, D, L be as in Lemma 5. Then $\mathbf{Z} = DL\mathbf{Y}'$ is smaller than $X_{M'}$ for some M' of cardinal m.

PROPOSITION 2. *Let X be a* HD_n *space. Let* $\mathbf{X}$ *and* $\mathbf{X}'$ *be fundamental* HD_n *subspaces of X. Then there exists a fundamental* HD_n *subspace* $\mathbf{Y}$ *of* $\mathbf{X}$ *(resp.* $\mathbf{Y}'$ *of* $\mathbf{X}'$*) such that* $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are isomorphic.*

Proof. Write $\mathbf{X} = \bigoplus_{i=1}^n X_i$, $\mathbf{X}' = \bigoplus_{i=1}^n X_i'$. We first find a sum $\mathbf{Y}$ smaller than $\mathbf{X}$ that $\mathrm{Id}+S$-embeds in $\mathbf{X}'$. As $\mathbf{X}'$ is fundamental in X, it is quasi-maximal: applying Corollary 1 for every i in $\{1,\ldots,n\}$, we find a subspace Y_i of X_i that embeds into $\mathbf{X}'$ by an $\mathrm{Id}+S$-isomorphism. The sum $\mathbf{Y} = \bigoplus_{i=1}^n Y_i$ is smaller than $\mathbf{X}$ and embeds into $\mathbf{X}'$ by an $\mathrm{Id}+S$-operator B, which we may assume to be an isomorphism by passing to finite-codimensional subspaces. Let $\mathbf{Z} = B(\mathbf{Y})$. As $\mathbf{Y} \subset \mathbf{X}$, we may apply Corollary 3: without loss of generality, taking a new restriction of the Y_i's, and up to relabelling, we may assume that there exists an isomorphism L such that for every i, $L(Z_i) \subset X_i'$. The sum $\mathbf{Y}' = L(\mathbf{Z}) = LB(\mathbf{Y})$ is smaller than $\mathbf{X}'$ and is isomorphic to $\mathbf{Y}$.

DEFINITION. It follows from the filter structure on the set of fundamental subspaces of X (Proposition 2), and from the fact that both $s(\mathbf{X})$ and $n(\mathbf{X})$ are preserved when taking a smaller sum or an isomorphism (Remark after Definition 3) that for X a HD_n space and $\mathbf{X}$ a fundamental subspace of X, neither $s(\mathbf{X})$ nor $n(\mathbf{X})$ depends on the choice of the decomposition of $\mathbf{X}$ as a direct sum, or even on the choice of $\mathbf{X}$. We define $s(X)$ (resp. $n(X)$) to be this common value.

We are now able to study the space of operators on a fundamental complex HD_n space, and then on a general complex HD_n space.

5. Operators on a complex HD_n space.

We first recall a theorem from [F1].

THEOREM 1. *Let X be a complex H.I. space and $Y \subset X$. Then every operator from Y to X is of the form $\lambda I_{YX} + S$, where λ is complex, I_{YX} is the canonical inclusion map from Y to X, and S is strictly singular.*

Let $\mathbf{X} = \bigoplus_{i=1}^n X_i$ be a fundamental complex HD_n space. Recall that p_{ij} denotes $\alpha_{ij}p_i$, where p_i is the projection on X_i, and α_{ij} a given isomorphism from X_i onto X_j. We denote by $\sum_{i\simeq j}$ the sum $\sum_C \sum_{i,j\in C}$, where C runs over the classes of indices for $\simeq$.

