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A condition implying boundedness and VMO for a function f

by

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Dedicated to the memory of my friend Filippo Chiarenza

Abstract. Some boundedness and VMO results are proved for a function f integrable on a cube Q_0, starting from an integral bound.

1. Introduction. In [12] the following condition, intimately connected with the idea of sharp function ([1]) was introduced:

$$\fint_Q \Big| f - \fint_Q f \Big| \, dx \le \varepsilon \fint_Q |f| \, dx, \tag{1.1}$$

$Q \subseteq Q_0$ (for notation and hypotheses see Section 2).

From condition (1.1), extraintegrability for f was obtained with different methods and in more and more general situations ([12], [3], [10], [14], e.g.). The general approach using maximal functions and rearrangements as in [10], [14] seems to be the best. In particular, in [10], [12] it is proved that there exists a constant γ depending only on n such that if $0 < \varepsilon < 1/\gamma$, then f belongs to $L^p(Q_0)$ for any $1 \le p < 1/(\varepsilon\gamma)$ and the order of the optimal integrability exponent is exact. Using this result the following integrability and continuity result is proved in [11]. Suppose that instead of (1.1), f satisfies

$$\fint_Q \Big| f - \fint_Q f \Big| \, dx \le \varepsilon(\sigma) \fint_Q |f| \, dx, \tag{1.2}$$

for any Q with $|Q| \le \sigma$ and $\lim_{\sigma\to 0} \varepsilon(\sigma) = 0$. Then f belongs to $L^p(Q_0)$ for any $1 \le p < \infty$. Moreover, if

$$\fint_Q \Big| f - \fint_Q f \Big| \, dx \le \sigma^\beta \fint_Q |f| \, dx, \tag{1.3}$$

$\beta > 0$, for any Q with $|Q| \le \sigma$ and for σ tending to zero, then f is Hölder-continuous of order β.

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In this paper we introduce an integral condition for f generalizing (1.2) that implies boundedness for f. This condition also implies that f is in the VMO space, but it is not sufficient for the continuity of the function.

At this point we want to emphasize that functions belonging to $L^\infty \cap VMO$ but not necessarily continuous have recently been isolated and considered in very important regularity problems connected with elliptic differential equations and systems with rough coefficients ([4], [6]–[9], e.g.).

The condition that we introduce is

$$\text{(F)} \qquad \int_0^{|Q_0|} \frac{f^{**}(t)-f^*(t)}{f^{**}(t)} \cdot \frac{1}{t}\, dt < \infty.$$

We shall note that this condition is more general than (1.2) and the condition in [16]. In particular, in [16] from a condition like (F), but stronger, the boundedness of f is deduced, but with a completely different method.

We point out that condition (F) seems different and more general than that in [16], just because it does not involve continuity.

Finally, we remark that condition (1.3) and (F) are in the same order of ideas of [5], [18].

2. Some notations and results. From now on Q is a cube in $\mathbb{R}^n$, i.e. a translate of $[0,s]^n$, $0 < s < \infty$, and we fix a cube Q_0 and consider subcubes Q of Q_0. We denote by f a function belonging to $L^1(Q_0)$. For any measurable subset E of Q_0 we denote by $|E|$ its Lebesgue measure, and set

$$\fint_E f\, dx = \frac{1}{|E|}\int_E |f|\, dx$$

where the integrals are with respect to the Lebesgue measure.

The nonincreasing rearrangement of f is denoted by f^* and is defined by ([1], e.g.)

$$f^*(t) = \sup_{|E|=t} \operatorname*{ess\,inf}_{x\in E} f(x), \qquad t \in \,]0, |Q_0|[.$$

We then set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds \equiv \fint_0^t f^*(s)\, ds, \qquad 0 < t < |Q_0|[,$$

$$Mf(x) = \sup_{\substack{Q\subseteq Q_0\\ x\in Q}} \fint_Q f\, dx, \qquad x \in Q_0,$$

$$f_t^{\#}(x) = \sup_{\substack{Q\subseteq Q_0\\ x\in Q\\ |Q|\le t}} \fint_Q \Big| f - \fint_Q f\Big|\, dx, \qquad 0 < t < |Q_0|.$$

The following definitions are used.

The function f is in the space $\mathrm{BMO}(Q_0)$ ([15]) if

$$\sup_{Q\subseteq Q_0} \fint_Q \Big| f - \fint_Q f\Big|\, dx < \infty;$$

then, setting for any $0 < r < |Q_0|$,

$$\eta(r) = \sup_{\substack{|Q|\le r\\ Q\subseteq Q_0}} \fint_Q \Big| f - \fint_Q f\Big|\, dx,$$

a function f belonging to the space BMO is in VMO ([17]) if

$$\lim_{r\to 0} \eta(r) = 0.$$

We need the following results.

