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Spectral sets

by

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Dedicated to the memory of Petr Pácalt

Abstract. The paper studies spectral sets of elements of Banach algebras as the zeros of holomorphic functions and describes them in terms of existence of idempotents. A new decomposition theorem characterizing spectral sets is obtained for bounded linear operators.

Introduction. Traditionally, spectral sets of bounded linear operators are studied with the help of restrictions of operators to invariant subspaces [2, 4, 13]. An alternative approach which works also for elements of Banach algebras is studied in [9] for the case of isolated spectral points. In the present paper we study spectral sets as the zeros of suitable holomorphic functions, and give a characterization in terms of existence of idempotents satisfying certain conditions. The main tools for the study of this characterization are the result of the present author [9, Theorem 1.1] and a theorem of Dunford and Schwartz [4, Theorem VII.3.19].

A denotes a complex unital Banach algebra with unit e. For any $a \in A$, $\sigma(a)$ denotes the spectrum of a, $\varrho(a)$ its resolvent set, and $\lambda \mapsto R(\lambda; a)$ its resolvent. An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$, and *idempotent* if $a^2 = a$. By $\mathrm{Inv}(A)$ and $\mathrm{qNil}(A)$ we denote the sets of all invertible and quasinilpotent elements of A, respectively. Poles and essential singularities of the resolvent of a will be referred to as poles and essential singularities of the element a. If K is a compact subset of the complex plane, then $H(K)$ is the set of all complex-valued functions, each holomorphic in some open neighbourhood of K. The holomorphic functional calculus for an element a of A is defined for functions in $H(\sigma(a))$. If $f \in H(\sigma(a))$ is defined on an open neighbourhood Ω of $\sigma(a)$, then

$$f(a) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda; a)\, d\lambda,$$

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where γ is a cycle in $\Omega\setminus\sigma(a)$ with $\operatorname{ind}(\lambda,\gamma)=0$ for all $\lambda\notin\Omega$ and $\operatorname{ind}(\lambda,\gamma)=1$ for all $\lambda\in\sigma(a)$. The set of all elements of A of the form $f(a)$, where $a\in A$ and $f\in H(\sigma(a))$, will be denoted by $\mathcal{F}(a)$, and its closure in A by $\overline{\mathcal{F}(a)}$.

A *spectral set* of $a\in A$ (or an *isolated part of the spectrum* of a) is a subset of $\sigma(a)$ which is both open and closed in $\sigma(a)$. The *spectral idempotent* $p(\sigma;a)$ corresponding to a spectral set σ of a is defined to be the element $p(\sigma;a)=e(a)$, where $e\in H(\sigma(a))$ is equal to 1 in a neighbourhood of σ and to 0 in a neighbourhood of $\sigma(a)\setminus\sigma$. For an isolated spectral point μ of a we write $p(\mu;a)$ instead of $p(\{\mu\};a)$.

By $L(X)$ we denote the Banach algebra of all bounded linear operators on a complex Banach space X. If $T\in L(X)$, we write $N(T)$ for the nullspace $T^{-1}(\{0\})$ and $R(T)$ for the range $T(X)$ of T. If σ is a spectral set of $T\in L(X)$, we write $P(\sigma;T)$ for the spectral projection of T corresponding to σ.

1. A general spectral set. In [9] the following characterization was obtained for an isolated spectral point of an element $a\in A$.

THEOREM 1.1. *0 is an isolated spectral point of $a\in A$ if and only if there exists a nonzero idempotent $p\in A$ commuting with A such that*

$$ap\in \operatorname{qNil}(A),\qquad a+p\in\operatorname{Inv}(A). \tag{1.1}$$

The element p is the spectral idempotent of a corresponding to 0.

Let us remark that the second condition, $a+p\in\operatorname{Inv}(A)$, can be replaced by $p\in\overline{\mathcal{F}(a)}$.

