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# Moment inequalities for sums of certain independent symmetric random variables

by

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**Abstract.** This paper gives upper and lower bounds for moments of sums of independent random variables $(X_k)$ which satisfy the condition $P(|X|_k \ge t) = \exp(-N_k(t))$, where $N_k$ are concave functions. As a consequence we obtain precise information about the tail probabilities of linear combinations of independent random variables for which $N(t) = |t|^r$ for some fixed $0 < r \le 1$. This complements work of Gluskin and Kwapień who have done the same for convex functions $N$.

**1. Introduction.** Let $X_1, X_2, \ldots$ be a sequence of independent random variables. In this paper we will be interested in obtaining estimates on the $L_p$-norm of sums of $(X_k)$, i.e. we will be interested in the quantity

$$\Big\|\sum X_k\Big\|_p = \Big(E\Big|\sum X_k\Big|^p\Big)^{1/p}, \qquad 2 \le p < \infty.$$

Although using standard symmetrization arguments our results carry over to more general cases, we will assume for the sake of simplicity that $X_k$ have symmetric distributions, i.e. $P(X_k \le t) = P(-X_k \le t)$ for all $t \in \mathbb{R}$.

Let us start by considering the case of linear combinations of identically distributed, independent (i.i.d.) random variables, that is, $X_k = a_k Y_k$, where $Y, Y_1, \ldots$ are i.i.d. We can assume without loss of generality that all $a_k$'s are nonnegative. Also, since the $Y_k$'s are i.i.d., we can rearrange the terms of $(a_k)$ arbitrarily without affecting the sum $\sum a_k Y_k$. Therefore, for notational convenience, we will adopt the following convention throughout this paper: whenever we are dealing with a sequence of real numbers we will *always*

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assume that its terms are nonnegative and form a nonincreasing sequence. In other words, we identify a sequence $(a_k)$ with the decreasing rearrangement of $(|a_k|)$.

Of course, a huge number of inequalities concerning the $L_p$-norm of a sum $\sum a_k Y_k$ are known. Let us recall two of them. The first, called Khinchin's inequality, deals with the Rademacher sequence, i.e. $Y=\varepsilon$, where $\varepsilon$ takes values $\pm 1$, each with probability 1/2. For $p\ge 2$ it can be formulated as follows (see e.g. Ledoux and Talagrand (1991, Lemma 4.1)): there exists an absolute constant $K$ such that

$$\Big(\sum a_k^2\Big)^{1/2} \le \Big\|\sum a_k\varepsilon_k\Big\|_p \le K\sqrt{p}\Big(\sum a_k^2\Big)^{1/2}.$$

The value of the smallest constant that can be put in place of $K\sqrt{p}$ is known (see Haagerup (1982)). The second inequality was proved by Rosenthal (1970), and in our generality says that for $2\le p<\infty$ there exists a constant $B_p$ such that for a sequence $a=(a_k)\in\ell_2$,

$$\max\{\|a\|_2\|Y\|_2,\|a\|_p\|Y\|_p\} \le \Big\|\sum a_kY_k\Big\|_p \le B_p\max\{\|a\|_2\|Y\|_2,\|a\|_p\|Y\|_p\},$$

where $\|a\|_s$ denotes the $\ell_s$-norm of the sequence $a$. The best possible constant $B_p$ is known (cf. Utev (1985) for $p\ge 4$, and Figiel, Hitczenko, Johnson, Schechtman and Zinn (1995) for $2<p<4$). We refer the reader to the latter paper and references therein for more information on the best constants in Rosenthal's inequality. For our purpose, we only note that Johnson, Schechtman and Zinn (1983) showed that $B_p\le Kp/\log p$ for some absolute constant $K$, and that up to the value of $K$, this bound cannot be further improved. This and other related results were extended in various directions by Pinelis (1994). When specialized to our generality, his inequality reads as

$$\max\{\|a\|_2\|Y\|_2,\|a\|_p\|Y\|_p\} \le \Big\|\sum a_kY_k\Big\|_p \le \inf_{1\le c\le p}\max\{\sqrt{c}\,e^{p/c}\|a\|_2\|Y\|_2, c\|a\|_p\|Y\|_p\}.$$

A common feature of these two inequalities is that they express the $L_p$-norm of a sum $\sum a_kY_k$ in terms of norms of individual terms $a_kY_k$. This is very useful and both inequalities found numerous applications. However, upper and lower bounds in these inequalities differ by a factor that depends on $p$, and sometimes are quite insensitive to the structure of the coefficient sequence. (Rosenthal's inequality is also insensitive to the common distribution of $Y_k$'s.) Consider, for example, the coefficient sequence $a_k=1/k$, $k\ge 1$. Then the only information on $\|\sum a_k\varepsilon_k\|_p$ given by Khinchin's inequality is that it is essentially between 1 and $\sqrt{p}$. Practically the same conclusion is given for two quite different sequences, namely $a_k=2^{-k}$, $k\ge 1$, and $a_k=1/\sqrt{n}$ or 0 according to whether $k\le n$ or $k>n$, $n\in\mathbb{N}$. (The "true values" of $\|\sum a_k\varepsilon_k\|_p$ are rather different in each of these three cases, and are of order $\log(1+p)$, 1, and $\sqrt{p\wedge n}$, respectively.)



From this point of view, it is natural to ask whether more precise information on the size of $\|\sum a_kY_k\|_p$ can be obtained. Although, in general, the answer to this question may be difficult, there are cases for which there is a satisfactory answer. First of all, if $Y_k$ is a standard Gaussian random variable, then $\|\sum a_kY_k\|_p=\|a\|_2\|Y\|_p$ so that

$$c\sqrt{p}\,\|a\|_2 \le \Big\|\sum a_kY_k\Big\|_p \le C\sqrt{p}\,\|a\|_2,$$

for some absolute constants $c$ and $C$. Next consider the case when $(Y_k)=(\varepsilon_k)$, a Rademacher sequence. We have

$$c\Big\{\sum_{k\le p}a_k+\sqrt{p}\Big(\sum_{k>p}a_k^2\Big)^{1/2}\Big\} \le \Big\|\sum a_k\varepsilon_k\Big\|_p \le C\Big\{\sum_{k\le p}a_k+\sqrt{p}\Big(\sum_{k>p}a_k^2\Big)^{1/2}\Big\}.$$

(Recall that according to our convention, $a_1\ge a_2\ge\ldots\ge 0$.) The above inequality has been established in Hitczenko (1993), although the proof drew heavily on a technique from Montgomery-Smith (1990). (In fact, the inequality for Rademacher variables can be deduced from the results obtained in the latter paper.) The next step was done by Gluskin and Kwapień (1995) who dealt with the case of random variables with logarithmically concave tails. To describe their result precisely, suppose that $Y$ is a symmetric random variable such that for $t>0$ one has $P(|Y|\ge t)=\exp(-N(t))$, where $N$ is an Orlicz function (i.e. convex, nondecreasing, and $N(0)=0$). Recall that if $M$ is an Orlicz function and $(a_k)$ is a sequence of scalars then the Orlicz norm of $(a_k)$ is defined by $\|(a_k)\|_M=\inf\{u>0:\sum M(a_k/u)\le 1\}$. Let $N'$ be the function conjugate to $N$, i.e. $N'(t)=\sup\{st-N(s):s>0\}$, and put $M_p(t)=N'(pt)/p$. Then Gluskin and Kwapień (1995) proved that

$$c\{\|(a_k)_{k\le p}\|_{M_p}+\sqrt{p}\,\|(a_k)_{k>p}\|_2\} \le \Big\|\sum a_kY_k\Big\|_p \le C\{\|(a_k)_{k\le p}\|_{M_p}+\sqrt{p}\,\|(a_k)_{k>p}\|_2\}.$$

In the special case $N(t)=|t|^r$, $r\ge 1$, this gives

$$c\{\|(a_k)_{k\le p}\|_{r'}\|Y\|_p+\sqrt{p}\|(a_k)_{k>p}\|_2\|Y\|_2\} \le \Big\|\sum a_kY_k\Big\|_p \le C\{\|(a_k)_{k\le p}\|_{r'}\|Y\|_p+\sqrt{p}\|(a_k)_{k>p}\|_2\|Y\|_2\},$$

where $1/r'+1/r=1$, and $c$, $C$ are absolute constants. From now on, we will refer to random variables corresponding to $N(t)=|t|^r$ as *symmetric Weibull random variables with parameter* $r$.

Now let us describe the results of this paper. We will say that a symmetric random variable has a *logarithmically convex tail* if $P(|X| \ge t) = \exp(-N(t))$ for $t \ge 0$, where $N : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave function with $N(0) = 0$. We will show the following result.

THEOREM 1.1. *There exist absolute constants $c$ and $K$ such that if $(X_k)$ is a sequence of independent random variables with logarithmically convex tails, then for each $p \ge 2$,*

$$c\Big\{\Big(\sum \|X_k\|_p^p\Big)^{1/p} + \sqrt{p}\Big(\sum \|X_k\|_2^2\Big)^{1/2}\Big\} \le \Big\|\sum X_k\Big\|_p \le K\Big\{\Big(\sum \|X_k\|_p^p\Big)^{1/p} + \sqrt{p}\Big(\sum \|X_k\|_2^2\Big)^{1/2}\Big\}.$$

This result includes the special case of linear combinations of symmetric Weibull random variables with fixed parameter $r \le 1$, and in this case we are also able to obtain information about the tail distributions of the sums. In fact, we will give a second proof of the moment inequalities in this special case. We include this second proof, because we believe that the methods will be of great interest to the specialists.

Let us mention that the main difficulty is to obtain tight upper bounds. Once this is done the lower bounds are rather easy to prove. For this reason, the main emphasis of this paper is on the proofs of upper bounds.