PROPOSITION 3. *Let* $\mathbf{X} = \bigoplus_{i=1}^n X_i$ *be a fundamental complex* HD_n *space. Let* $Y \subset \mathbf{X}$*. Then every operator from Y to* $\mathbf{X}$ *has the form* $\sum_{i\simeq j} \lambda_{ij} p_{ij} + S$*, where λ_{ij} is a constant and S is strictly singular.*

Proof. In the following, S always stands for a strictly singular operator. The space Y is HD_m with $m \le n$. Let $\mathbf{Y} = \bigoplus_{i=1}^m Y_i$ be a fundamental HD_m subspace of Y. According to Lemma 5, we may, renumbering the X_i's, assume that the matrix $I_{\mathbf{Y},\mathbf{X}}$ of the injection of $\mathbf{Y}$ into $\mathbf{X}$ is of the form $\binom{D+S}{V}$, where D is the (m,m) block-diagonal matrix of an isomorphism, S is the (m,m)-matrix of a strictly singular operator, and V some $(n-m,m)$-matrix. We may also assume (passing to a smaller sum) that for all i in $\{1,\ldots,m\}$, $p_{i|i}$ is an isomorphism. For all such i, let $H_i = p_i(Y_i)$. Let $p_{i|i}^{-1}$ be the inverse operator of $p_{i|i}$ (defined on H_i).

Let now T be any operator from Y to $\mathbf{X}$. For every $j \simeq i$, H_i is isomorphic by α_{ij} to a subspace of X_j, so by Theorem 1, every operator from H_i to X_j is of the form $\lambda\alpha_{ij} + S$; while for every $j \not\simeq i$, H_i and X_j are totally incomparable, so every operator from H_i to X_j is strictly singular. As the operator $Tp_{i|i}^{-1}$ is from H_i to $\mathbf{X}$, it follows that $Tp_{i|i}^{-1}$ is of the form $\sum_{i\simeq j}\lambda_{ij}\alpha_{ij}+S_i$, where S_i is strictly singular. It follows that for every $i \le m$, $T_{|Y_i}$ is of the form $(\sum_{j\simeq i}\lambda_{ij}p_{ij})_{|Y_i}+S_i$. If we let $\lambda_{ij}=0$ for $i \ge m+1$, then

$$T_{|\mathbf{Y}} = \sum_{i\simeq j}\lambda_{ij}p_{ij} + S.$$

So the restriction to $\mathbf{Y}$ of the operator S defined by $S = T - \sum_{i\simeq j}\lambda_{ij}p_{ij}$ is strictly singular. By Lemma 2, S is also strictly singular on Y.

PROPOSITION 4. *Let X be a complex* HD_n *space. Let $Y \subset X$ and let $m \le n$ be such that Y is* HD_m*. Then* $\dim(\mathcal{L}(Y,X)/\mathcal{S}(Y,X)) \le n(Y)n(X) \le mn$.

Proof. Let $\mathbf{Y} = \bigoplus_{i=1}^m Y_i$ be a fundamental HD_m subspace of Y. Let $\mathbf{X} = \bigoplus_{i=1}^n X_i$ be a fundamental HD_n subspace of X containing $\mathbf{Y}$. Without loss of generality, we may also assume that the injection from $\mathbf{Y}$ to $\mathbf{X}$ is of the form given by Lemma 5.

Let $T \in \mathcal{L}(\mathbf{Y},X)$. As $\mathbf{X}$ is quasi-maximal in X, for $1 \le i \le m$, we may apply Lemma 3 with $Z = Y_i$; we obtain $Y_i' \subset Y_i$, and T_i' a strictly singular perturbation of $T_{|Y_i'}$ with $\mathrm{Im}\,T_i' \subset \mathbf{X}$. Let $\mathbf{Y}' = \bigoplus_{i=1}^n Y_i'$. Passing to subspaces, we may assume that $\mathbf{Y}'$ is fundamental. The unique operator T' on $\mathbf{Y}'$ such that for all i, $T'_{|Y_i'} = T_i'$ takes its values in $\mathbf{X}$ and is a strictly singular perturbation of $T_{|\mathbf{Y}'}$. The form of T' is then the one given by Proposition 3, and $T_{|\mathbf{Y}'}$ is of the same form. As $\mathbf{Y}'$ is quasi-maximal in $\mathbf{Y}$, Lemma 2 implies that T is of the same form on the whole of $\mathbf{Y}$. Now because of the form of $\mathbf{Y}$, the elements of any equivalence class in $\alpha(\mathbf{Y})$ embed into the elements of one and only one equivalence class in $\alpha(\mathbf{X})$. This defines a quotient map e from $\alpha(\mathbf{Y})$ into $\alpha(\mathbf{X})$. It follows that