THEOREM 2.1 ([2], [10]). *Let Ω be a relatively open subset of Q_0 such that*

$$|\Omega| < |Q_0|/2.$$

Then there exists a family $(Q_j)_{j\in\mathbb{N}}$ of cubes with pairwise disjoint interiors such that:

(1) $|\Omega \cap Q_j| \le |Q_j|/2 \le |C\Omega \cap Q_j|$,
(2) $\Omega \subset \bigcup_j Q_j \subset Q_0$,
(3) $|\Omega| \le \sum_j |Q_j| \le 2^{n+1}|\Omega|$.

THEOREM 2.2 ([13]). *For any $0 \le t \le |Q_0|$,*

$$3^{-n}(Mf)^*(t) \le f^{**}(t) \le (1+2^n)(Mf)^*(t).$$

3. An integral bound for f implying boundedness. Suppose f is as in Section 2 and satisfies the condition

$$\text{(F)} \qquad \int_0^{|Q_0|} \frac{f^{**}(t)-f^*(t)}{f^{**}(t)} \cdot \frac{1}{t}\, dt < \infty.$$

We prove the following:

THEOREM 3.1. *If f satisfies* (F), *then f belongs to $L^\infty(Q_0)$.*

Proof. Set

$$\eta(t) = \frac{f^{**}(t)-f^*(t)}{f^{**}(t)}, \qquad 0 < t < |Q_0|.$$

Then, obviously, $f^{**}(t) = f^*(t)/(1-\eta(t))$. From this we deduce

$$\text{(3.1)} \qquad \frac{f^*(t)}{\int_0^t f^*(s)\, ds} = \frac{1-\eta(t)}{t}.$$

Integrating (3.1) from t to $\bar t$, $0 < t < \bar t < |Q_0|$, we obtain

$$\int_t^{\bar t} \frac{f^*(t)}{\int_0^s f^*(\sigma)\,d\sigma}\,ds = \int_t^{\bar t} \frac{1-\eta(s)}{s}\,ds = \int_t^{\bar t} \frac{1}{s}\,ds - \int_t^{\bar t} \frac{\eta(s)}{s}\,ds.$$

Hence, for a.a. $0 < t < \bar t < |Q_0|$,

$$\log \frac{\int_0^{\bar t} f^*(s)\,ds}{\int_0^t f^*(s)\,ds} = \log \frac{\bar t}{t} - \int_t^{\bar t} \frac{\eta(s)}{s}\,ds. \tag{3.2}$$

Taking the exponential in (3.2) gives

$$\frac{\int_0^{\bar t} f^*(s)\,ds}{\int_0^t f^*(s)\,ds} = \frac{\bar t/t}{\exp \int_t^{\bar t} \frac{\eta(s)}{s}\,ds}$$

and so

$$\int_0^t f^*(s)\,ds = t f^{**}(\bar t) \exp \int_t^{\bar t} \frac{\eta(s)}{s}\,ds. \tag{3.3}$$

But f^* is decreasing, which yields

$$t f^*(t) \le \int_0^t f^*(s)\,ds \le t f^{**}(\bar t) \exp \int_0^{\bar t} \frac{\eta(s)}{s}\,ds,$$

that is,

$$f^*(t) \le f^{**}(\bar t) \exp \int_0^{\bar t} \frac{\eta(s)}{s}\,ds. \tag{3.4}$$

Using (F) we deduce immediately from (3.4) that f^* is bounded, and then f belongs to $L^\infty(Q_0)$.

We note at this point that (F) does not imply continuity for f. In fact, consider a nonnegative strictly decreasing and continuous function f of a real variable defined in the interval $]0, b[$. Suppose that f satisfies condition (F). Then, by Theorem 3.1, f is bounded. Now we can consider a function $\widetilde f$ that is equal to f in $]0, a[$ where $0 < a < b$, but is discontinuous in a subset of $]a, b[$ of positive measure and $\widetilde f(x) \le f(a)$ for any $a < x < b$. The function $\widetilde f$ satisfies (F) and hence is bounded, but it is not continuous.

Now we want to consider a condition introduced in [16] from which, with a method different from the one used in this paper, boundedness for f is deduced. We shall prove that this condition implies (F).

To do this, we prove a rearranged inequality for f. This inequality is a local version of a theorem in [2], [10].