We will now outline the main steps in both proofs of the upper bounds, and at the same time describe the organization of this paper. In Section 2, we describe a weaker version of the result that depends upon a certain moment condition. In particular, if we specialize to linear combinations of symmetric Weibull random variables, we obtain constants that become unbounded as the parameter $r$ tends to zero (but are still universal in $p \ge 2$). The proof is based on hypercontractive methods that were suggested to us by Kwapień. In Section 3 we will show that moments of sums of symmetric random variables with logarithmically convex tails are dominated by the linear combinations of suitably chosen multiples of standard exponential random variables by independent symmetric 3-valued random variables. We are then able to obtain the main result.

The main idea of the second proof is to reduce the problem to the case of an i.i.d. sum. This is done in Section 4. In Section 5 we deal with the problem of finding an upper bound for the $L_p$-norm of a sum of i.i.d. random variables. We accomplish this by estimating from above a decreasing rearrangement of the sum in terms of a decreasing rearrangement of an individual summand. In Section 6 we apply the results from the preceding two sections to obtain an upper bound on the $L_p$-norm of $\sum a_k X_k$, where $(X_k)$ are i.i.d. symmetric with $P(|X| > t) = \exp(-t^r)$, $0 < r \le 1$.



**2. Random variables satisfying the moment condition.** In this section we will prove a variant of Theorem 1.1, where the random variables satisfy a certain moment condition. We use hypercontractive methods, similar to those in Kwapień and Szulga (1991). This approach was suggested to us by S. Kwapień. It should be emphasized that if the result is specialized to symmetric Weibull random variables then the constant $B$ in Theorem 2.2 below depends on $r$. Let us begin with the following.

DEFINITION 2.1. We say that a random variable $X$ *satisfies the moment condition* if it is symmetric, all moments of $X$ are finite and there exist positive constants $b, c$ such that for all even natural numbers $n \ge k \ge 2$,

$$\frac{1}{b}\cdot\frac{n}{k} \le \frac{\|X\|_n}{\|X\|_k} \le c^{n-k}.$$

It is easy to check that the existence of such a $c$ is equivalent to the finiteness of

$$\sup_{n\in\mathbb{N}} (E|X|^n)^{1/n^2}$$

or

$$\limsup_{t\to\infty} (\ln t)^{-2} \ln P(|X| > t) < 0.$$

Here is the main result of this section.

THEOREM 2.2. *If independent real random variables $X_1, X_2, \ldots$ satisfy the moment condition with the same constants $b$ and $c$ then there exist positive constants $A$ and $B$ such that for any $p \ge 2$, any natural number $m$, and $S = \sum_{i=1}^m X_i$,*

$$A\Big(\sqrt{p}\,\|S\|_2 + \Big(\sum_{k=1}^m \|X_k\|_p^p\Big)^{1/p}\Big) \le \|S\|_p \le B\Big(\sqrt{p}\,\|S\|_2 + \Big(\sum_{k=1}^m \|X_k\|_p^p\Big)^{1/p}\Big).$$

We will prove a lemma first.

LEMMA 2.3. *If $X$ satisfies the moment condition then there exists a positive constant $K$ such that for any $q \ge 1$ and any real $x$,*

$$(E|x+X|^{2q})^{1/q} \le 4eEX^2q + (|x|^{2q} + K^{2q}E|X|^{2q})^{1/q}.$$

Proof. Put $K = \sqrt{2}\,ebc^4 + 1$. Let $n$ be the least natural number greater than $q$. We consider two cases:

(i) $|x| \le (K-1)\|X\|_{2q}$. Then $\|x + X\|_{2q} \le |x| + \|X\|_{2q} \le K\|X\|_{2q}$.

(ii) $|x| \ge (K-1)\|X\|_{2q} \ge \frac{K-1}{c^2}\|X\|_{2n}$. Then

$$E(x+X)^{2n}-x^{2n} = \sum_{k=1}^{n} \frac{2n(2n-1)\ldots(2n-2k+1)}{(2k)!} x^{2n-2k} EX^{2k}$$
$$\le 4n^2x^{2n-2}\sum_{k=1}^{n}\frac{(2n)^{2k-2}\|X\|_{2k}^{2k}}{(2k)!x^{2k-2}}$$
$$\le 4n^2x^{2n-2}\sum_{k=1}^{n}\|X\|_{2k}^{2}\frac{(2n)^{2k-2}}{(2k)!\left(\frac{K-1}{c^2}\right)^{2k-2}}\left(\frac{\|X\|_{2k}}{\|X\|_{2n}}\right)^{2k-2}$$
$$\le 4n^2x^{2n-2}\sum_{k=1}^{n}(\|X\|_2c^{2k-2})^2\frac{(2n)^{2k-2}\left(\frac{bk}{n}\right)^{2k-2}}{(2k)!\left(\frac{K-1}{c^2}\right)^{2k-2}}$$
$$= 4n^2x^{2n-2}\sum_{k=1}^{n}\frac{EX^2}{(2k)!}\left(\frac{2bc^4k}{K-1}\right)^{2k-2}$$
$$\le n^2x^{2n-2}eEX^2\sum_{k=1}^{n}\left(\frac{ebc^4}{K-1}\right)^{2k-2} \le 2eEX^2n^2x^{2n-2}.$$

Hence

$$\|x+X\|_{2q} \le \|x+X\|_{2n} \le \sqrt{x^2+2eEX^2n} \le \sqrt{x^2+4eEX^2q}.$$

This completes the proof of Lemma 2.3.

P r o o f o f T h e o r e m 2.2. We begin with the left hand side inequality. The inequality

$$\|S\|_p \ge \Big(\sum_{k=1}^{m} E|X_k|^p\Big)^{1/p}$$

follows from Rosenthal's inequality (Rosenthal (1970)), or by an easy induction argument and the inequality $2(|x|^p+|y|^p) \le |x+y|^p+|x-y|^p$. To complete the proof of this part we will show that if $X$ is a random variable satisfying the left hand side inequality in Definition 2.1 with constant $b$ then

$$\|S\|_p \ge \frac{1}{2\sqrt{2}\,b}\|S\|_2\sqrt{p}.$$

To this end let $G$ be a standard Gaussian random variable. Then for every even natural number $n$ we have

$$\frac{\|X\|_2}{2b}\|G\|_n = ((n-1)!!)^{1/n}\frac{\|X\|_2}{2b} \le n\frac{\|X\|_2}{2b} \le \|X\|_n.$$

Hence, by the binomial formula, for every even natural number $n$ and any real number $x$,

$$E\left|x+\frac{\|X\|_2}{2b}G\right|^n \le E|x+X|^n.$$



If $G_k$'s are i.i.d. copies of $G$ then, by an easy induction argument, we get

$$E\left|x+\frac{\|S\|_2}{2b}G\right|^n = E\left|x+\sum\frac{\|X_k\|_2}{2b}G_k\right|^n \le E|x+S|^n.$$

Letting $n$ be the largest even number not exceeding $p$, and putting $x=0$, we obtain

$$\|S\|_p \ge \|S\|_n \ge \frac{\|S\|_2}{2b}\|G\|_n \ge \frac{\|S\|_2}{2b}\sqrt{n} \ge \frac{1}{2\sqrt{2}\,b}\|S\|_2\sqrt{p},$$

which completes the proof of the first inequality of Theorem 2.2.

To prove the second inequality we will proceed by induction. For $N=1,\ldots,m$, let $S_N=\sum_{k=1}^N X_k$, $h_N = K(\sum_{k=1}^N E|X_k|^p)^{1/p}$, and $q=p/2$. We will show that for any real $x$,

$$(E|x+S_N|^{2q})^{1/q} \le 4eqES_N^2 + (|x|^{2q}+h_N^{2q})^{1/q}.$$

For $N=1$ this is just Lemma 2.3. Assume that the above inequality is satisfied for $N<m$. Let $E'=E(\cdot\mid X_1,\ldots,X_N)$, $E''=E(\cdot\mid X_{N+1})$. Then

$$(E|x+S_{N+1}|^{2q})^{1/q} = (E''E'|(x+S_N)+X_{N+1}|^{2q})^{1/q}$$
$$\le (E'(4eqEX_{N+1}^2+(|x+S_N|^{2q}+K^{2q}E|X_{N+1}|^{2q})^{1/q})^q)^{1/q}$$
$$\le 4eqEX_{N+1}^2+(E'|x+S_N|^{2q}+K^{2q}E|X_{N+1}|^{2q})^{1/q} \qquad \text{(by Lemma 2.3)}$$
$$\le 4eqEX_{N+1}^2+((4eqES_N^2+(|x|^{2q}+h_N^{2q})^{1/q})^q+K^{2q}E|X_{N+1}|^{2q})^{1/q}$$
$$\text{(by the inductive assumption)}$$
$$= 4eqEX_{N+1}^2+\|(4eqES_N^2+(|x|^{2q}+h_N^{2q})^{1/q}, K^2(E|X_{N+1}|^{2q})^{1/q})\|_q$$
$$\le 4eqEX_{N+1}^2+\|(4eqES_N^2,0)\|_q$$
$$+\|((|x|^{2q}+h_N^{2q})^{1/q}, K^2(E|X_{N+1}|^{2q})^{1/q})\|_q$$
$$= 4eqES_{N+1}^2+(|x|^{2q}+h_{N+1}^{2q})^{1/q}.$$

Our induction is finished. Taking $N=m$ and $x=0$ we have

$$\|S\|_p \le \sqrt{2epES^2+h_m^2} \le \sqrt{2e}\,\|S\|_2\sqrt{p} + K\Big(\sum_{k=1}^m E|X_k|^p\Big)^{1/p}.$$

This completes the proof of Theorem 2.2.

**3. Random variables with logarithmically convex tails.** The aim of this section is to prove the main result, Theorem 1.1. Although random variables with logarithmically convex tails do not have to satisfy the moment condition we have the following.