Also, we need the following result of Dunford and Schwartz (interpreted for elements of a Banach algebra):

THEOREM 1.2 [4, Theorem VII.3.19]. *Let $f\in H(\sigma(a))$. If τ is a spectral set of $f(a)$, then $\sigma=\sigma(a)\cap f^{-1}(\tau)$ is a spectral set of a, and $p(\sigma;a)=p(\tau;f(a))$.*

We use Theorems 1.1 and 1.2 to give a characterization of an arbitrary spectral set as the inverse image of 0 under a holomorphic function provided there exists a suitable idempotent.

THEOREM 1.3. *Let $\sigma=\sigma(a)\cap f^{-1}(\{0\})$ for some $f\in H(\sigma(a))$. Then σ is a spectral set of a if and only if there is an idempotent $p\in A$ commuting with a such that*

$$f(a)p\in\operatorname{qNil}(A),\qquad f(a)+p\in\operatorname{Inv}(A)\quad (or\ p\in\overline{\mathcal{F}(a)}). \tag{1.2}$$

p is then the spectral idempotent $p(\sigma;a)=p(0;f(a))$.

Proof. Suppose that there is $p\in A$ satisfying the conditions of the theorem. If $p=0$, then $f(a)$ is invertible, and $\sigma=\emptyset$. Assume that $p\neq0$. By Theorem 1.1, 0 is an isolated spectral point of $f(a)$. By Theorem 1.2 for the special case of $\tau=\{0\}$ we conclude that σ is spectral for a, and that $p(\sigma;a)=p(0;f(a))$. Theorem 1.1 also implies that $p=p(\sigma;a)$.

Conversely, suppose that σ is a spectral set of a. If $\sigma=\emptyset$, then $p(\sigma;a)=0$, and (1.2) hold with $p=0$. Let $\sigma\neq\emptyset$. The spectral mapping theorem and the compactness of $\sigma(a)$ ensure that 0 is an isolated spectral point of $f(a)$. The conclusion follows from Theorem 1.1 with a replaced by $f(a)$.

We introduce the following notation: If $\sigma=\sigma(a)\cap f^{-1}(\{0\})$ for some $f\in H(\sigma(a))$, we define

$$\sigma_1=\{\mu\in\sigma: f^{(m)}(\mu)\neq0 \text{ for some } m>0\},\quad \sigma_2=\sigma\setminus\sigma_1,\quad \sigma_3=\sigma(a)\setminus\sigma.$$

Since zeros of finite order of a holomorphic function are isolated and since $\sigma(a)$ is compact, each point of σ_1 is isolated in the compact set σ, and σ_1 is finite, say $\sigma_1=\{\mu_1,\ldots,\mu_k\}$. The sets σ_2 and σ_3 can be separated by open sets as f vanishes in some neighbourhood of σ_2 and is nonzero in some neighbourhood of σ_3.

If each point of σ_1 is isolated in $\sigma(a)$, then the sets $\sigma=\sigma_1\cup\sigma_2$ and σ_3 can be separated by open sets, which shows that σ is a spectral set of a. Conversely, if σ is a spectral set of a, then the sets σ and σ_3 can be separated by open sets; since each point of σ_1 is isolated in σ, it is also isolated in $\sigma(a)$. This proves the following criterion for a spectral set.

THEOREM 1.4. *Let $\sigma=\sigma(a)\cap f^{-1}(\{0\})$ for some $f\in H(\sigma(a))$. Then σ is a spectral set of a if and only if, for each zero $\mu\in\sigma$ of f of finite order, μ is an isolated spectral point of a.*

2. A finite spectral set. In this section we are concerned with a partial converse to the following result from Bonsall and Duncan [1]. We are particularly interested in the case when all but one of the σ_i are singletons.

