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On the semi-Browder spectrum

by

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Abstract. An operator in a Banach space is called upper (lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (descent). We extend this notion to n-tuples of commuting operators and show that this notion defines a joint spectrum. Further we study relations between semi-Browder and (essentially) semiregular operators.

Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators in a complex Banach space X and by I the identity operator in X. For T in $\mathcal{L}(X)$ denote by $N(T)=\{x\in X: Tx=0\}$ and $R(T)=\{Tx: x\in X\}$ its kernel and range, respectively. Set further $R^\infty(T)=\bigcap_{k=0}^\infty R(T^k)$ and $N^\infty(T)=\bigcup_{k=0}^\infty N(T^k)$.

The sets of all *upper* (*lower*) *semi-Fredholm operators* in X will be denoted by $\Phi_+(X)$ and $\Phi_-(X)$. Recall that $T\in\Phi_+(X)$ if and only if $\dim N(T)<\infty$ and $R(T)$ is closed; $T\in\Phi_-(X)$ if and only if $\operatorname{codim} R(T)<\infty$ (then $R(T)$ is closed automatically). The *ascent* and *descent* of T are defined by $a(T)=\min\{n: N(T^n)=N(T^{n+1})\}$ and $d(T)=\min\{n: R(T^n)=R(T^{n+1})\}$.

We say that an operator $T\in\mathcal{L}(X)$ is *upper* (*lower*) *semi-Browder* if it is upper (lower) semi-Fredholm and has a finite ascent (descent). The set of all upper (lower) semi-Browder operators in X will be denoted by $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$. Semi-Browder operators were studied by many authors (see e.g. [4], [12], [14], [18], [20]–[22], [24]). The name was introduced in [6].

We extend the notion of semi-Browder operators to n-tuples of commuting operators. We discuss the lower semi-Browder case; the upper case is dual.

Let $T=(T_1,\ldots,T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We use the standard multiindex notation. Denote

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by $\mathbb{Z}_+$ the set of all non-negative integers. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ then $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $T^\alpha = T_1^{\alpha_1} \dots T_n^{\alpha_n}$.

For $k = 0, 1, 2, \dots$, define $M_k(T) = R(T_1^k) + \dots + R(T_n^k)$ and let $M_k'(T)$ be the smallest subspace of X containing the set $\bigcup\{R(T^\alpha) : \alpha \in \mathbb{Z}_+^n$ and $|\alpha| = k\}$. Clearly $X = M_0(T) \supset M_1(T) \supset M_2(T) \supset \dots$ and $X = M_0'(T) \supset M_1'(T) \supset M_2'(T) \supset \dots$ Further

$$M'_{n(k-1)+1}(T) \subset M_k(T) \subset M_k'(T). \tag{1}$$

Indeed, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\alpha| = n(k-1)+1$ then there exists i, $1 \le i \le n$, such that $\alpha_i \ge k$, so that $R(T^\alpha) \subset R(T_i^k) \subset M_k(T)$. This proves the first inclusion of (1) and the second inclusion is clear.

Set $R^\infty(T) = \bigcap_{k=0}^\infty M_k(T) = \bigcap_{k=0}^\infty M_k'(T)$.

If $M_k'(T) = M_{k+1}'(T)$ for some k then it is easy to see by induction that $M_m'(T) = M_k'(T)$ for every $m \ge k$, so that $R^\infty(T) = M_k'(T)$.

As usual we say that an n-tuple $T = (T_1, \dots, T_n)$ of mutually commuting operators in X is *lower semi-Fredholm* ($T \in \Phi_-^{(n)}(X)$) if

$$\operatorname{codim} M_1(T) = \operatorname{codim}(R(T_1) + \dots + R(T_n)) < \infty.$$

Clearly $T = (T_1, \dots, T_n)$ is lower semi-Fredholm if and only if the operator $\widehat{T} : X^n \to X$ defined by $\widehat{T}(x_1, \dots, x_n) = T_1x_1 + \dots + T_nx_n$ is lower semi-Fredholm.

We say that $T = (T_1, \dots, T_n)$ is *lower semi-Browder* if $\operatorname{codim} R^\infty(T) < \infty$. The set of all lower semi-Browder n-tuples will be denoted by $\mathcal{B}_-^{(n)}(X)$. Clearly $\Phi_-^{(n)}(X) \subset \mathcal{B}_-^{(n)}(X)$.