PROPOSITION 3.1. *Let $\Gamma$ be an exponential random variable with density $e^{-x}$ for $x \ge 0$. If $X$ is a symmetric random variable with logarithmically* *convex tails then for any $p \ge q > 0$,*

$$\frac{\|X\|_p}{\|\Gamma\|_p} \ge \frac{\|X\|_q}{\|\Gamma\|_q}.$$

*In particular, random variables with logarithmically convex tails satisfy the first inequality of Definition 2.1 with constant $b = e/\sqrt{2}$.*

Proof. Let $F = N^{-1}$. Then $|X|$ has the same distribution as $F(\Gamma)$. Since $N$ is a concave function and $N(0) = 0$, it follows that $N(x)/x$ is nonincreasing. Therefore, $f(x) = F(x)/x$ is nondecreasing. By a standard inequality for the measure $x^p e^{-x} dx$ we have

$$\left(\frac{\int_0^\infty f(x)^p x^p e^{-x}\,dx}{\int_0^\infty x^p e^{-x}\,dx}\right)^{1/p} \ge \left(\frac{\int_0^\infty f(x)^q x^p e^{-x}\,dx}{\int_0^\infty x^p e^{-x}\,dx}\right)^{1/q},$$

so it suffices to prove that

$$\frac{\int_0^\infty f(x)^q x^p e^{-x}\,dx}{\int_0^\infty x^p e^{-x}\,dx} \ge \frac{\int_0^\infty f(x)^q x^q e^{-x}\,dx}{\int_0^\infty x^q e^{-x}\,dx}.$$

Since $f(x)^q$ is a nondecreasing function, it is enough to show that

$$\frac{\int_a^\infty x^p e^{-x} dx}{\Gamma(p+1)} \ge \frac{\int_a^\infty x^q e^{-x} dx}{\Gamma(q+1)}, \quad a \ge 0.$$

Let

$$h(a) = \frac{\int_a^\infty x^p e^{-x} dx}{\Gamma(p+1)} - \frac{\int_a^\infty x^q e^{-x} dx}{\Gamma(q+1)}.$$

Then $h(0) = 0$ and $\lim_{a\to\infty} h(a) = 0$. Since $h'(a)$ is positive on $(0, a_0)$, and negative on $(a_0, \infty)$, where $a_0 = (\Gamma(p+1)/\Gamma(q+1))^{1/(p-q)}$, we conclude that $h(a) \ge 0$. To see that $b = e/\sqrt{2}$, notice that the sequence $(n/\|\Gamma\|_n) = ((n^n/n!)^{1/n})$ is increasing to $e$. Therefore, since $2/\|\Gamma\|_2 = \sqrt{2}$,

$$\frac{\|X\|_n}{\|X\|_k} \ge \frac{\|\Gamma\|_n}{\|\Gamma\|_k} \ge \frac{n}{k} \cdot \frac{\sqrt{2}}{e}.$$

This completes the proof.

Since the left hand side inequality in Theorem 1.1 follows by exactly the same argument as in Theorem 2.3, we will concentrate on the right hand side inequality. We first establish a comparison result which may be of its own interest. Let $\Gamma$ be an exponential random variable with mean 1. For a random variable $X$ with logarithmically convex tails, and $p > 2$, we denote by $\mathcal{E}_p(X)$ a random variable distributed like $a\Theta\Gamma$, where $\Theta$ and $\Gamma$ are independent, $\Theta$ is symmetric, 3-valued (i.e. $P(\Theta = \pm 1) = \alpha/2$, $P(\Theta = 0) = 1-\alpha$), and $a$, $\alpha$ are chosen so that $\|X\|_2 = \|\mathcal{E}_p(X)\|_2$ and $\|X\|_p = \|\mathcal{E}_p(X)\|_p$. Proposition 3.1 guarantees that such $a$ and $\alpha$ exist. We begin with the following lemma.



LEMMA 3.2. *There exists an absolute constant $c$ such that for any random variable with logarithmically convex tails $X$ and for every $2 \le q < p < \infty$ one has*

$$\|X\|_q \le c\|\mathcal{E}_p(X)\|_q.$$

Proof. Replacing $X$ by $X/a$ we can assume $a = 1$. Assume first that $4 \le q \le p/2$. We have

$$\|X\|_q^q = q\int_0^\infty t^{q-1} e^{-N(t)}\,dt = q\Big(\int_0^2 + \int_2^p + \int_p^\infty\Big) t^{q-1} e^{-N(t)}\,dt.$$

We will estimate each integral separately. By Markov's inequality, for every $t > 0$ we have

$$(*) \qquad t^p e^{-N(t)} \le \|X\|_p^p = \|\Theta\|_p^p \|\Gamma\|_p^p.$$

Hence,

$$\int_p^\infty t^{q-1} e^{-N(t)}\,dt \le \|\Theta\|_p^p \|\Gamma\|_p^p \int_p^\infty t^{q-1-p}\,dt \le \|\Theta\|_p^p \Gamma(p+1)\frac{p^{q-p}}{p-q}$$
$$\le \|\Theta\|_q^q \frac{2^p p^q}{e^p} \le K^q \|\Theta\|_q^q q^q \le K^q \|\Theta\|_q^q \|\Gamma\|_q^q.$$

We estimate the first integral in a similar way; since for every $t > 0$,

$$(\ast\atop\ast) \qquad t^2 e^{-N(t)} \le \|\Theta\|_2^2 \|\Gamma\|_2^2,$$

we get

$$\int_0^2 t^{q-1} e^{-N(t)}\,dt \le \|\Theta\|_2^2 \|\Gamma\|_2^2 \int_0^2 t^{q-3}\,dt \le C^q \|\Theta\|_2^2 \|\Gamma\|_q^q = C^q \|\Theta\|_q^q \|\Gamma\|_q^q.$$

It remains to estimate the middle integral. Notice that $(*)$ with $t = p$ and $(\ast\atop\ast)$ with $t = 2$ imply that

$$N(p) \ge p\ln(p/\|\Gamma\|_p) - \ln\alpha \ge p\ln(e/2) - \ln\alpha$$

and

$$N(2) \ge 2\ln(e/2) - \ln\alpha.$$

Hence, by concavity of $N$, we infer that $N(t) \ge t\ln(e/2) - \ln\alpha$ for $2 \le t \le p$. Consequently,

$$\int_2^p t^{q-1} e^{-N(t)} dt \le \alpha \int_0^\infty t^{q-1} e^{-t\ln(e/2)}\,dt \le K^q \|\Theta\|_q^q \|\Gamma\|_q^q.$$

Now consider the case $2 \le q < 4$. Since for $q < p$, $\|\Gamma\|_q \ge kq/p\|\Gamma\|_p$ for some absolute constant $k > 0$, and for $\Theta$ Hölder's inequality is in fact equality, we obtain the following. If $0 < s < 1$ is chosen so that $1/q = s/(2q) + (1-s)/2$, then

$$\|\Theta\Gamma\|_q = \|\Theta\|_{2q}^s\|\Gamma\|_q^s\|\Theta\|_2^{1-s}\|\Gamma\|_q^{1-s} \geq \|\Theta\|_{2q}^s\Big(\frac{k}{2}\|\Gamma\|_{2q}\Big)^s\|\Theta\|_2^{1-s}\|\Gamma\|_2^{1-s}$$
$$\geq \frac{k}{2}\|\Theta\Gamma\|_{2q}^s\|\Theta\Gamma\|_2^{1-s} \geq \frac{k}{2}(\|X\|_{2q}/c_1)^s\|X\|_2^{1-s} \geq \frac{k}{2}c_1\|X\|_q.$$

Here $c_1$ denotes an absolute constant obtained in the first part of proof (as $2q \geq 4$). Since the similar argument (with $1/q = s/p + (1-s)/2$) works for $p/2 \leq q \leq p$, the proof is complete.

In the remainder of this paper we will assume that if $(X_k)$ are independent, then the corresponding variables $\mathcal{E}_k$ are independent, too. We have the following.

PROPOSITION 3.3. *There exists an absolute constant $K$ such that if $(X_k)$ is a sequence of independent random variables with logarithmically convex tails and for $p > 2$, $(\mathcal{E}_k) = (\mathcal{E}_p(X_k))$ is the corresponding sequence of $(\Theta_k\Gamma_k)$, then*

$$\Big\|\sum X_k\Big\|_p \leq K\Big\|\sum \mathcal{E}_k\Big\|_p.$$

Proof. Let $q \in \mathbb{N}$ be an integer, $p/2 \leq 2q < p$. It follows from Kwapień and Woyczyński (1992, Proposition 1.4.2) (cf. Hitczenko (1994, Proposition 4.1) for details) that if $2q \leq p$ and $(Z_k)$ is any sequence of independent, symmetric random variables, then

$$\Big\|\sum Z_k\Big\|_p \leq K\frac{p}{2q}\Big\{\Big\|\sum Z_k\Big\|_{2q} + \|\max|Z_k|\|_p\Big\}$$
$$\leq K\frac{p}{2q}\Big\{\Big\|\sum Z_k\Big\|_{2q} + \Big(\sum\|Z_k\|_p^p\Big)^{1/p}\Big\}.$$

Therefore,

$$\Big\|\sum X_k\Big\|_p \leq K\Big\{\Big\|\sum X_k\Big\|_{2q} + \Big(\sum\|X_k\|_p^p\Big)^{1/p}\Big\}.$$

It follows from the binomial formula and Lemma 3.2 that $\|\sum X_k\|_{2q} \leq c\|\sum \mathcal{E}_k\|_{2q}$. Hence we obtain

$$\Big\|\sum X_k\Big\|_p \leq K\Big\{\Big\|\sum \mathcal{E}_k\Big\|_{2q} + \Big(\sum\|\mathcal{E}_k\|_p^p\Big)^{1/p}\Big\} \leq K\Big\|\sum \mathcal{E}_k\Big\|_p,$$

as required.