PROPOSITION 2.1 (see [1, Proposition I.7.9]). *Let $\sigma(a)$ be the disjoint union of nonempty spectral sets $\sigma_1,\ldots,\sigma_m$. If $p_i=p(\sigma_i;a)$, the elements $p_1,\ldots,p_m$ commute with a and form a complete set of idempotents, that is,*

$$p_i^2=p_i,\qquad p_ip_j=0\quad(i\neq j),\qquad p_1+\ldots+p_m=e.$$

If $\sigma_i=\{\mu_i\}$, then $b_i=(a-\mu_ie)p_i$ is quasinilpotent.

We will also need the following lemma, stated without proof:

LEMMA 2.2. *Let $p_1,\ldots,p_m$ be a complete set of idempotents in a Banach algebra A. If $u_1,\ldots,u_m$ are invertible elements of A commuting with each p_i, then*

$$\Big(\sum_{i=1}^m u_ip_i\Big)^{-1}=\sum_{i=1}^m u_i^{-1}p_i.$$

The following result is a multipoint version of Theorem 1.1 of [9]. It will be convenient, for the case of several isolated spectral points, to define the element c slightly differently than in [9].

THEOREM 2.3. *Suppose that $a \in A$, that $\mu_1, \ldots, \mu_k$ are distinct complex numbers, and that the following conditions are satisfied:*

(a) *There are nonzero idempotents p_i $(i = 1, \ldots, k)$ in A commuting with a such that $p_i p_j = 0$ if $i \neq j$.*

(b) *Each element $b_i = (a - \mu_i e)p_i$ is quasinilpotent.*

If $\{\alpha_1, \ldots, \alpha_k\}$ is a set of complex numbers disjoint from $\{\mu_1, \ldots, \mu_k\}$ and if

$$c = \sum_{i=1}^{k} \alpha_i p_i + a(e - p), \tag{2.1}$$

where $p = p_1 + \ldots + p_k$, then

$$\sigma(c) \cup \{\mu_1, \ldots, \mu_k\} = \sigma(a) \cup \{\alpha_1, \ldots, \alpha_k\}. \tag{2.2}$$

Furthermore, if $f \in H(\sigma(a) \cup \{\alpha_1, \ldots, \alpha_k\})$, then

$$f(a) = f(c)(e - p) + \sum_{i=1}^{k} \sum_{n=0}^{\infty} \frac{f^{(n)}(\mu_i)}{n!}(a - \mu_i e)^n p_i, \tag{2.3}$$

and

$$f(c) = f(a)(e - p) + \sum_{i=1}^{k} f(\alpha_i)p_i. \tag{2.4}$$

Proof. Note that $e - p, p_1, \ldots, p_k$ is a complete set of idempotents in the sense of Proposition 2.1. We calculate that

$$\lambda e - a = (\lambda e - c)(e - p) + \sum_{i=1}^{k}((\lambda - \mu_i)e - b_i)p_i.$$

If $\lambda \notin \sigma(c) \cup \{\mu_1, \ldots, \mu_k\}$, we use Lemma 2.2 to conclude that $\lambda e - a$ is invertible. Similarly,

$$\lambda e - c = (\lambda e - a)(e - p) + \sum_{i=1}^{k}(\lambda - \alpha_i)p_i;$$

if $\lambda \notin \sigma(a) \cup \{\alpha_1, \ldots, \alpha_k\}$, we conclude that $\lambda e - c$ is invertible using Lemma 2.2. Equations (2.2)–(2.4) then follow by a standard argument.

Note 2.4. We observe that, for $k = 1$, we define c in the preceding theorem by $c = \alpha p + a(e - p)$ for $\alpha \neq \mu$, whereas in [9] we have $c = \xi p + a$ for $\xi \neq 0$.

THEOREM 2.5. *A set $\{\mu_1, \ldots, \mu_k\}$ is spectral for an element $a \in A$ if and only if conditions* (a) *and* (b) *of Theorem 2.3 are satisfied together with one of the following:*

(c) *For a set $\{\alpha_1, \ldots, \alpha_k\}$ of complex numbers disjoint from $\{\mu_1, \ldots, \mu_k\}$, we have $\mu_i \in \varrho(c)$, where c is given by* (2.1).