Define

$$\sigma_{\Phi_-}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \Phi_-^{(n)}(X)\},$$

and

$$\sigma_{\mathcal{B}_-}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \mathcal{B}_-^{(n)}(X)\}.$$

It is well known that σ_{Φ_-} has the spectral mapping property [1]. In particular, $(T_1, \dots, T_n) \in \Phi_-^{(n)}(X)$ if and only if $(T_1^k, \dots, T_n^k) \in \Phi_-^{(n)}(X)$. Thus $\operatorname{codim} M_1(T) < \infty$ implies $\operatorname{codim} M_k(T) < \infty$ for every k.

THEOREM 1. *Let $T = (T_1, \dots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. The following statements are equivalent:*

(a) $T \in \mathcal{B}_-^{(n)}(X)$.

(b) $T \in \Phi_-^{(n)}(X)$ *and there exists k such that $M_k'(T) = M_{k+1}'(T)$.*

(c) $T \in \Phi_-^{(n)}(X)$ *and there exists k such that $M_k(T) = M_{k+1}(T)$.*

(d) *There exists a subspace $Y \subset X$ invariant with respect to every T_i ($i = 1, \dots, n$) such that $\operatorname{codim} Y < \infty$ and $T_1Y + \dots + T_nY = Y$. We can take $Y = R^\infty(T)$.*

Proof. (c)⇒(b). Let $M_k(T) = M_{k+1}(T)$ for some k. Using the same argument as in the proof of (1) we can show

$$M'_{n(k-1)+1}(T) = M'_{n(k-1)+2}(T).$$

(b)⇒(a). Let $M_k'(T) = M_{k+1}'(T)$ for some k. Then $M_k(T) \subset M_k'(T) = R^\infty(T)$. Further $T \in \Phi_-^{(n)}(X)$ implies $\operatorname{codim} M_k(T) < \infty$, so that $T \in \mathcal{B}_-^{(n)}(X)$.

(a)⇒(d). Set $Y = R^\infty(T)$. Clearly Y is invariant with respect to T_i ($i = 1, \dots, n$), $\operatorname{codim} Y < \infty$ and $Y = M_k(T) = M_{k+1}(T)$ for some k. If $y \in Y$ then for some $x_1, \dots, x_n \in X$ we have

$$y = \sum_{i=1}^n T_i^{k+1}x_i = \sum_{i=1}^n T_i(T_i^k x_i) \in T_1Y + \dots + T_nY.$$

(d)⇒(c). Since $M_1(T) \supset M_1(T|_Y) = Y$ we have $\operatorname{codim} M_1(T) < \infty$ so that $T \in \Phi_-^{(n)}(X)$. Further $M_1'(T|_Y) = M_0'(T|_Y) = Y$ implies $R^\infty(T|_Y) = Y$ and $M_k(T) \supset M_k(T|_Y) \supset Y$ for every k. Thus the sequence $M_k(T)$ is constant for k large enough.

COROLLARY 2. *Let $T = (T_1, \dots, T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\varepsilon > 0$ such that $(T_1 - \lambda_1, \dots, T_n - \lambda_n) \in \mathcal{B}_-^{(n)}(X)$ for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^n |\lambda_i| < \varepsilon$. Moreover,*

$$\operatorname{codim} R^\infty(T_1 - \lambda_1, \dots, T_n - \lambda_n) \le \operatorname{codim} R^\infty(T_1, \dots, T_n).$$

Proof. Set $Y = R^\infty(T)$. Then $\operatorname{codim} Y < \infty$ and $T_1Y + \dots + T_nY = Y$. There exists $\varepsilon > 0$ such that $(T_1 - \lambda_1)Y + \dots + (T_n - \lambda_n)Y = Y$ if $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $\sum_{i=1}^n |\lambda_i| < \varepsilon$, so that $R^\infty(T_1 - \lambda_1, \dots, T_n - \lambda_n) \supset Y = R^\infty(T_1, \dots, T_n)$.