Remark 3.4. The above proposition is similar in spirit to a result of Utev (1985), who proved that if $p > 4$, $(Y_k)$ is a sequence of independent symmetric random variables, and $(Z_k)$ is a sequence of independent symmetric 3-valued random variables whose second and $p$th moments are the same as $Y_k$'s, then $\|\sum Y_k\|_p \leq \|\sum Z_k\|_p$.

We do not know what is the best constant $K$ in Proposition 3.3. We do know, however, that if $p$ is an even number, then $K \leq e$.

In order to finish the proof of Theorem 1.1 we need one more lemma.



LEMMA 3.5. *There exists a constant $K$ such that if $\Theta$ is a symmetric 3-valued random variable with $P(|\Theta| = 1) = \delta = 1 - P(\xi = 0)$, independent of $\Gamma$, then for every $q \geq 1$ and $x \in \mathbb{R}$ we have*

$$(E|x + \Theta\Gamma|^{2q})^{1/q} \leq eqE(\Theta\Gamma)^2 + (|x|^{2q} + K^{2q}E|\Theta\Gamma|^{2q})^{1/q}.$$

Proof. We need to prove that

$$(|x|^{2q} + \delta(E|x+\Gamma|^{2q} - |x|^{2q}))^{1/q} \leq 2eq\delta + (|x|^{2q} + \delta K^{2q}E\Gamma^{2q})^{1/q}.$$

Set for simplicity $A = E|x+\Gamma|^{2q} - |x|^{2q}$ and $B = K^{2q}E\Gamma^{2q}$. Since the case $A \leq B$ is trivial, assume $A > B$. It suffices to show that

$$\frac{\phi(A\delta) - \phi(B\delta)}{\delta} \leq 2eq,$$

where $\phi(t) = (x^{2q} + t)^{1/q}$. Since $\phi$ is a concave function, the left hand side above is decreasing in $\delta$, so it suffices to check that

$$2eq \geq \lim_{\delta\to 0}\frac{\phi(A\delta) - \phi(B\delta)}{\delta} = A\phi'(0) - B\phi'(0) = \frac{1}{q}(A-B)x^{2-2q}.$$

But this is equivalent to

$$E|x+\Gamma|^{2q} - x^{2q} \leq 2eq^2x^{2q-2} + K^{2q}E\Gamma^{2q} = eq^2x^{2q-2}E\Gamma^2 + K^{2q}E\Gamma^{2q}.$$

The latter inequality follows from the proof of Lemma 2.3 since an exponential variable satisfies the moment condition. (A direct proof giving $K = e^2$ can also be given.) The proof is complete.

The proof of Theorem 1.1 is now very easy. By Proposition 3.3 it suffices to prove the result for the sequence $(\mathcal{E}_k)$. But, in view of the previous lemma and the proof of Theorem 2.2, we get

$$\Big\|\sum \mathcal{E}_k\Big\|_p \leq K\Big\{\Big(\sum\|\mathcal{E}_k\|_p^p\Big)^{1/p} + \sqrt{p}\Big(\sum\|\mathcal{E}_k\|_2^2\Big)^{1/2}\Big\},$$

which completes the proof.

**4. $L_p$-domination of linear combinations by i.i.d. sums.** The aim of this section is to prove the following

THEOREM 4.1. *Let $(Y_k)$ be i.i.d. symmetric random variables. For a sequence of scalars $(a_k) \in \ell_2$, let $m = \lceil(\|a\|_2/\|a\|_p)^{2p/(p-2)}\rceil$, where $\lceil x\rceil$ denotes the smallest integer no less than $x$. Then*

$$\Big\|\sum a_kY_k\Big\|_p \le K\frac{\|a\|_p}{m^{1/p}}\Big\|\sum_{k=1}^{m\vee p} Y_k\Big\|_p$$

*for some absolute constant $K$.*

Remark 4.2. (i) This result is related to an inequality obtained by Figiel, Iwaniec and Pełczyński (1984, Proposition 2.2′). They proved that if $(a_k)$ is a sequence of scalars and $(Y_k)$ are symmetrically exchangeable then

$$\Big\|\sum_{k=1}^n a_kY_k\Big\|_p \le \frac{(\sum_{k=1}^n |a_k|^p)^{1/p}}{n^{1/p}}\Big\|\sum_{k=1}^n Y_k\Big\|_p.$$

Although we do not get constant 1, our $m$ is generally smaller than $n$. Also (cf. Marshall and Olkin (1979, Chapter 12.G)), for certain random variables $(Y_k)$, including the Rademacher sequence, one has

$$\Big\|\sum_{k=1}^n a_kY_k\Big\|_p \le \frac{(\sum_{k=1}^n |a_k|^2)^{1/2}}{\sqrt{n}}\Big\|\sum_{k=1}^n Y_k\Big\|_p.$$

Our $m$ is chosen so that the $\ell_2$ and $\ell_p$ norms of a new coefficient sequence are essentially the same as those of the original one.

(ii) If, roughly, the first $p$ coefficients in the sequence $(a_k)$ are equal, then automatically $m \ge cp$, and thus $m\vee p$ in the upper limit of summation can be replaced by $m$. This follows from the fact that if $a_1 \ge a_2 \ge \ldots \ge 0$, then the ratio

$$\frac{\sum_{j=1}^k a_j^2}{(\sum_{j=1}^k a_j^p)^{2/p}}$$

is increasing in $k$. This observation will be important in Section 6.

We will break up the proof of Theorem 4.1 into several propositions. Recall that for a random variable $Z \in L_{2q}$, $q \in \mathbb{N}$, we have $EZ^{2q} = (-1)^q\phi_Z^{(2q)}(0)$, where $\phi_Z$ is the characteristic function of $Z$ and $\phi_Z^{(2q)}$ its $2q$th derivative.

Proposition 4.3. *Let $(Y_k)$ be a sequence of i.i.d. symmetric random variables such that*

$$(-1)^l(\ln\phi_Y)^{(2l)}(0) \ge 0 \quad \text{for } l = 1,\ldots,q.$$

*Suppose that $a$ and $b$ are two sequences of real numbers such that $\|a\|_{2l} \le \|b\|_{2l}$ for $l = 1,\ldots,q$. Then*

$$E\Big(\sum_{k=1}^n a_kY_k\Big)^{2q} \le E\Big(\sum_{k=1}^n b_kY_k\Big)^{2q}.$$



Proof. Let $S = \sum_{k=1}^n a_kY_k$. We will show that $(-1)^q(\phi_S)^{(2q)}(0)$ is an increasing function of $\|a\|_{2l}$, $l = 1,\ldots,q$. We have

$$\phi_S(t) = \exp\{\ln\phi_S(t)\} = \exp\Big\{\sum_k \ln\phi_{a_kY}(t)\Big\}.$$

Differentiating once and then $2q-1$ times using the Leibniz formula we get

$$\begin{aligned}\phi_S^{(2q)} &= \sum_k (\phi_S \ln'\phi_{a_kY})^{(2q-1)}\\ &= \sum_k\sum_{j=0}^{2q-1}\binom{2q-1}{j}(\ln\phi_{a_kY})^{(j+1)}\phi_S^{(2q-1-j)}.\end{aligned}$$

Hence, evaluating at zero, and using $\phi_S^{(j)}(0) = 0$, for odd $j$'s, we get

$$\begin{aligned}\phi_S^{(2q)}(0) &= \sum_k\sum_{j=0}^{2q-1}\binom{2q-1}{j}(\ln\phi_{a_kY})^{(j+1)}(0)\phi_S^{(2q-1-j)}(0)\\ &= \sum_k\sum_{j=1}^{q}\binom{2q-1}{2j-1}a_k^{2j}(\ln\phi_Y)^{(2j)}(0)\phi_S^{(2(q-j))}(0),\end{aligned}$$

and it follows that

$$\|S\|_{2q}^{2q} = (-1)^q\phi_S^{(2q)}(0) = \sum_{j=1}^q \|a\|_{2j}^{2j}\binom{2q-1}{2j-1}(-1)^j(\ln\phi_Y)^{(2j)}(0)\|S\|_{2(q-j)}^{2(q-j)}.$$

Since we assume that $(-1)^j(\ln\phi_Y)^{(2j)}(0) \ge 0$, the result follows by induction on $q$.

Proposition 4.4. *Let $p \ge 2$. Suppose that $a$ and $b$ are two sequences of real numbers such that $\|a\|_s \le \|b\|_s$ for $2 \le s \le p$. Let $(Y_k)$ be i.i.d. symmetric such that $(-1)^l(\ln\phi_Y)^{(2l)}(0) \ge 0$ for all $l \le p$. Then*

$$\Big\|\sum_{k=1}^n a_kY_k\Big\|_p \le K\Big\|\sum_{k=1}^n b_kY_k\Big\|_p,$$

*for some absolute constant $K$.*

Proof. It follows from Kwapień and Woyczyński (1992, Proposition 1.4.2) (cf. Hitczenko (1994, Proposition 4.1) for details) that if $2q \le p$ and $(Z_k)$ is any sequence of independent, symmetric random variables, then

$$\begin{aligned}\Big\|\sum Z_k\Big\|_p &\le K\frac{p}{2q}\Big\{\Big\|\sum Z_k\Big\|_{2q} + \|\max|Z_k|\|_p\Big\}\\ &\le K\frac{p}{2q}\Big\{\Big\|\sum Z_k\Big\|_{2q} + \Big(\sum\|Z_k\|_p^p\Big)^{1/p}\Big\}.\end{aligned}$$

Applying this inequality to $Z_k = a_kY_k$, using Proposition 4.3, and the inequalities

$$\Big\|\sum Z_k\Big\|_{2q} \le \Big\|\sum Z_k\Big\|_p \quad\text{and}\quad \Big(\sum \|Z\|_p^p\Big)^{1/p} \le \|\sum Z_k\|_p,$$

we get the desired result, provided $p/q \le C$.