(d) *For a function $f \in H(\sigma(a))$ whose only zeros in $\sigma(a)$ are $\mu_1, \ldots, \mu_k$, the element $p + f(a)$ is invertible, where $p = p_1 + \ldots + p_k$.*

For each $i \in \{1, \ldots, k\}$, p_i is then the spectral idempotent corresponding to the spectral set $\{\mu_i\}$, (c) *holds for any set $\{\alpha_1, \ldots, \alpha_k\}$ disjoint from $\{\mu_1, \ldots, \mu_k\}$, and* (d) *holds for any $f \in H(\sigma(a))$ whose only zeros in $\sigma(a)$ are $\mu_1, \ldots, \mu_k$.*

Proof. First assume that $\{\mu_1, \ldots, \mu_k\}$ is a spectral set for a. In Proposition 2.1 set $m = k + 1$, $\sigma_i = \{\mu_i\}$, $p_i = p(\sigma_i; a)$ for $i = 1, \ldots, k$, and $\sigma_{k+1} = \sigma(a) \setminus \{\mu_1, \ldots, \mu_k\}$; (a) and (b) then follow. The proof of (c) and (d), based on the holomorphic calculus for a, is routine.

Conversely, suppose that (a)–(c) are satisfied for some $\{\alpha_1, \ldots, \alpha_k\}$ disjoint from $\{\mu_1, \ldots, \mu_k\}$. By (2.2), each μ_i is an isolated spectral point of a.

Next suppose that (a), (b) and (d) are satisfied. For each $i = 1, \ldots, k$ there is $g_i \in H(\sigma(a))$ such that $f(\lambda) = g_i(\lambda)(\lambda - \mu_i)$. Since $f(a)p = \sum_{i=1}^{k} g_i(a)(a - \mu_i e)p_i$ is a sum of commuting quasinilpotent elements, $f(a)p$ is quasinilpotent. Thus the set $\sigma = \{\mu_1, \ldots, \mu_k\}$ is a spectral set for a by Theorem 1.3, and each μ_i is an isolated spectral point of a.

To show that $p_i = p(\mu_i; a)$ in both cases, choose a set $\{\alpha_1, \ldots, \alpha_k\}$ of complex numbers disjoint from $\{\mu_1, \ldots, \mu_k\}$, and define c by (2.1). There is a function $e_i \in H(\sigma(a) \cup \{\alpha_1, \ldots, \alpha_k\})$ equal to 1 in a neighbourhood of μ_i and to 0 in a neighbourhood of $(\sigma(a) \setminus \{\mu_i\}) \cup \{\alpha_1, \ldots, \alpha_k\}$. By (2.2), e_i vanishes on $\sigma(c)$. By (2.4),

$$p(\mu_i; a) = e_i(a) = e_i(c)(e - p) + \sum_{j=1}^{k} e_i(\mu_j)p_j = p_i.$$

THEOREM 2.6. *A set $\{\mu_1, \ldots, \mu_k\}$ is spectral for an element $a \in A$ if and only if a admits a splitting*

$$a = c + b + \sum_{i=1}^{k}(\mu_i - \alpha_i)p_i, \tag{2.5}$$

where

(i) *$\{\alpha_1, \ldots, \alpha_k\}$ is a set disjoint from $\{\mu_1, \ldots, \mu_k\}$,*

(ii) *$p_1, \ldots, p_k$ are nonzero idempotents with $p_i p_j = 0$ if $i \neq j$,*

(iii) *$\mu_i \in \varrho(c)$ and $cp_i = \alpha_i p_i$ for all i,*

(iv) *b is quasinilpotent and $b(p_1 + \ldots + p_k) = b$,*

(v) *the elements* $c, b, p_1, \ldots, p_k$ *all commute.*

For each $i \in \{1, \ldots, k\}$, p_i *is then the spectral idempotent corresponding to* μ_i.