$$\dim(\mathcal{L}(\mathbf{Y},X)/\mathcal{S}(\mathbf{Y},X)) = \sum_{\alpha\in\alpha(\mathbf{Y})} |\alpha|\cdot|e(\alpha)|,$$

so by the Cauchy–Schwarz inequality,

$$\dim(\mathcal{L}(\mathbf{Y},X)/\mathcal{S}(\mathbf{Y},X)) \le n(\mathbf{Y})n(\mathbf{X}).$$

Now for $Z \subset X$ and $U \in \mathcal{L}(Z,X)$, let $\widetilde{U}$ be the class of U modulo strictly singular operators. We define a map $\delta : \mathcal{L}(Y,X)/\mathcal{S}(Y,X) \to \mathcal{L}(\mathbf{Y},X)/\mathcal{S}(\mathbf{Y},X)$ by $\delta(\widetilde{T}) = \widetilde{T_{|\mathbf{Y}}}$. It is clear that δ is a linear mapping and, by Lemma 2, it is an injection. It follows that

$$\dim(\mathcal{L}(Y,X)/\mathcal{S}(Y,X)) \le n(Y)n(X) \le mn.$$

COROLLARY 4. *Let X be a complex* HD$_n$ *space. Then*

$$\dim(\mathcal{L}(X)/\mathcal{S}(X)) \le n(X)^2 \le n^2.$$

Remarks. These inequalities are sharp because, by Proposition 3, if X is for example a HD$_n$ space of the form $\bigoplus_{i=1}^n Y$, we have $\dim(\mathcal{L}(X)/\mathcal{S}(X)) = n(X)^2 = n^2$.

However, the inequalities can be strict. It is clear that $n(X)^2$ can be strictly less than n^2, as soon as we know the existence of totally incomparable H.I. spaces. One can see examples of totally incomparable H.I. spaces in [F2], but the simplest examples are probably a class of different versions of X_{GM} with different values for the function f (see [GM] for an explicit definition of X_{GM}).

Also, $\dim(\mathcal{L}(X)/\mathcal{S}(X))$ is not necessarily equal to $n(X)^2$: take a H.I. space Y, a subspace Z of Y of infinite codimension, and let X be the HD$_2$ space $Y \oplus Z$; then $Z \oplus Z$ is fundamental in X, so $n(X)^2 = 4$, while $\dim(\mathcal{L}(X)/\mathcal{S}(X)) = 3$ because $\mathcal{L}(Y,Z) = \mathcal{S}(Y,Z)$.

It is an open problem whether each value between 1 and n^2 can actually be obtained for $\dim(\mathcal{L}(X)/\mathcal{S}(X))$; we do not even have an example of a non-H.I. space for which this dimension is 1.

6. Spectral theory in HD$_n$ spaces

DEFINITIONS. Let X be a Banach space, T be an operator on X. Let $S(T)$ denote the essential spectrum of T: by definition, it is the spectrum of the class of T in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Let $\delta S(T)$ be the boundary of $S(T)$. We say that the operator T is *infinitely singular* if for every finite-codimensional subspace X' of X, the restriction of T to X' is not an isomorphism. By Proposition 2.c.4 of [LT], this is equivalent to saying that for every $\varepsilon > 0$, there exists a subspace Y of X such that $\|T_{|Y}\| \le \varepsilon$. A scalar λ is *infinitely singular for* T if $T - \lambda\,\mathrm{Id}$ is infinitely singular. Let $I(T)$ be the set of infinitely singular values for T.

We also recall that T is *Fredholm* if its image is closed, and its kernel and cokernel are finite-dimensional. Fredholm operators are exactly the operators that are invertible modulo compact (resp. strictly singular) operators. We say that T is *semi-Fredholm* if its image is closed, and its kernel *or* cokernel is finite-dimensional. The *generalized Fredholm index*, defined by $i(T) = \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T)$, is defined and continuous over the set of semi-Fredholm operators (see [LT] about Fredholm operators).