THEOREM 3.2. *Let f be a nonnegative function belonging to $L^1(Q_0)$. Then, for any $0 < t < |Q_0|/(3\cdot 2^{n+1})$,*

$$f^{**}(t) - f^*(t) \le 3\cdot 2^{n+2} (f^{\#}_{3\cdot 2^{n+1}t})^*(t).$$

Proof. Fix $0 < t < |Q_0|/(3\cdot 2^{n+1})$ and set

$$E = \{x \in Q_0 : f(x) > f^*(t)\},$$
$$F = \{x \in Q_0 : f^{\#}_{3\cdot 2^{n+1}t}(x) > (f^{\#}_{3\cdot 2^{n+1}t})^*(t)\}.$$

Obviously, $|E \cup F| \le 2t$, and we can find a set Ω relatively open in Q_0 such that $|\Omega| \le 3t$, $E \cup F \subset \Omega \subset Q_0$, from which $|\Omega| < |Q_0|/2$. From a well known property of decreasing rearrangements ([1], e.g.) we have

$$t(f^{**}(t) - f^*(t)) = \int_E (f(x) - f^*(t))\,dx. \tag{3.5}$$

Let $\{Q_j\}$ be the covering of Ω furnished by Theorem 2.1. Then, by (3) of Theorem 2.1,

$$|Q_j| \le 3\cdot 2^{n+1} t \tag{3.6}$$

and by (3.5), (3.6) and (1) of Theorem 2.1 we obtain

$$\begin{aligned} t(f^{**}(t) - f^*(t)) &\le \sum_j \int_{Q_j} \Big| f(x) - \fint_{Q_j} f \Big|\,dx \\ &\quad + \sum_j |E \cap Q_j| \Big(\fint_{Q_j} f - f^*(t) \Big) \\ &\le 2 \sum_j \int_{Q_j} \Big| f(x) - \fint_{Q_j} f \Big|\,dx \le 2 \sum_j |Q_j| f^{\#}_{3\cdot 2^{n+1}t}(x_j), \end{aligned} \tag{3.7}$$

for any $x_j \in Q_j$. From (1) of Theorem 2.1 we note that $Q_j \cap CF$ is nonempty for any j. Choosing $x_j \in Q_j \cap CF$ we obtain from (3.7),

$$t(f^{**}(t) - f^*(t)) \le 2 \sum_j |Q_j| (f^{\#}_{3\cdot 2^{n+1}t})^*(t).$$

Finally, using (3) of Theorem 2.1 we deduce

$$t(f^{**}(t) - f^*(t)) \le 3\cdot 2^{n+2} t (f^{\#}_{3\cdot 2^{n+1}t})^*(t).$$

For f nonnegative and integrable in Q_0 in [16] the following quantity is considered:

$$v(f, \sigma) = \sup \frac{\fint_Q |f - \fint_Q f|\,dx}{\fint_Q f\,dx}, \quad 0 < \sigma < |Q_0|, \tag{K$_1$}$$

where the sup is extended over all $Q \subseteq Q_0$ with $|Q| \le \sigma$ such that $\fint_Q f > 0$.

From the condition

$$(\mathrm{K}_2) \qquad \int_0^{|Q_0|} v(f,\sigma)\frac{1}{\sigma}\,d\sigma < \infty$$

it is deduced that f belongs to $L^\infty(Q_0)$.

Obviously, from (K_1) we can deduce

$$f^\#_\sigma(x) \le v(f,\sigma)Mf(x)$$

for any $x \in Q_0$ and $0 < \sigma < |Q_0|$ and consequently,

$$(3.8) \qquad (f^\#_\sigma)^*(t) \le v(f,\sigma)(Mf)^*(t),$$

for any $0 < \sigma < |Q_0|$ and $0 < t < |Q_0|$.

From Theorem 3.2 and (3.8) we obtain

$$f^{**}(t) - f^*(t) \le 3\cdot 2^{n+2}v(f, 3\cdot 2^{n+1}t)(Mf)^*(t),$$

and using Theorem 2.2,

$$(3.9) \qquad \frac{f^{**}(t)-f^*(t)}{f^{**}(t)} \le 3^{n+1}2^{n+2}v(f,3\cdot 2^{n+1}t).$$

From (3.9) it is clear that condition (K_2) implies condition (F).

4. The VMO result. Suppose f to be as in Section 2 and satisfy condition (F). Then, by Theorem 3.1, f belongs to $L^\infty(Q_0)$, and (F) implies

$$(4.1) \qquad \int_0^{|Q_0|} \frac{f^{**}(t)-f^*(t)}{t}\,dt < \infty.$$

Let us prove:

THEOREM 4.1. *If f satisfies* (4.1) *then f belongs to* VMO.

Proof. Fix $\varepsilon > 0$. Condition (4.1) assures the existence of $\overline{t} > 0$ such that

$$\fint_0^t (f^*(s) - f^*(t))\,ds < \varepsilon$$

for any $t \in]0,\overline{t}[- E$, where E is some subset of $]0,\overline{t}[$ with $|E| = 0$.