Proof. If a admits the splitting (2.5), then $(a - \mu_i e)p_i = bp_i$, which is a quasinilpotent element as p_i commutes with b. We can check that c satisfies (2.1). Then each μ_i is an isolated spectral point of a in view of (2.2).

The converse follows from Theorem 2.3 if we set $p_i = p(\mu_i; a)$, $b = \sum_{i=1}^k (a - \mu_i e)p_i$ and $c = \sum_{i=1}^k \alpha_i p_i + a(e - p)$.

If $\sigma(a) = \{\mu_1, \ldots, \mu_k\}$, then $p = e$. Theorem 2.6 then yields the following Banach algebra version of the Jordan form:

COROLLARY 2.7. *Let* $a \in A$. *Then* $\sigma(a) = \{\mu_1, \ldots, \mu_k\}$ *if and only if* a *admits a splitting*

$$a = b + \sum_{i=1}^{k} \mu_i p_i, \tag{2.6}$$

where $p_1, \ldots, p_k$ *form a complete set of idempotents*, b *is quasinilpotent*, *and* $bp_i = p_i b$ *for all* i.

3. Examples. We give three examples illustrating the theorems of the previous section.

EXAMPLE 3.1 (see [7] and [5]). Let D be the closed unit disc in the complex plane, D^0 the interior of D and ∂D the boundary of D. The following conditions on $a \in A$ are equivalent:

(i) The monothetic semigroup $\{a^n : n \in \mathbb{N}\}$ is relatively compact in A.
(ii) $\{a^n : n \in \mathbb{N}\}$ is weakly relatively compact in A.
(iii) $\sigma(a) \subset D$ and $\sigma(a) \cap \partial D$ is a finite (possibly empty) set of simple poles of a.
(iv) Either $r(a) < 1$ or a has a unique decomposition $a = \mu_1 p_1 + \ldots + \mu_k p_k + c$, where $\mu_i \in \partial D$, p_i are nonzero idempotents with $p_i p_j = 0$ if $i \neq j$, $cp_i = p_i c = 0$ and $\sigma(c) \subset D^0$.

The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are proved in [5].

(iii)$\Rightarrow$(iv). Let $\sigma(a) \cap \partial D = \{\mu_1, \ldots, \mu_k\}$ be simple poles of a. Set $p_i = p(\mu_i; a)$ and $\alpha_i = 0$, $i = 1, \ldots, k$. Then (2.5) holds with $b = \sum_{i=1}^k (a - \mu_i e)p_i = 0$ and $c = a(e - p)$, where $p = p_1 + \ldots + p_k$. The inclusion $\sigma(c) \subset D^0$ follows from (2.2).

(iv)$\Rightarrow$(i). Using the splitting (iii) for a, we deduce that

$$a^n = \mu_1^n p_1 + \ldots + \mu_k^n p_k + c^n,$$

where $\mu_1^n p_1 + \ldots + \mu_k^n p_k$ is a bounded sequence in a finite-dimensional subspace of A, and $c^n \to 0$. So the set $\{a^n : n \in \mathbb{N}\}$ has compact closure in A. The uniqueness of the decomposition follows from the fact that p_i is the spectral idempotent corresponding to μ_i (Theorem 2.6).

We give two more examples with a sketch of proofs.

EXAMPLE 3.2. Let H be the closed left half-plane of the complex plane, H^0 the interior and ∂H the boundary of H. The following conditions on $a \in A$ are equivalent:

(i) The semigroup $\{\exp(sa) : s \geq 0\}$ is relatively compact in A.
(ii) $\{\exp(sa) : s \geq 0\}$ is weakly relatively compact in A.
(iii) $\sigma(a) \subset H$ and $\sigma(a) \cap \partial H$ is a finite (possibly empty) set of simple poles of a.
(iv) Either $\sigma(a) \subset H^0$ or a has a unique decomposition $a = (\mu_1 + 1)p_1 + \ldots + (\mu_k + 1)p_k + c$, where $\mu_i \in \partial H$, p_i are nonzero idempotents with $p_i p_j = 0$ if $i \neq j$, $cp_i = c_i p = -p_i$ and $\sigma(c) \subset H^0$.