PROPOSITION 3. *Suppose $T_1, \dots, T_n, S_1, \dots, S_n$ are mutually commuting operators in X such that $\sum_{i=1}^n T_iS_i = I$. Then $(T_1, \dots, T_n) \in \mathcal{B}_-^{(n)}(X)$.*

Proof. Clearly $M_1(T_1, \dots, T_n) = X = M_0(T_1, \dots, T_n)$ so that $(T_1, \dots, T_n) \in \mathcal{B}_-^{(n)}(X)$.

COROLLARY 4. *$\sigma_{\mathcal{B}_-}(T)$ is a compact subset of $\mathbb{C}^n$.*

Proof. $\sigma_{\mathcal{B}_-}(T)$ is closed by Corollary 2. Further $\sigma_{\mathcal{B}_-}(T) \subset \sigma^{\langle T\rangle}(T)$ where $\langle T\rangle$ denotes the smallest closed subalgebra of $\mathcal{L}(X)$ containing $T_1, \dots, T_n$ and the identity operator and $\sigma^{\langle T\rangle}(T)$ denotes the spectrum in the commutative Banach algebra $\langle T\rangle$. Thus $\sigma_{\mathcal{B}_-}(T)$ is bounded and hence compact.

LEMMA 5. *Let $T_1, \dots, T_n, T_{n+1}$ be mutually commuting operators in a Banach space X. Suppose $\operatorname{codim} R^\infty(T_1, \dots, T_n) = \infty$ and let $k \in \mathbb{N}$. Then* *there exists a complex number* λ *such that*

$$\operatorname{codim}[R(T_1^k)+\ldots+R(T_n^k)+R((T_{n+1}-\lambda)^k)]\ge k. \tag{2}$$

Proof. Using condition (c) of Theorem 1 we can distinguish two cases:

(a) $(T_1,\ldots,T_n)\notin \Phi_-^{(n)}(X)$ so that $(0,\ldots,0)\in\sigma_{\Phi_-}(T_1,\ldots,T_n)$. By the projection property for σ_{Φ_-} there exists $\lambda\in\mathbb{C}$ such that $(0,\ldots,0,\lambda)\in\sigma_{\Phi_-}(T_1,\ldots,T_n,T_{n+1})$, i.e., $\operatorname{codim}[R(T_1^k)+\ldots+R(T_n^k)+R((T_{n+1}-\lambda)^k)]=\infty$. Hence we have (2).

(b) $\operatorname{codim} M_m(T)<\infty$ and $M_m(T)\ne M_{m+1}(T)$ for every $m\ge 1$ where $T=(T_1,\ldots,T_n)$. Fix $k\in\mathbb{N}$. Then there exists $i, 1\le i\le n$, such that $R(T_i^{k-1})\not\subset M_k(T)$ (otherwise $M_{k-1}(T)=M_k(T)$). Set $Y=X/M_k(T)$, so that $\dim Y<\infty$ and let $S:Y\to Y$ be defined by $S(x+M_k(T))=T_ix+M_k(T)$. Clearly $S^k=0$ and $S^{k-1}\ne 0$.

Consider the operator $U:Y\to Y$ defined by $U(x+M_k(T))=T_{n+1}x+M_k(T)$. Clearly $US=SU$. Let Z be a subspace of Y satisfying $Z\oplus N(S^{k-1})=Y$. In this decomposition U can be written as

$$U=\begin{pmatrix}U_{11} & 0\\ U_{12} & U_{22}\end{pmatrix}.$$

Choose a complex number λ such that $U_{11}-\lambda$ is singular, i.e., there exists a non-zero $z\in Z$ with $(U-\lambda)z\in N(S^{k-1})$. Since $z\in N(S^k)\setminus N(S^{k-1})$ we have

$$S^{k-m}z\in N(S^m)\setminus N(S^{m-1})\quad (m=1,\ldots,k).$$

Further

$$(U-\lambda)S^{k-m}z=S^{k-m}(U-\lambda)z\in S^{k-m}N(S^{k-1})\subset N(S^{m-1}).$$