If t belongs to $]0,\overline{t}[-E$ then by the property of decreasing rearrangements ([1], e.g.) we have

$$\fint_Q |f(x) - f^*(t)|\,dx \le \fint_0^t (f(x)-f^*(t))^*(s)\,ds < \varepsilon$$

and so

$$(4.2) \qquad \fint_Q \Big|f(x) - \fint_Q f\Big|\,dx \le 2\varepsilon$$

for any $Q \subseteq Q_0$ with $|Q| = t$.

Now suppose that $t \in E$. For any $\delta > 0$ such that $]t-\delta, t+\delta[\subset]0,\overline{t}[$ there exists $t_\delta \in]t-\delta,t+\delta[- E$. We can select a δ such that if Q is a cube with $|Q| = t$ and $\widetilde{Q}$ is the cube with the same center and $|\widetilde{Q}| = t_\delta$, $Q \subset Q_0$, $\widetilde{Q} \subset Q_0$, then

$$(4.3) \qquad \Big|\fint_Q \Big|f(x) - \fint_Q f\Big|\,dx - \fint_{\widetilde{Q}}\Big|f(x) - \fint_{\widetilde{Q}} f\Big|\,dx\Big| < \varepsilon.$$

Then, by (4.2) and (4.3), for any $0 < t < \overline{t}$ and any cube $Q \subseteq Q_0$ with $|Q| = t$,

$$\fint_Q \Big|f(x) - \fint_Q f\Big|\,dx \le 3\varepsilon$$

and f is in VMO.

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Higher-dimensional weak amenability

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Abstract. Bade, Curtis and Dales have introduced the idea of weak amenability. A commutative Banach algebra $\mathfrak{A}$ is weakly amenable if there are no non-zero continuous derivations from $\mathfrak{A}$ to $\mathfrak{A}^*$. We extend this by defining an alternating n-derivation to be an alternating n-linear map from $\mathfrak{A}$ to $\mathfrak{A}^*$ which is a derivation in each of its variables. Then we say that $\mathfrak{A}$ is n-dimensionally weakly amenable if there are no non-zero continuous alternating n-derivations on $\mathfrak{A}$. Alternating n-derivations are the same as alternating Hochschild cocycles. Since such a cocycle is a coboundary if and only if it is 0, the alternating n-derivations form a subspace of $H^n(\mathfrak{A},\mathfrak{A}^*)$. The hereditary properties of n-dimensional weak amenability are studied; for example, if J is a closed ideal in $\mathfrak{A}$ such that $\mathfrak{A}/J$ is m-dimensionally weakly amenable and J is n-dimensionally weakly amenable then $\mathfrak{A}$ is $(m+n-1)$-dimensionally weakly amenable. Results of Bade, Curtis and Dales are extended to n-dimensional weak amenability. If $\mathfrak{A}$ is generated by n elements then it is $(n+1)$-dimensionally weakly amenable. If $\mathfrak{A}$ contains enough regular elements a with $\|a^m\| = o(m^{n/(n+1)})$ as $m \to \pm\infty$ then $\mathfrak{A}$ is n-dimensionally weakly amenable. It follows that if $\mathfrak{A}$ is the algebra $\operatorname{lip}_\alpha(X)$ of Lipschitz functions on the metric space X and $\alpha < n/(n+1)$ then $\mathfrak{A}$ is n-dimensionally weakly amenable. When X is the product of n copies of the circle then $\mathfrak{A}$ is n-dimensionally weakly amenable if and only if $\alpha < n/(n+1)$.

1. Introduction. Throughout this paper $\mathfrak{A}$ denotes a commutative Banach algebra and $\mathfrak{X}$ a symmetric Banach $\mathfrak{A}$-bimodule, that is, we have $ax = xa$ for all $a \in \mathfrak{A}$, $x \in \mathfrak{X}$. Following [1], $\mathfrak{A}$ is *weakly amenable* if, for all $\mathfrak{X}$, all derivations from $\mathfrak{A}$ into $\mathfrak{X}$ are zero. In this paper we extend this by saying that $\mathfrak{A}$ is *n-dimensionally weakly amenable* [Definition 2.1] if, for all $\mathfrak{X}$, all alternating n-cocycles from $\mathfrak{A}$ into $\mathfrak{X}$ are zero. By an *n-cocycle* we mean a continuous n-linear map from $\mathfrak{A}$ into $\mathfrak{X}$ whose Hochschild coboundary is 0 (cf. [5]). For $n = 1$ this reduces to weak amenability in the sense of Bade, Curtis and Dales. In Section 2 we show that an alternating n-cocycle is the same as an alternating linear map which is a derivation in each of its variables. This enables us to show how the values of an alternating n-cocycle are related to its values on the generators of an algebra and show in particular that if $\mathfrak{A}$ has n-generators then it is $(n+1)$-dimensionally

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