(i)$\Rightarrow$(ii)$\Rightarrow$(iii). This follows from Example 3.1 and the observation that the monothetic semigroup $\{w^n : n \in \mathbb{N}\}$, where $w = \exp(a)$, is relatively compact whenever $\{\exp(sa) : s \geq 0\}$ is.

(iii)$\Rightarrow$(iv). Let $\sigma(a) \cap \partial H = \{\mu_1, \ldots, \mu_k\}$ be a set of simple poles of a. Let $p_i = p(\mu_i; a)$, $\alpha_i = -1$ for $i = 1, \ldots, k$. Then (2.5) holds with $b = 0$, $c = -p + a(e - p)$, where $p = p_1 + \ldots + p_k$, and $\mu_i \in \varrho(c)$ (Theorem 2.6). The inclusion $\sigma(c) \subset H^0$ holds in view of (2.2).

(iv)$\Rightarrow$(i). If a admits the splitting (iii), then (2.1) holds with $\alpha_i = -1$ for all i. Setting $f(\lambda) = e^{s\lambda}$ in (2.3), we get

$$\exp(sa) = \exp(sc)(e - p) + \sum_{i=1}^{k} e^{s\mu_i} p_i, \quad s \geq 0,$$

where $p = p_1 + \ldots + p_k$, and where $\{\sum_{i=1}^k e^{s\mu_i} p_i : s \geq 0\}$ is a bounded set in a finite-dimensional subspace of A; also, $\exp(sc) \to 0$ (as $s \to \infty$) since $\sigma(c) \subset H^0$. Then $\{\exp(sa) : s \geq 0\}$ is a compact subset of A.

The elements u_m of the following example are of interest as $\lim_{m \to \infty} u_m$, if it exists, is a generalized inverse of a (see [8]).

EXAMPLE 3.3. Let M be the set $\{\lambda : |\lambda^2 - 1| \leq 1\}$, M^0 the interior of M and ∂M the boundary of M. For $a \in A$ define $u_0 = a$ and

$$u_m = \sum_{n=0}^{m} a(e - a^2)^n, \quad m = 1, 2, \ldots$$

Then the following conditions are equivalent:

(i) $\{u_m : m \in \mathbb{N}\}$ is relatively compact in A.

(ii) $\{u_m : m \in \mathbb{N}\}$ is weakly relatively compact in A.

(iii) $\sigma(a) \subset M$ and $\sigma(a) \cap \partial M$ is a finite (possibly empty) set of simple poles of a.

(iv) Either $\sigma(a) \subset M^0$ or a has a unique decomposition $a = (\mu_1 - 1)p_1 + \ldots + (\mu_k - 1)p_k + c$, where $\mu_i \in \partial M$, p_i are nonzero idempotents with $p_i p_j = 0$ for $i \neq j$, $cp_i = p_i c = p_i$, and $\sigma(c) \subset M^0$.

(i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(iii). If $w = e - a^2$, then the monothetic semigroup $\{w^m : m \in \mathbb{N}\}$ is weakly relatively compact as $w^m = e - au_m$. The result is then deduced from Example 3.1, the spectral mapping theorem applied to $g(\lambda) = 1 - \lambda^2$ and Theorem 2.5.