We first prove a well-known lemma in spectral theory.

LEMMA 6. *Let X be a Banach space, and T be an operator on X. Then $\delta S(T) \subset I(T)$.*

Proof. By definition, $S(T)$ is the set of λ such that $T - \lambda\,\mathrm{Id}$ is not invertible modulo compact operators, i.e. is not Fredholm. Now let λ be in $\delta S(T)$. As $S(T)$ is closed, $T - \lambda\,\mathrm{Id}$ is not Fredholm. Neither is it semi-Fredholm of infinite index, for otherwise, by continuity, $T - \lambda'\,\mathrm{Id}$ would be semi-Fredholm of infinite index for λ' in a neighbourhood of λ, contradicting the fact that λ is in the boundary of $S(T)$. It follows that either $\dim(\operatorname{Ker}(T - \lambda\,\mathrm{Id})) = \infty$, or $(T - \lambda\,\mathrm{Id})(X)$ is not closed; in both cases, $T - \lambda\,\mathrm{Id}$ is infinitely singular.

PROPOSITION 5. *Let X be a complex* HD$_n$ *space, and T be an operator on X. Then the cardinality of $S(T)$ satisfies $|S(T)| \le n$.*

Proof. It suffices to prove that $|I(T)| \le n$: indeed, it then follows from Lemma 6 that $|\delta S(T)| \le n$, so that $S(T)$ is a set of at most n isolated points.

For every λ in $I(T)$ and every $\varepsilon > 0$, there exists a space $Y_\lambda(\varepsilon)$ on which $T - \lambda\,\mathrm{Id}$ is of norm at most ε. For every λ, using a normalized basic sequence $(y_n)_{n\in\mathbb{N}}$ of vectors such that $\|T(y_n) - \lambda y_n\| \le 2^{-n}$, we may assume that the map $\varepsilon \mapsto Y_\lambda(\varepsilon)$ is increasing. We now prove that if N is a finite subset of $I(T)$, then for some $\varepsilon > 0$, the spaces $(Y_\lambda(\varepsilon))_{\lambda\in N}$ form a direct sum; then from the fact that X is HD$_n$ it follows clearly that $|I(T)| \le n$.

Assume that the property is false, and let N be a finite subset of $I(T)$ of minimum cardinality among those contradicting the property. It is clear that $|N| \ge 2$. Let $c = \min_{\lambda\ne\lambda'\in N} |\lambda - \lambda'|$ and $C = \max_{\lambda\in N} |\lambda|$. Let $\varepsilon > 0$. There are vectors y_λ with $y_\lambda \in Y_\lambda(\varepsilon)$, $\max_{\lambda\in N} \|y_\lambda\| = 1$ and $\|\sum_{\lambda\in N} y_\lambda\| \le \varepsilon$. Applying T to this inequality, we get

$$\Big\|\sum_{\lambda\in N} \lambda y_\lambda\Big\| \le (\|T\| + |N|)\varepsilon.$$

Let λ_ε in N be such that $\max_{\lambda\ne\lambda_\varepsilon} \|y_\lambda\| = 1$ (such a number exists because $|N| \ge 2$). We have

$$\Big\|\sum_{\lambda\ne\lambda_\varepsilon} (\lambda - \lambda_\varepsilon) y_\lambda\Big\| \le (|\lambda_\varepsilon| + \|T\| + |N|)\varepsilon,$$

while

$$\max_{\lambda\ne\lambda_\varepsilon} \|(\lambda - \lambda_\varepsilon) y_\lambda\| \ge c.$$

Since $\varepsilon \mapsto Y_\lambda(\varepsilon)$ is increasing for every λ, and λ_ε takes a finite number of values, we may assume that λ_ε is constantly equal to some λ_0, so that we get vectors y'_λ for $\lambda \neq \lambda_0$ with $y'_\lambda \in Y_\lambda(\varepsilon)$, $\|\sum_{\lambda\neq\lambda_0} y'_\lambda\| \le (C+\|T\|+|N|)\varepsilon$, and $\max_{\lambda\neq\lambda_0}\|y'_\lambda\| \ge c$.