Remark 4.5. It is natural to ask whether we can drop the assumption $(-1)^l(\ln\phi_Y)^{(2l)}(0) \ge 0$. In general it is not possible. To see this let $(\varepsilon_k)$ be a Rademacher sequence, and for $p \in \mathbb{N}$ let $a_k = 1$ or 0 according to whether $k \le p$ or $k > p$. If $b_1 = \sqrt{p}$ and $b_k = 0$ for $k \ge 2$, then $\|a\|_s \le \|b\|_s$ for $2 \le s \le p$, but $\|\sum a_k\varepsilon_k\|_p \approx p$ and $\|\sum b_k\varepsilon_k\|_p = \sqrt{p}$.

Before we proceed, we need some more notation. For a random variable $Z$ we let $\mathrm{Pois}(Z) = \sum_{k=1}^N Z_k$, where $N, Z_1, Z_2, \ldots$, are independent random variables, $N$ is a Poisson random variable with parameter 1, and $(Z_k)$ are i.i.d. copies of $Z$. Moreover, if the $Z_k$ are independent, then $\mathrm{Pois}(Z_k)$ will always be chosen so that they are independent. Since $P(|Y| > t) \le (1/2)(1-e^{-1})P(|\mathrm{Pois}(Y)| > t)$, the next proposition is a consequence of the contraction principle.

PROPOSITION 4.6. *Let $(Y_k)$ be a sequence of independent symmetric random variables. Then*

$$\Big\|\sum Y_k\Big\|_p \le C\Big\|\sum \mathrm{Pois}(Y_k)\Big\|_p$$

*for some absolute constant $C$.*

Next we note that, for an arbitrary random variable $Z$,

$$\phi_{\mathrm{Pois}(Z)} = \exp\{\phi_Z - 1\},$$

and hence, in particular, if $Z$ is symmetric, then $(-1)^k(\ln\phi_{\mathrm{Pois}(Z)})^{(2k)}(0) \ge 0$ for all $k$ and thus, by the above discussion, if $(Z_k)$ are i.i.d. copies of $Z$, then

$$\Big\|\sum a_k\mathrm{Pois}(Y_k)\Big\|_p \le C\Big\|\sum b_k\mathrm{Pois}(Y_k)\Big\|_p$$

whenever $\|a\|_s \le \|b\|_s$, for $2 \le s \le p$.

Now fix a sequence $a$, and let $m = \lceil(\|a\|_2/\|a\|_p)^{2p/(p-2)}\rceil$ as in Theorem 4.1. Define a sequence $b$ by

$$b_k = \begin{cases}\beta & \text{if } k \le m,\\ 0 & \text{otherwise,}\end{cases}$$

where $\beta = \|a\|_p/m^{1/p}$. Note that $\|b\|_2 \ge \|a\|_2$ and $\|b\|_p = \|a\|_p$. Hence by Hölder's inequality $\|a\|_s \le \|b\|_s$ for all $2 \le s \le p$. Therefore, for an arbitrary sequence $(Y_k) \subset L_p$ of i.i.d. symmetric random variables we have

$$\Big\|\sum a_kY_k\Big\|_p \le K\frac{\|a\|_p}{m^{1/p}}\Big\|\sum_{k=1}^m \mathrm{Pois}(Y_k)\Big\|_p = K\frac{\|a\|_p}{m^{1/p}}\Big\|\sum_{k=1}^{N_m} Y_k\Big\|_p,$$



where $N_m$ is a Poisson random variable with parameter $m$, independent of the sequence $(Y_k)$.

Our final step is to estimate the quantity $\|\sum_{k=1}^{N_m} Y_k\|_p$. The following computation is rather straightforward; a very similar calculation can be found, for example, in Klass (1976, proof of Proposition 1).

PROPOSITION 4.7. *For $m \in \mathbb{N}$, and $(Y_k)$ and $N_m$ as above, we have*

$$\Big\|\sum_{k=1}^{N_m} Y_k\Big\|_p \le K\Big\|\sum_{k=1}^{m\vee p} Y_k\Big\|_p.$$

Proof. For $j \ge 1$, let $S_j = \sum_{k=1}^j Y_k$. Note that, since $(S_j/j)$ is a reversed martingale (or by an application of the triangle inequality), $\|S_j/j\|_p$ is decreasing in $j$. Therefore, for $j_0$ to be specified later, we have

$$\begin{aligned}\Big\|\sum_{k=1}^{N_m} Y_k\Big\|_p^p &= \sum_{j=1}^{\infty}\|S_j\|_p^p\frac{e^{-m}m^j}{j!}\\ &\le \|S_{j_0}\|_p^p\sum_{j=1}^{j_0}\frac{e^{-m}m^j}{j!} + \frac{\|S_{j_0}\|_p^p}{j_0^p}\sum_{j>j_0}\frac{j^pe^{-m}m^j}{j!}\\ &\le \|S_{j_0}\|_p^p\Big(1 + \frac{e^{-m}}{j_0^p}\sum_{j>j_0}\frac{j^pm^j}{j!}\Big).\end{aligned}$$

The ratio of two consecutive terms in the last sum is equal to

$$\Big(\frac{j+1}{j}\Big)^p\frac{m}{j+1} \le \frac{e^{p/j}m}{j+1} \le \frac{1}{2}$$

whenever $j \ge \max\{2em, p\}$. Therefore, choosing $j_0 \approx \max\{2em, p\}$, we obtain

$$\Big\|\sum_{k=1}^{N_m} Y_k\Big\|_p^p \le \|S_{j_0}\|_p^p\Big(1 + \frac{2e^{-m}}{j_0^p}j_0^p\Big(\frac{m}{j_0}\Big)^{j_0}\Big) \le 2\|S_{j_0}\|_p^p \le K^p\|S_{m\vee p}\|_p^p.$$

This completes the proof of Proposition 4.7.

Theorem 4.1 now follows immediately from the above results.

**5. Distribution of a sum of i.i.d. random variables.** In this section, $Y_1, \ldots, Y_n$ are independent copies of a symmetric random variable $Y$. We fix $n$, and let $S = S_n = \sum_{i=1}^n Y_i$ and $M = M_n = \sup_{1\le i\le n} Y_i$. Our aim is to calculate $\|S\|_p$, and as we will see below, this is equivalent to finding $\|S\|_{p,\infty} + \|M\|_p$. Let us recall that for a random variable $Z$, $\|Z\|_{p,\infty}$ is defined by

$$\|Z\|_{p,\infty} = \sup_{s>0} s^{1/p}Z^*(s),$$

where $Z^*$ denotes the decreasing rearrangement of the distribution function of $Z$, i.e.

$$Z^*(s) = \sup\{t : P(|Z| > t) > s\}, \quad 0 < s < 1.$$

The following simple observation will be useful:

LEMMA 5.1. *Let $p \geq 2$. For any sequence $(Z_k) \subset L_p$ of independent, symmetric random variables we have*

$$\frac{1}{4}\Big\{\Big\|\sum Z_k\Big\|_{p,\infty} + \|\max|Z_k|\|_p\Big\} \leq \Big\|\sum Z_k\Big\|_p \leq K\Big\{\Big\|\sum Z_k\Big\|_{p,\infty} + \|\max|Z_k|\|_p\Big\}$$

*for some absolute constant $K$.*

Proof. Since for $p > q \geq 1$ and for any random variable $W$,

$$\|W\|_{q,\infty} \leq \|W\|_q \leq \left(\frac{p}{p-q}\right)^{1/q} \|W\|_{p,\infty},$$

the first inequality is trivial. For the second inequality we first use a result of Kwapień and Woyczyński (see proof of Proposition 4.4 above)

$$\Big\|\sum Z_k\Big\|_p \leq K\frac{p}{q}\Big\{\Big\|\sum Z_k\Big\|_q + \|\max|Z_k|\|_p\Big\},$$

and then the right estimate above with $q = p/2$. This completes the proof.

Throughout the rest of this section, by $f(x) \preceq g(x)$ we mean that $f(cx) \leq Cg(x)$ for some constants $c, C$. We will write $f(x) \approx g(x)$ if $f(x) \preceq g(x)$ and $g(x) \preceq f(x)$. With this convention we have

THEOREM 5.2. *For $0 \leq \theta \leq 1$ we have*

$$S^*(\theta) \preceq T_1(\theta) + T_2(\theta),$$

*where*

$$T_1(\theta) = \frac{\log(1/\theta)}{\sqrt{\theta^{-1/n} - 1}} \|Y^*|_{[\theta/n \vee (\theta^{-1/n}-1),1]}\|_2$$

*and*

$$T_2(\theta) = \log(1/\theta) \sup_{\theta/n \leq t \leq \theta^{-1/n}-1} \frac{Y^*(t)}{\log(1 + (\theta^{-1/n} - 1)/t)}.$$

Proof. Let $t > 0$. Following the argument in Hahn and Klass (1995), for $a > 0$ we have

$$\begin{aligned} P(S \geq t) &\leq P(M \geq a) + \inf_{\lambda>0} E(e^{\lambda(S-t)} \mid M < a)P(M < a) \\ &= 1 - P^n(Y < a) + (\inf_{\lambda>0} E(e^{\lambda(Y-t/n)} \mid Y < a)P(Y < a))^n. \end{aligned}$$



If we choose $a$ so that these two terms are equal, and then we let

$$\theta = 1 - P^n(Y < a) = (\inf_{\lambda>0} E(e^{\lambda(Y-t/n)} \mid Y < a)P(Y < a))^n,$$

then, by the above computation,

$$P(S \geq t) \leq 2\theta.$$

The whole idea now is to invert the above relationship, i.e. starting with $\theta$ we will find a relatively small $t$ so that $S^*(\theta) \preceq t$. Since the values of $S^*(\theta)$ are of no importance for $\theta$ close to 1 we can assume that $\theta$ is bounded away from 1, $\theta \leq 1/10$, say. Then $P(Y < a) = (1-\theta)^{1/n} \approx 1 - \theta/n$, so that $P(Y \geq a) \approx \theta/n$, which, in turn, means that $Y^*(\theta/n) \approx a$.