For $m=1,\ldots,k$ we have

$$\begin{aligned}\dim[N(S^m)/(U-\lambda)^mN(S^m)]&=\dim N((U-\lambda)^m|_{N(S^m)})\\ &\ge \dim N((U-\lambda)^m|_M),\end{aligned}$$

where $M=N(S^{m-1})\vee\{S^{k-m}z\}$ and $(U-\lambda)^mM\subset(U-\lambda)^{m-1}N(S^{m-1})$. Further

$$\begin{aligned}&\dim N((U-\lambda)^m|_M)=\dim[M/(U-\lambda)^mM]\\ &\ge\dim[M/(U-\lambda)^{m-1}N(S^{m-1})]=\dim[N(S^{m-1})/(U-\lambda)^{m-1}N(S^{m-1})]+1,\end{aligned}$$

since $S^{k-m}z\notin N(S^{m-1})$. Thus, by induction,

$$\dim[N(S^m)/(U-\lambda)^mN(S^m)]\ge m\quad (m=1,\ldots,k).$$

In particular, $\dim(Y/(U-\lambda)^kY)\ge k$. Consequently,

$$\operatorname{codim}[R(T_1^k)+\ldots+R(T_n^k)+R((T_{n+1}-\lambda)^k)]\ge k.$$

COROLLARY 6. *Let* $T_1,\ldots,T_n,T_{n+1}$ *be mutually commuting operators in a Banach space* X. *Suppose that* $\operatorname{codim} R^\infty(T_1,\ldots,T_n)=\infty$. *Then there exists* $\lambda\in\mathbb{C}$ *such that*

$$\operatorname{codim} R^\infty(T_1,\ldots,T_n,T_{n+1}-\lambda)=\infty.$$

Proof. For each $k\ge 1$ we can find $\lambda_k\in\mathbb{C}$ such that

$$\begin{aligned}&\operatorname{codim} R^\infty(T_1,\ldots,T_n,T_{n+1}-\lambda_k)\\ &\qquad\ge\operatorname{codim}[R(T_1^k)+\ldots+R(T_n^k)+R((T_{n+1}-\lambda_k)^k)]\ge k.\end{aligned}$$

Clearly $\lambda_k\in\sigma(T_{n+1})$ for every k. Thus we may assume (by passing to a subsequence if necessary) that the sequence $\{\lambda_k\}$ is convergent, $\lambda_k\to\lambda\in\sigma(T_{n+1})$. We have

$$\lim_{k\to\infty}\operatorname{codim} R^\infty(T_1,\ldots,T_n,T_{n+1}-\lambda_k)=\infty.$$

By Corollary 2 this implies that $\operatorname{codim} R^\infty(T_1,\ldots,T_n,T_{n+1}-\lambda)=\infty$.

COROLLARY 7. *If* $T_1,\ldots,T_n,T_{n+1}$ *are mutually commuting operators, then*

$$\sigma_{\mathcal{B}_-}(T_1,\ldots,T_n)=P\sigma_{\mathcal{B}_-}(T_1,\ldots,T_{n+1}),$$

where $P:\mathbb{C}^{n+1}\to\mathbb{C}^n$ *is the projection onto the first* n *coordinates.*

Proof. The inclusion $\subset$ was proved in Corollary 6. If $(T_1,\ldots,T_n)\in\mathcal{B}_-^{(n)}(X)$ then clearly

$$R^\infty(T_1,\ldots,T_n,T_{n+1}-\lambda)\supset R^\infty(T_1,\ldots,T_n),$$

so that $(T_1,\ldots,T_n,T_{n+1}-\lambda)\in\mathcal{B}_-^{(n+1)}(X)$ for every $\lambda\in\mathbb{C}$. This proves the other inclusion.

COROLLARY 8. $\sigma_{\mathcal{B}_-}$ *is a subspectrum in the sense of Żelazko* [25]. *Consequently, by* [17], $\sigma_{\mathcal{B}_-}$ *has the spectral mapping property*:

$$f\sigma_{\mathcal{B}_-}(T)=\sigma_{\mathcal{B}_-}f(T)$$

for every n*-tuple* $T=(T_1,\ldots,T_n)$ *of mutually commuting operators and every* m*-tuple* $f=(f_1,\ldots,f_m)$ *of functions analytic in a neighbourhood of the Taylor spectrum of* $(T_1,\ldots,T_n)$.

The following lemma is a well-known stability result for semi-Fredholm operators.