(iii)$\Rightarrow$(iv). Let $\sigma(a) \cap \partial M = \{\mu_1, \ldots, \mu_k\}$ be simple poles of a. Set $p_i = p(\mu_i; a)$ and $\alpha_i = 1$ for all i in Theorem 2.6. Then (2.5) holds with $b = 0$, $c = p + a(e - p)$, where $p = p_1 + \ldots + p_k$, and $\mu_i \in \varrho(c)$. So $\sigma(c) \subset M^0$ by (2.2).

(iv)$\Rightarrow$(i). If a admits the splitting (iii), then (2.1) holds with $\alpha_i = 1$ for all i. Let $f_m(\lambda) = \sum_{n=0}^{m} \lambda(1 - \lambda^2)^n$. By (2.4),

$$u_m = f_m(c)(e - p) + \sum_{i=1}^{k} f_m(\mu_i) p_i.$$

Furthermore, $f_m(c)(e-p) \to c^{-1}(e-p)$, and $\sum_{i=1}^{k} f_m(\mu_i)p_i$ lie in a bounded subset of a finite-dimensional subspace of A. Then the set $\{u_m - c^{-1}(e - p)\}$ has a compact closure, and so does $\{u_m\}$.

4. Decomposition theorems for operators. Let T be a bounded linear operator on a Banach space X. It is then well known [6, 13] that μ is an isolated spectral point of T if and only if X can be decomposed into a topological direct sum $X = M \oplus N$ such that $M \neq 0$, $T|M$ is quasinilpotent and $T|N$ is invertible. Mbekhta [10, 11] gave an explicit description of the subspaces M and N in terms of T:

$$M = H_0(T - \mu I), \qquad N = K(T - \mu I),$$

where

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\},$$

$$K(T) = \{x \in X : (\exists x_n \in X)(\forall n \in \mathbb{N})\, Tx_{n+1} = x_n,\quad Tx_1 = x,\ \sup_n \|x_n\|^{1/n} < \infty\}.$$

For a general spectral set of T we get the following result, which follows from Mbekhta's theorem applied to $f(T)$ and from Theorems 1.3 and 1.4.

THEOREM 4.1. *Let $f \in H(\sigma(T))$ and let $\sigma = \sigma(T) \cap f^{-1}(\{0\})$. Then the following are equivalent:*

(i) *X is a topological sum $X = H_0(f(T)) \oplus K(f(T))$.*

(ii) *There is a projection P commuting with T such that $f(T)P$ is quasinilpotent and $P + f(T)$ is invertible (or $P \in \overline{\mathcal{F}(T)}$).*

(iii) *The set σ is spectral for T, and*

$$R(P(\sigma;T)) = H_0(f(T)), \qquad N(P(\sigma;T)) = K(f(T)). \tag{4.1}$$

(iv) *If $\mu \in \sigma$ is a zero of f of a finite order, then μ is an isolated spectral point of T.*

The theorem reduces to the characterization of an isolated spectral point μ of T when we take $f(\lambda) = \lambda - \mu$. As in Schmoeger [12] it is possible to relax condition (i) in the preceding theorem by assuming that only $K(f(T))$ in the direct sum is closed. By Koliha [9], (i) can be further relaxed to

(i′) $X = M \oplus N$, where M, N are invariant under T, $N(f(T)) \subset M \subset H_0(f(T))$, N is closed and $N \subset R(f(T))$.

Added in proof. The author and Pak Wai Poon proved that it is enough to assume in (i) that only one of the spaces $H_0(f(T))$, $K(f(T))$ is closed.

The next theorem is a special case of the preceding decomposition. It generalizes a well-known result for operators (see [3, Theorem 2.23]) in two ways: instead of a polynomial it considers an arbitrary holomorphic function f, and it adds a new condition (ii). If $f(\lambda) = (\lambda - \mu)^m$, the theorem can be used to characterize a pole of the resolvent $R(\lambda;T)$ as a point μ for which the ascent and descent of $T - \mu I$ are both finite. See [2, 1.54], [13, V.10.2] and [6, Proposition 50.2]. We give a proof of the result independent of Theorem 4.1, a proof which is shorter and we hope more transparent than proofs based on the restriction of operators to subspaces.