In order to find a $t$, we will use the relation

$$\theta = \inf_{\lambda} E(e^{\lambda(Y-t/n)} \mid Y < a)^n = \inf_{\lambda}(Ee^{\lambda(\widetilde{Y}-t/n)})^n,$$

where $\widetilde{Y}$ denotes $YI_{Y<a}$. Then for $\lambda > 0$ we have $\theta \leq e^{-\lambda t}(Ee^{\lambda\widetilde{Y}})^n$, and taking logarithms on both sides we get

$$t \leq \frac{1}{\lambda}\ln(1/\theta) + n\ln(Ee^{\lambda\widetilde{Y}})^{1/\lambda}.$$

Hence, we can take

$$t = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln(1/\theta) + n\ln(Ee^{\lambda\widetilde{Y}})^{1/\lambda}\right\}.$$

Note that the second term, being equal to $n\ln\|e^{\widetilde{Y}}\|_\lambda$, is an increasing function of $\lambda$. Since the first term is decreasing in $\lambda$, it follows that, up to a factor of 2, the infimum is attained when both terms are equal. Thus $t \approx (1/\lambda)\ln(1/\theta)$, where $\lambda$ is chosen so that $\ln(1/\theta) = n\ln(Ee^{\lambda\widetilde{Y}})$, i.e. $Ee^{\lambda\widetilde{Y}} = \theta^{-1/n}$. Since $\widetilde{Y}$ is symmetric, $Ee^{\lambda\widetilde{Y}} = E\cosh(\lambda|\widetilde{Y}|)$, and the above equality is equivalent to

$$E\left(\frac{\cosh(\lambda\widetilde{Y}) - 1}{\theta^{-1/n} - 1}\right) = 1.$$

But this simply means that

$$t \approx \|\widetilde{Y}\|_\Phi \ln(1/\theta), \quad \text{where} \quad \Phi(s) = \frac{\cosh(s) - 1}{\theta^{-1/n} - 1}.$$

(Recall that if $\Psi$ is an Orlicz function and $Z$ is a random variable, then $\|Z\|_\Psi = \inf\{u > 0 : E\Psi(Z/u) \leq 1\}$.)

In order to complete the proof, it suffices to show that, with $\Phi$ as above, we have

$$\|\widetilde{Y}\|_\Phi \approx \sup_{\theta/n\le t\le \theta^{-1/n}-1} \frac{Y^*(t)}{\log(1+(\theta^{-1/n}-1)/t)} + \frac{1}{\sqrt{\theta^{-1/n}-1}}\|Y^*|_{[\theta/n\vee(\theta^{-1/n}-1),1]}\|_2.$$

We will need the following two observations. For an Orlicz function $\Psi$ and a random variable $Z$, define the weak Orlicz norm by

$$\|Z\|_{\Psi,\infty} = \sup_{0<x<1}\left\{\frac{1}{\Psi^{-1}(1/x)}Z^*(x)\right\}.$$

LEMMA 5.3. *Let $\psi(s)=e^s-1$. Then*

$$\|Z\|_{\psi,\infty} \le \|Z\|_\psi \le 3\|Z\|_{\psi,\infty}.$$

Proof. Only the second inequality needs justification. Suppose $\|Z\|_{\psi,\infty} \le 1$. This means that $Z^*(x)\le \ln(1+1/x)$ for each $0<x<1$. Hence

$$E\psi(Z/u) = Ee^{\{Z^*/u\}}-1 \le \int_0^1 e^{(1/u)\ln(1+1/x)}\,dx - 1$$
$$\le \int_0^1 (2/x)^{1/u}\,dx - 1 = \frac{2^{1/u}u}{u-1}-1\le 1$$

whenever $u\ge 3$.

Remark 5.4. The above relationship is true for more general Orlicz functions as long as they grow fast enough; see Theorem 4.6 in Montgomery-Smith (1992) for more details.

LEMMA 5.5. *Suppose that $\Phi_1,\Phi_2:[0,\infty)\to[0,\infty)$ are nondecreasing continuous functions that map zero to zero, and such that there exists a constant $c>1$ such that $\Phi_i(x/c)\le\Phi_i(x)/2$ $(i=1,2)$. Suppose further that there are numbers $a,A>0$ such that $\Phi_1(a)=\Phi_2(a)=A$. Define $\Phi:[0,\infty)\to[0,\infty)$ by*

$$\Phi(x)=\begin{cases}\Phi_1(x) & \text{if } x\le a,\\ \Phi_2(x) & \text{if } x>a.\end{cases}$$

*Then for any measurable scalar-valued function $f$, we have*

$$\|f\|_\Phi \approx \|f^*|_{[0,A^{-1}]}\|_{\Phi_2}+\|f^*|_{[A^{-1},\infty)}\|_{\Phi_1}.$$

*The constants of approximation depend only upon $c$.*

Proof. Notice that $\Phi(x/c^2)\le\Phi(x)/2$. This is clear if both $x/c^2$ and $x$ lie on the same side of $a$, and otherwise it follows by considering the cases where $x/c$ is greater or less than $a$.



Let us first suppose that the right hand side is less than 1. Then

$$\int_0^{A^{-1}}\Phi_2(f^*(x))\,dx\le 1 \quad\text{and}\quad \int_{A^{-1}}^\infty \Phi_1(f^*(x))\,dx\le 1.$$

From the first inequality, we see that $A^{-1}\Phi_2(f^*(A^{-1}))\le 1$, from which it follows that $f^*(A^{-1})\le a$. Let $B$ be such that $f^*(B^+)\le a\le f^*(B^-)$, so that $B\le A$. (Here, and in the rest of the proof, we define $f(A^+)=\lim_{x\searrow A}f(x)$ and $f(A^-)=\lim_{x\nearrow A}f(x)$.) Then

$$\int_0^B\Phi(f^*(x))\,dx=\int_0^B\Phi_2(f^*(x))\,dx\le 1,$$
$$\int_B^{A^{-1}}\Phi(f^*(x))\,dx\le A^{-1}\Phi(f^*(B^+))\le 1,$$
$$\int_{A^{-1}}^\infty\Phi(f^*(x))\,dx=\int_{A^{-1}}^\infty\Phi_1(f^*(x))\,dx\le 1.$$

Hence

$$\int_0^\infty \Phi(f^*(x))\,dx\le 3,$$

and so

$$\int_0^\infty\Phi(f^*(x)/c^4)\,dx\le 1,\quad\text{that is,}\quad \|f\|_\Phi\le c^4.$$

Now suppose that $\|f\|_\Phi\le 1$, so that $\int_0^\infty\Phi(f^*(x))\,dx\le 1$. Then $A^{-1}\Phi(f^*(A^{-1}))\le 1$, which implies that $f^*(A^{-1})\le a$. Let $B$ be such that $f^*(B^+)\le a\le f^*(B^-)$, so that $B\le A$. Then

$$\int_0^B\Phi_2(f^*(x))\,dx=\int_0^B\Phi(f^*(x))\,dx\le 1$$

and

$$\int_B^{A^{-1}}\Phi_2(f^*(x))\,dx\le A^{-1}\Phi_2(f^*(B^+))\le 1.$$

Hence

$$\int_0^{A^{-1}}\Phi_2(f^*(x))\,dx\le 2,$$

from which it follows that $\|f|_{[0,A^{-1}]}\|_{\Phi_2} \le c$. Also

$$\int_{A^{-1}}^{\infty} \Phi_1(f^*(x))\,dx = \int_{A^{-1}}^{\infty} \Phi(f^*(x))\,dx \le 1,$$

and hence $\|f|_{[A^{-1},\infty)}\|_{\Phi_1} \le 1$. This completes the proof of Lemma 5.5.

Now, in order to finish the proof of Theorem 5.2, let

$$\Phi(s) = \frac{\cosh(s)-1}{\theta^{-1/n}-1}.$$

Then

$$\Phi(s) = \begin{cases} \Phi_1(s) & \text{if } 0 \le s \le 1,\\ \Phi_2(s) & \text{if } s > 1,\end{cases}$$

where

$$\Phi_1(s) \approx \frac{s^2}{\theta^{-1/n}-1} \quad\text{and}\quad \Phi_2(s) \approx \frac{e^s-1}{\theta^{-1/n}-1}.$$

Since $\widetilde{Y}^* = (YI_{Y<a})^* \approx (Y^*|_{[\theta/n,1]})^*$, by Lemma 5.5 we obtain

$$\begin{aligned}\|\widetilde{Y}\|_{\Phi} &\approx \|Y^*|_{[0,\theta^{-1/\theta}-1]}\|_{\Phi_2} + \|Y^*|_{[\theta^{-1/\theta}-1,1]}\|_{\Phi_1}\\ &\approx \|Y^*|_{[0,\theta^{-1/\theta}-1]}\|_{\Phi_2,\infty} + \frac{1}{\sqrt{\theta^{-1/\theta}-1}}\|Y^*|_{[\theta^{-1/\theta}-1,1]}\|_2\\ &\approx \sup_{\theta/n\le t\le \theta^{-1/n}-1} \frac{Y^*(t)}{\log(1+(\theta^{-1/n}-1)/t)}\\ &\quad + \frac{1}{\sqrt{\theta^{-1/n}-1}}\|Y^*|_{[\theta/n \vee (\theta^{-1/n}-1),1]}\|_2.\end{aligned}$$

This completes the proof of Theorem 5.2.