LEMMA 9. *Let* $T=(T_1,\ldots,T_n)\in\Phi_-^{(n)}(X)$. *Then there exists* $\varepsilon>0$ *such that* $\operatorname{codim} M_1(S)\le\operatorname{codim} M_1(T)$ *for every commuting* n*-tuple* $S=(S_1,\ldots,S_n)\in\mathcal{L}(X)^n$ *with* $\sum_{i=1}^n\|S_i-T_i\|<\varepsilon$.

The previous lemma enables us to generalize the result of [12] to n-tuples of operators.

THEOREM 10. *Let $T = (T_1, \ldots, T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\varepsilon > 0$ such that $S \in \mathcal{B}_-^{(n)}(X)$ for every commuting n-tuple $S = (S_1, \ldots, S_n) \in \mathcal{L}(X)^n$ with $\sum_{i=1}^n \|S_i - T_i\| < \varepsilon$.*

Proof. Choose k such that $M_k(T) = R^\infty(T)$ and $\operatorname{codim} R^\infty(T) \le k$. Then $(T_1^{k+1}, \ldots, T_n^{k+1}) \in \Phi_-^{(n)}(X)$. By the previous lemma there exists $\varepsilon > 0$ with the following property: if $S = (S_1, \ldots, S_n)$ is a commuting n-tuple of operators in X with $\sum_{i=1}^n \|S_i - T_i\| < \varepsilon$ then $(S_1^{k+1}, \ldots, S_n^{k+1}) \in \Phi_-^{(n)}(X)$ and

$$\begin{aligned}\operatorname{codim} M_1(S_1^{k+1}, \ldots, S_n^{k+1}) &\le \operatorname{codim} M_1(T_1^{k+1}, \ldots, T_n^{k+1}) \\ &= \operatorname{codim} M_{k+1}(T) = \operatorname{codim} R^\infty(T) \le k.\end{aligned}$$

Since $M_1(S) \supset M_2(S) \supset \ldots \supset M_{k+1}(S)$ and $\operatorname{codim} M_{k+1}(S) \le k$, there exists $j \le k$ such that $M_j(S) = M_{j+1}(S)$. Consequently, $S \in \mathcal{B}_-^{(n)}(X)$.

From the general theory of joint spectrum it is easy to deduce the following consequences:

(a) The mapping $(T_1, \ldots, T_n) \mapsto \sigma_{\mathcal{B}_-}(T_1, \ldots, T_n)$ is upper semicontinuous. In particular, if $T_1 \in \mathcal{L}(X)$ and U is a neighbourhood of $\sigma_{\mathcal{B}_-}(T_1)$, then $\sigma_{\mathcal{B}_-}(S_1) \subset U$ for every operator S_1 close enough to T_1.

(b) $\sigma_{\mathcal{B}_-}$ is continuous on commuting elements (see [11, Theorem 1.9]). More precisely, if $\{T_k\}_{k=1}^\infty \subset \mathcal{L}(X)$, $T \in \mathcal{L}(X)$, $\lim T_k = T$ and $T_k T = T T_k$, $k = 1, 2, \ldots$, then $\lambda \in \sigma_{\mathcal{B}_-}(T)$ if and only if there exist $\lambda_k \in \sigma_{\mathcal{B}_-}(T_k)$ such that $\lambda_k \to \lambda$.

(c) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then (cf. [11, Proposition 1.8])

$$\delta(\sigma_{\mathcal{B}_-}(T), \sigma_{\mathcal{B}_-}(S)) \le r_e(T - S),$$

where δ denotes the Hausdorff distance and r_e the *essential spectral radius*,

$$r_e(T) = \max\{|\lambda| : T - \lambda \text{ is not Fredholm}\} = \max\{|\lambda| : T - \lambda \notin \mathcal{B}_-(X)\}$$

(see [7]).

(d) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then

$$TS \in \mathcal{B}_-(X) \Leftrightarrow T, S \in \mathcal{B}_-(X)$$

(see [6] and [16, Theorem 2.1]).

(e) Let T and Q be commuting operators acting in X, let $T \in \mathcal{B}_-(X)$ and let Q be a quasinilpotent. Then $T + Q \in \mathcal{B}_-(X)$ (see e.g. [11, Remark after Theorem 1.9], [18, Theorem 4.1] or [21, Corollary 2]).