Again because $\varepsilon \mapsto Y_\lambda(\varepsilon)$ is increasing, the sum of the $(Y_\lambda(\varepsilon))$ for $\lambda \neq \lambda_0$ is not direct, and this holds for any $\varepsilon > 0$, a contradiction with the minimality of N.

Remark. Notice that the inequality in Proposition 5 is sharp; for example, it is an equality in a fundamental HD_n space, for an operator of the form $\sum_{i=1}^n \lambda_i p_i$, with $\lambda_i \neq \lambda_j$ if $i \neq j$.

LEMMA 7. *Let X be a finitely hereditarily decomposable space. Then every semi-Fredholm operator on X is Fredholm with index* 0.

Proof. Let T be a semi-Fredholm operator on X. First assume that X is complex. If T is semi-Fredholm with infinite index, then so is $T-\lambda\,\mathrm{Id}$ for λ small enough, a contradiction with Lemma 6. So T is Fredholm. As $S(T)$ is finite, $\mathbb{C}\setminus S(T)$ is connected, so $\mathrm{ind}(T-\lambda\,\mathrm{Id})$ is constantly equal to $\mathrm{ind}(T)$ on $\mathbb{C}\setminus S(T)$. Furthermore, when λ tends to infinity, $T-\lambda\,\mathrm{Id} = -\lambda(\mathrm{Id}-T/\lambda)$ is close to $-\lambda\,\mathrm{Id}$, so it is Fredholm with index 0, hence $\mathrm{ind}(T)=0$.

Assume now that X is real and consider its complexification $X_{\mathbb{C}}$ (that is, $X_{\mathbb{C}} = X \oplus X$ with the law $i(x,y)=(-y,x)$). Then $X_{\mathbb{C}}$ is a real HD_{2n} space, and as every complex subspace of $X_{\mathbb{C}}$ is also a real subspace, $X_{\mathbb{C}}$ is a complex HD_m space for some $m \le 2n$. Let i_T be the generalized Fredholm index of T. It is easy to check that $T\oplus T$, the complexification of T on $X_{\mathbb{C}}$, is also semi-Fredholm with index i_T (the dimensions in the definition of the index of $T\oplus T$ are over $\mathbb{C}$). By the first paragraph of this proof, and as $X_{\mathbb{C}}$ is finitely hereditarily decomposable, i_T is equal to 0.

COROLLARY 5. *Let X be a finitely hereditarily decomposable space. Then X is not isomorphic to any proper subspace.*

Proof. Every isomorphism from X into X is semi-Fredholm; but by Lemma 7, it is Fredholm with index 0, so it must be an onto isomorphism.

7. Operators on a real H.I. space

THEOREM 2. *Let X be a real H.I. space. Then for all $Y \subset X$,*

$$\dim(\mathcal{L}(Y,X)/\mathcal{S}(Y,X)) \le 4.$$

Furthermore, $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Proof. The first part of the proof uses a result of [F1]. It was shown in that article that all the spaces $E_Y = \mathcal{L}(Y,X)/\mathcal{S}(Y,X)$ embed in a limit space E which is a Banach algebra and a division ring. By Gelfand's theorem

(whose proof can be found in [R]), the space E is of dimension at most 4 (and must be isomorphic either to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). It follows that $\dim E_Y \le 4$.

The statement relative to the case $Y = X$ can be proved directly as follows. Let T be a non-strictly singular operator on X. Then by Lemma 2, no restriction of T to a subspace of X is strictly singular; by Proposition 2.c.4 of [LT], the restriction of T to some finite-codimensional subspace of X is an into isomorphism, and so T is semi-Fredholm. Finally, by Lemma 7, T is Fredholm. This means that every operator on X is either strictly singular, or Fredholm. As Fredholm operators are invertible modulo strictly singular operators, this is equivalent to saying that $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring, and so by Gelfand's theorem it is either real, complex or quaternionic.

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