**6. Moments of linear combinations of symmetric Weibull random variables.** In this section we will apply our methods to obtain an upper bound for the $L_p$-norm of a linear combination of i.i.d. symmetric Weibull random variables with parameter $0<r<1$. We will also show that this upper bound is tight. A random variable $X$ will be called *symmetric Weibull* with parameter $r$, $0<r<\infty$, if $X$ is symmetric and $|X|$ has density given by

$$f_{|X|}(t) = rt^{r-1}e^{-t^r}, \quad t>0.$$

We refer to Johnson and Kotz (1970, Chapter 20) for more information on Weibull random variables. Here we only note that by a change of variables,

$$\|X\|_p^p = \Gamma(1+p/r), \quad p>0,$$

so that using Stirling formula and elementary estimates we have the following.



LEMMA 6.1. *If $X$ is a symmetric Weibull random variable with parameter $0<r<1$, and $p\ge 2$, then*

$$p\|X\|_2 \le C\|X\|_p,$$

*where $C$ is a constant not depending on $p$ or $r$.*

As we mentioned in the introduction, Gluskin and Kwapień (1995) established a two-sided inequality for the $L_p$-norm of a linear combinations of i.i.d. symmetric random variables with logarithmically concave tails. In the special case where $P(|\xi|>t)=\exp(-t^r)$, $r\ge 1$, their result reads as follows:

$$\begin{aligned} &c\Big\{\Big(\sum_{k\le p} a_k^{r'}\Big)^{1/r'}\|\xi\|_p + \sqrt{p}\,\|\xi\|_2\Big(\sum_{k>p}a_k^2\Big)^{1/2}\Big\}\\ &\quad\le \Big\|\sum a_k\xi_k\Big\|_p \le C\Big\{\Big(\sum_{k\le p} a_k^{r'}\Big)^{1/r'}\|\xi\|_p + \sqrt{p}\,\|\xi\|_2\Big(\sum_{k>p}a_k^2\Big)^{1/2}\Big\},\end{aligned}$$

where $r'$ is the exponent conjugate to $r$, i.e. $1/r+1/r'=1$, and $c$ and $C$ are absolute constants. In this section we complement the result of Gluskin and Kwapień, by treating the case $r<1$. Here is the main result of this section.

THEOREM 6.2. *There exist absolute constants $c$, $C$ such that if $(X_i)$ is a sequence of i.i.d. symmetric Weibull random variables with parameter $r$, where $0<r<1$, and $(a_k)\in \ell_2$, then*

$$\begin{aligned}&c\max\{\sqrt{p}\,\|a\|_2\|X\|_2, \|a\|_p\|X\|_p\}\\ &\quad\le \Big\|\sum a_kX_k\Big\|_p \le C\max\{\sqrt{p}\,\|a\|_2\|X\|_2, \|a\|_p\|X\|_p\}.\end{aligned}$$

Proof. We begin with the following result.

PROPOSITION 6.3. *Let $p\ge 2$, and let $(X_i)$ be a sequence i.i.d. symmetric Weibull random variables with parameter $r$ where $0<r<1$. Then*

$$a_1\|X\|_p \le \Big\|\sum_{k\le p} a_kX_k\Big\|_p \le Ca_1\|X\|_p.$$

Proof. The first inequality is trivial. To prove the second, note that

$$\Big\|\sum_{k\le p} a_kX_k\Big\|_p \le a_1\Big\|\sum_{k\le p} X_k\Big\|_p \le a_1\Big\|\sum_{k\le p} |X_k|\Big\|_p.$$

Assume first that $r$ is bounded away from 0, say $r>1/2$. Then letting $(\Gamma_i)$

be a sequence of i.i.d. exponential distributions with mean 1 we can write

$$E\Big(\sum_{k\le p}|X_k|\Big)^p = E\Big(\Big[\sum_{k\le p}(|X_k|^r)^{1/r}\Big]^r\Big)^{p/r} = E\Big(\Big[\sum_{k\le p}\Gamma_k^{1/r}\Big]^r\Big)^{p/r}$$
$$\le E\Big(\sum_{k\le p}\Gamma_k\Big)^{p/r} \le p\frac{\Gamma(p+p/r)}{\Gamma(p)}$$
$$\le p\frac{(p(1+1/r))^p}{\Gamma(p)}\Gamma(1+p/r) \le K^p\Gamma(1+p/r),$$

since $r$ is bounded away from zero. This shows the upper bound for $r > 1/2$. To handle the case $r \le 1/2$ we will apply a method that was used in Schechtman and Zinn (1990). Let $(X_{(k)})$ denote the nonincreasing rearrangement of $(|X_k|)_{k\le p}$. For a number $t > 0$ we have

$$P\Big(\sum_{k\le p}|X_k| \ge t\Big) = P\Big(\sum_{k\le p}X_{(k)} \ge t\Big) \le \sum_{k\le p}P(X_{(k)} \ge q_k t),$$

where $(q_k)$ is any sequence of nonnegative numbers such that $\sum_{k\le p}q_k \le 1$. Now, using the inequality

$$P(Y_{(k)} \ge s) \le \frac{(\sum_{j\le p}P(Y_j\ge s))^k}{k!},$$

valid for any sequence of independent random variables $(Y_k)$ (cf. e.g. Talagrand (1989, Lemma 9)) we get

$$P(X_{(k)} \ge q_k t) \le \frac{(p\exp\{-q_k^r t^r\})^k}{k!}.$$

Therefore,

$$E\Big(\sum_{k\le p}|X_k|\Big)^p \le \sum_{k\le p}\frac{p^k}{k!}\cdot p\int_0^\infty t^{p-1}\exp\{-kq_k^r t^r\}\,dt.$$

Substituting $u = (k^{1/r}q_k t)^r$ into the $k$th term and integrating we get

$$\int_0^\infty t^{p-1}\exp\{-(k^{1/r}q_k t)^r\}\,dt = \frac{\Gamma(p/r)}{r(k^{1/r}q_k)^p}.$$

Therefore,

$$E\Big|\sum_{k\le p}X_k\Big|^p \le (p/r)\Gamma(p/r)\sum_{k\le p}\frac{p^{k+1}}{k!k^{p/r}q_k^p}$$
$$\le (p/r)\Gamma(p/r)\frac{1}{(\inf_k\{k^{1/r}q_k\})^p}\sum_{k=1}^\infty\frac{p^k}{k!},$$



which is bounded by $C^p\|X\|_p^p$, as long as $k^{1/r}q_k$ is bounded away from zero. The choice $q_k \approx k^{-1/r}$ concludes the proof of the proposition.

Now for the general case. By the previous proposition we can and do assume that $n > p$. We will establish the upper bound first.

We first observe that it suffices to prove the result under the additional assumption that the first $\lceil p\rceil$ entries in the coefficient sequence are equal. Indeed, suppose we know the result in that case. Then, for the general sequence $(a_k)$, we can write

$$\Big\|\sum a_k X_k\Big\|_p \le \Big\|\sum_{k\le p}a_1X_k + \sum_{k\ge1}a_kX_{\lceil p\rceil+k}\Big\|_p$$
$$\le C\max\{\sqrt{p}\,\|X\|_2(\sqrt{p}\,a_1+\|a\|_2), (p^{1/p}a_1+\|a\|_p)\|X\|_p\}$$
$$\le C\max\{\sqrt{p}\,\|a\|_2\|X\|_2, \|a\|_p\|X\|_p\},$$

since, by Lemma 6.1, we have

$$pa_1\|X\|_2 \le Ka_1\|X\|_p \le K\|a\|_p\|X\|_p.$$

Next we note that if the first $\lceil p\rceil$ coefficients are equal, and $m$ is defined as in Theorem 4.1, then automatically

$$m \approx (\|a\|_2/\|a\|_p)^{2p/(p-2)} \ge cp,$$

and the inequality of that theorem takes the form

$$\Big\|\sum a_kX_k\Big\|_p \le K\frac{\|a\|_p}{m^{1/p}}\Big\|\sum_{k=1}^m X_k\Big\|_p.$$

By definition $m \le 2(\|a\|_2/\|a\|_p)^{2p/(p-2)}$, so that

$$\frac{\|a\|_p}{m^{1/p}}\sqrt{pm} \le \sqrt{2p}\,\|a\|_2.$$

Therefore, in order to complete the proof it suffices to show that

$$\Big\|\sum_{k=1}^m X_k\Big\|_p \le K\max\{\sqrt{pm}\,\|X\|_2, m^{1/p}\|X\|_p\}.$$

In other words, the problem has been reduced to the special case when all coefficients are equal.

To prove the latter inequality we will use Theorem 5.2. Fix $n\in\mathbb{N}$. Let $S = \sum_{i=1}^n X_i$ and $M = \sup_{1\le i\le n}X_i$, that is, $M^*(t)\approx X^*(t/n)$. Our aim is to show

$$\|S\|_p \le K\{\sqrt{np}\,\|X\|_2 + \|M\|_p\}.$$

(Note that $\|M\|_p \le n^{1/p}\|X\|_p$.) By Lemma 5.1, it suffices to estimate $\|S\|_{p,\infty} + \|M\|_p$. Let us recall that $X^*(t) = (\log(1/t))^{1/r}$.

From Theorem 5.2, we know that

$$S^*(x) \preceq T_1(x) + T_2(x),$$

where

$$T_1(x) = \frac{\log(1/x)}{\sqrt{x^{-1/n}-1}} \| X^* \|_{[x/n \vee (x^{-1/n}-1),1]} \|_2$$

and

$$T_2(x) = \log(1/x) \sup_{x/n \le t \le x^{-1/n}-1} \frac{X^*(t)}{\log(1+(x^{-1/n}-1)/t)}.$$

Now, $\|S\|_{p,\infty} \approx \sup_{0<x<1} x^{1/p} T_1(x) + \sup_{0<x<1} x^{1/p} T_2(x)$.

To get a handle on these quantities, we use the following approximation:

$$x^{-1/n} - 1 \approx \begin{cases} (1/n)\log(1/x) & \text{if } x \ge e^{-n}, \\ x^{-1/n} & \text{if } x \le e^{-n}. \end{cases}$$

Now, we can see that if $x \le e^{-n}$, then $T_1(x) = 0$, and if $x \ge e^{-n}$, then

$$T_1(x) \le c\sqrt{n}\sqrt{\log(1/x)}\, \|X\|_2.$$

Hence, $\sup_{0<x<1} x^{1/p} T_1(x) \le c\sqrt{np}\, \|X\|_2$.