THEOREM 4.2. *Let $f \in H(\sigma(T))$ and let $\sigma = \sigma(T) \cap f^{-1}(\{0\})$. Then the following are equivalent:*

(i) *$X = N(f(T)) \oplus R(f(T))$.*

(ii) *There is a projection P commuting with T such that $f(T)P = 0$ and $P + f(T)$ is invertible (or $P \in \overline{\mathcal{F}(T)}$).*

(iii) *The set σ is spectral for T, and*

$$R(P(\sigma;T)) = N(f(T)), \qquad N(P(\sigma;T)) = R(f(T)). \tag{4.2}$$

(iv) *If $\mu \in \sigma$ is a zero of f of a finite order m, then μ is a pole of T of order at most m.*

Proof. (i)$\Rightarrow$(ii). If (i) holds, then $R(f(T))$ is closed by Heuser [6, Proposition 36.2]. Let P be the projection of X onto $N(f(T))$ associated with the direct sum. Then $P \in L(X)$ and P commutes with T since $N(f(T))$ and $R(f(T))$ are invariant under T. Also, $f(T)P = 0$.

Suppose that $(P + f(T))x = 0$. Then $Px = w = -f(T)x$, and $w = 0$ as $w \in N(f(T)) \cap R(f(T))$. So $x \in N(f(T))$, and $x = Px = 0$. Thus $P + f(T)$ is injective.

Let $x \in X$. There is $u \in X$ with $x = Px + f(T)u$. Then

$$(P + f(T))(Px + (I - P)u) = x,$$

and $P + f(T)$ is in fact bijective. This proves (ii).

(ii)⇒(iii). By Theorem 1.3, $\sigma = \sigma(T) \cap f^{-1}(\{0\})$ is a spectral set for T and $P = P(\sigma; T)$. We prove that

$$N(f(T)) = R(P), \quad R(f(T)) = N(P). \tag{4.3}$$

From $f(T)P = 0$ it follows that $R(P) \subset N(f(T))$ and $R(f(T)) \subset N(P)$. Write $S = P + f(T)$. Then

$$PS^{-1} + S^{-1}f(T) = I = f(T)S^{-1} + S^{-1}P.$$

This implies that $N(f(T)) \subset R(P)$ and $N(P) \subset R(f(T))$.

(iii)⇒(i) is clear.

The proof of (ii)⇔(iv) is fairly routine, and can be based on Theorems 1.3 and 1.4.

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References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.
[2] H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, London, 1978.
[3] N. Dunford, *Spectral theory I. Convergence to projections*, Trans. Amer. Math. Soc. 54 (1943), 185–217.
[4] N. Dunford and J. T. Schwartz, *Linear Operators I*, Interscience, New York, 1957.
[5] J. E. Galé, *Weakly compact homomorphisms and semigroups in Banach algebras*, J. London Math. Soc. 45 (1992), 113–125.
[6] H. Heuser, *Functional Analysis*, Wiley, New York, 1982.
[7] M. A. Kaashoek and T. T. West, *Locally Compact Semi-Algebras with Applications to Spectral Theory of Positive Operators*, North-Holland Math. Stud. 9, North-Holland, Amsterdam, 1974.
[8] J. J. Koliha, *Convergence of an operator series*, Aequationes Math. 16 (1977), 31–35.
[9] —, *Isolated spectral points*, Proc. Amer. Math. Soc. 124 (1996), 3417–3424.
[10] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. 29 (1987), 159–175.

[11] M. Mbekhta, *Sur la théorie spectrale locale et limite des nilpotents*, Proc. Amer. Math. Soc. 110 (1990), 621–631.
[12] C. Schmoeger, *On isolated points of the spectrum of a bounded linear operator*, ibid. 117 (1993), 715–719.
[13] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, Wiley, New York, 1980.

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