As for $T_2$, we use similar approximations, and we arrive at the following formula. If $x \ge e^{-n}$, then

$$T_2(x) \approx \sup_{x/n \le t \le (1/n)\log(1/x)} \frac{\log(1/x) X^*(t)}{1+\log(1/t) - \log(n/\log(1/x))},$$

and if $x \le e^{-n}$, then

$$T_2(x) \approx \sup_{x/n \le t \le x^{1/n}} \frac{\log(1/x) X^*(t)}{\log(1/t)}.$$

Now let us make the substitution $X^*(t) = (\log(1/t))^{1/r}$, where $0 < r < 1$. If $x \le e^{-n}$, then it is clear that the supremum that defines $T_2(x)$ is attained when $t$ is as small as possible, that is, $t = x/n$. But, since $x \le e^{-n}$, it follows that $\log(n/x) \approx \log(1/x)$, and hence

$$T_2(x) \approx X^*(x/n) \approx M^*(x).$$

Now consider the case $x \ge e^{-n}$. Then

$$T_2(x) \approx \sup_{x/n \le t \le (1/n)\log(1/x)} \log(1/x) H(\log(1/t)),$$

where

$$H(u) = \frac{u^{1/r}}{1 - \log(n/\log(1/x)) + u}.$$

Now, looking at the graph of $H(u)$, we see that the supremum of $H(u)$ over an interval on which $H(u)$ is positive is attained at one of the endpoints of the interval. Hence

$$T_2(x) \approx \log(1/x) \max\{H(\log(n/x)), H(\log(n/\log(1/x)))\}.$$

Now,

$$\log(1/x) H(\log(n/x)) = \frac{\log(1/x) X^*(x/n)}{1+\log(\log(1/x)/x)} \approx X^*(x/n) \approx M^*(x),$$

because $1 + \log(\log(1/x)/x) \approx \log(1/x)$. Also,

$$\log(1/x) H(\log(n/\log(1/x))) = \log(1/x)(\log(n/\log(1/x)))^{1/r}.$$

Putting all this together, we see that

$$\sup_{0<x<1} x^{1/p} T_2(x) \approx \|M\|_{p,\infty} + \sup_{x \ge e^{-n}} F(\log(1/x)),$$

where

$$F(u) = e^{-u/p} u (\log(n/u))^{1/r}.$$

Thus, the proof will be complete if we can show that $F(u) \le \sqrt{np}\, \|X\|_2$ for all $u \le n$. To this end, from Stirling's formula, we observe that $\|X\|_2 = \Gamma(1+2/r)^{1/2} \ge c^{-1}(2/(re))^{1/r}$.

Notice that

$$\frac{uF'(u)}{F(u)} = -\frac{u}{p} + 1 - \frac{1}{r\log(n/u)}.$$

This quantity is positive if $u$ is small, it is negative if $u = p$ or $u = n$, and it is decreasing. Hence, $F(u)$ attains its supremum at $u_0$, where $F'(u_0) = 0$ and $u_0 \le p$. But then

$$F(u_0) \le \sqrt{np}\, G(n/u_0), \quad \text{where} \quad G(v) = \frac{(\log v)^{1/r}}{\sqrt{v}}.$$

Simple calculus shows that

$$G(v) \le G(e^{2/r}) = \left(\frac{2}{re}\right)^{1/r} \le c\|X\|_2,$$

which proves the right inequality in Theorem 6.2.

To prove the left inequality, notice that by the original proof of Rosenthal's inequality (Rosenthal (1970)) we have

$$\Big\| \sum a_k X_k \Big\|_p \ge \Big(\sum a_k^p\Big)^{1/p} \|X\|_p,$$

so it suffices to show that

$$\Big\| \sum a_k X_k \Big\|_p \ge c\sqrt{p} \Big(\sum a_k^2\Big)^{1/2} \|X\|_2.$$

Assume without loss of generality that $p \geq 3$. Let $\delta = (\sum_{j\leq p} a_j^2)^{1/2}/\sqrt{p}$, and define a sequence $d = (d_k)$ by the formula

$$d_k = \begin{cases} \delta & \text{if } k \leq p, \\ a_k & \text{otherwise.} \end{cases}$$

Then

$$\Big\| \sum a_k X_k \Big\|_p \geq \kappa \Big\| \sum d_k X_k \Big\|_p$$

for some absolute constant $\kappa > 0$. Indeed, let $C$ be a constant such that $\| \sum_{k\leq p} X_k \|_p \leq C\|X\|_p$. Since $\delta \leq a_1$ we have

$$\Big\| \sum d_k X_k \Big\|_p \leq \delta \Big\| \sum_{k\leq p} X_k \Big\|_p + \Big\| \sum_{k>p} a_k X_k \Big\|_p \leq Ca_1 \|X\|_p + \Big\| \sum_{k>p} a_k X_k \Big\|_p$$
$$\leq (C+1) \Big\| \sum a_k X_k \Big\|_p,$$

so that one can take $\kappa = 1/(C+1)$. Let $(\varepsilon_k)$ be a sequence of Rademacher random variables, independent of the sequence $(X_k)$. Notice that $\max d_j/\|d\|_2 \leq 1/\sqrt{p}$. Therefore, using the minimality property of Rademacher functions (cf. Figiel, Hitczenko, Johnson, Schechtman and Zinn (1995, Theorem 1.1), or Pinelis (1994, Corollary 2.5)) (here we use $p \geq 3$) and then Hitczenko and Kwapień (1994, Theorem 1) we get

$$\Big\| \sum a_k X_k \Big\|_p \geq \kappa \Big\| \sum d_k X_k \Big\|_p \geq \kappa \Big\| \sum d_k \|X_k\|_2 \varepsilon_k \Big\|_p$$
$$\geq c\sqrt{p}\, \|d\|_2 \|X\|_2 = c\sqrt{p}\, \|a\|_2 \|X\|_2.$$

The proof is complete.

Remark 6.4. For $r = 1$ our formula gives $p\|a\|_p + \sqrt{p}\, \|a\|_2$, while Gluskin and Kwapień obtained $p \sup_{k\leq p} a_k + \sqrt{p}\, (\sum_{k>p} a_k^2)^{1/2}$. Although these two quantities look different, they are equivalent. Clearly $p \sup_{k\leq p} a_k + \sqrt{p}\, (\sum_{k>p} a_k^2)^{1/2} \leq p\|a\|_p + \sqrt{p}\, \|a\|_2$. To see that the opposite inequality holds with an absolute constant, notice that if $a_1$ and $\sum_{k>p} a_k^2$ are fixed, then $p\|a\|_p + \sqrt{p}\, \|a\|_2$ is maximized if the first several $a_k$'s are equal to $a_1$, the next one is between $a_1$ and $0$, and the rest are $0$. In this case it is very easy to check that the required inequality holds.

Theorem 6.2 implies the following result for tail probabilities.

COROLLARY 6.5. *Let $a = (a_k) \in \ell_2$, $a \neq 0$, and let $S = \sum a_k X_k$. Then*

$$\lim_{t\to\infty} \log_t \ln 1/P(|S| > t) = r.$$

Proof. Since $P(|S| > t) \geq \frac{1}{2} P(a_1|X_1| > t) = \frac{1}{2} \exp\{-(t/a_1)^r\}$, we immediately get

$$\limsup_{t\to\infty} \log_t \ln 1/P(|S| > t) \leq r.$$



To show the opposite inequality note that if $s < r$ then

$$E\exp\{|S|^s\} = 1 + \sum_{k=1}^{\infty} \frac{E|S|^{sk}}{k!} < \infty,$$

by Theorem 6.2. Hence $P(|S| > t) \leq \exp\{-t^s\} E\exp\{|S|^s\}$, which implies

$$\liminf_{t\to\infty} \log_t \ln 1/P(|S| > t) \geq s.$$

Since this is true for every $s < r$, the result follows.

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# The boundary Harnack principle for the fractional Laplacian

by

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**Abstract.** We study nonnegative functions which are harmonic on a Lipschitz domain with respect to symmetric stable processes. We prove that if two such functions vanish continuously outside the domain near a part of its boundary, then their ratio is bounded near this part of the boundary.

**1. Introduction.** The boundary Harnack principle (BHP) for nonnegative harmonic functions has important applications in probability theory and potential theory. Among these are approximations to excursion laws for the Brownian motion (see [6]), "3G Theorem" and "Conditional Gauge Theorem" (see [8]). BHP was first proved in [9] for Lipschitz domains by analytic methods (see also [12], [11]). Later, the classical link between harmonic functions and the Brownian motion in $\mathbb{R}^n$ was used to give a probabilistic proof of BHP ([2]). The result and generalizations of BHP to elliptic operators and Schrödinger operators have yielded stimulating interplay between probability theory, harmonic analysis and potential theory (see [7], [3], [8], [6], [1]).

The Brownian motion is a particular (and limiting) instance of the standard rotation invariant $\alpha$-stable process, $\alpha \in (0,2]$. The infinitesimal generator $\Delta^{\alpha/2}$ of the latter and the related class of $\alpha$-harmonic functions have simple homogeneity properties analogous to those of the classical Laplacian and harmonic functions ($\alpha = 2$) in $\mathbb{R}^n$. Also, the potential theory of $\Delta^{\alpha/2}$ ($n > \alpha$) enjoys an explicit formulation in terms of M. Riesz kernels, and is similar to that of the Laplacian in $\mathbb{R}^n$, $n > 2$ ([13]).

The main result of this paper is the following theorem which gives another extension of the classical theory ($\alpha = 2$) to the case $\alpha \in (0,2)$.

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