Analogously we can define the upper semi-Browder n-tuples. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We say that T is *upper semi-Fredholm* ($T \in \Phi_+^{(n)}(X)$) if the mapping $\widetilde{T} : X \to X^n$ defined by $\widetilde{T}x = (T_1 x, \ldots, T_n x)$ is upper semi-Fredholm.

We say that T is *upper semi-Browder* ($T \in \mathcal{B}_+^{(n)}(X)$) if $T \in \Phi_+^{(n)}(X)$ and $\dim N^\infty(T) < \infty$, where

$$N^\infty(T) = \bigcup_{k=1}^\infty [N(T_1^k) \cap \ldots \cap N(T_n^k)].$$

Define $T^* = (T_1^*, \ldots, T_n^*) \in \mathcal{L}(X^*)^n$.

THEOREM 11. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Then*

$$T \in \mathcal{B}_-^{(n)}(X) \Leftrightarrow T^* \in \mathcal{B}_+^{(n)}(X^*)$$

and

$$T \in \mathcal{B}_+^{(n)}(X) \Leftrightarrow T^* \in \mathcal{B}_-^{(n)}(X^*).$$

Proof. The corresponding equivalences are well-known for semi-Fredholm n-tuples. Further it is easy to check that

$$N(T_1^k) \cap \ldots \cap N(T_n^k) = {}^\perp[R(T_1^{*k}) + \ldots + R(T_n^{*k})].$$

and

$$[R(T_1^k) + \ldots + R(T_n^k)]^\perp = N(T_1^{*k}) \cap \ldots \cap N(T_n^{*k}).$$

The statement of Theorem 11 is now an easy consequence of these identities.

For a commuting n-tuple $T = (T_1, \ldots, T_n) \in \mathcal{L}(X)^n$ we define the *upper semi-Browder spectrum* of T by

$$\sigma_{\mathcal{B}_+}(T) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \ldots, T_n - \lambda_n) \notin \mathcal{B}_+^{(n)}(X)\}.$$

By the previous theorem it is easy to see that $\sigma_{\mathcal{B}_+}$ has the same properties as $\sigma_{\mathcal{B}_-}$.

Define further the *Browder spectrum* $\sigma_{\mathcal{B}}$ of a commuting n-tuple $T = (T_1, \ldots, T_n)$ by

$$\sigma_{\mathcal{B}}(T) = \sigma_{\mathcal{B}_-}(T) \cup \sigma_{\mathcal{B}_+}(T).$$

For a single operator T_1 this definition coincides with the usual definition of the Browder spectrum of T_1 as the union of $\sigma_e(T_1)$ and the limit points of $\sigma(T_1)$, where $\sigma_e(T_1)$ denotes the *essential spectrum* of T_1, i.e.,

$$\sigma_e(T_1) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

and $\sigma(T_1)$ denotes the ordinary spectrum of T_1. Again it is easy to see that $\sigma_{\mathcal{B}}$ has all the properties proved for $\sigma_{\mathcal{B}_-}$.

Remark. The possibility of extending the Browder spectrum to commuting n-tuples was proved in [3]. Our extension

$$\sigma_{\mathcal{B}}(T_1, \ldots, T_n) = \sigma_{\mathcal{B}_-}(T_1, \ldots, T_n) \cup \sigma_{\mathcal{B}_+}(T_1, \ldots, T_n)$$

exhibits similar properties to the spectrum

$$\sigma_{\mathrm{b}}(T_1,\dots,T_n)=\sigma_{\mathrm{Te}}(T_1,\dots,T_n)\cup(\sigma_{\mathrm{T}}(T_1,\dots,T_n))'$$

defined there. (Here σ_{T} and σ_{Te} denote the Taylor and the essential Taylor spectrum and M' denotes the set of all limit points of a set M.) However, these extensions differ for $n\geq 2$; an example will be given later.

The semi-Fredholm and semi-Browder operators are closely related to semiregular and essentially semiregular operators which were intensively studied (under various names; see e.g. [5], [9]–[11], [13], [15], [16], [19] and [23]). An operator $T\in\mathcal{L}(X)$ is called *semiregular* if it has closed range and $N(T)\subset R^\infty(T)$. T is *essentially semiregular* if $R(T)$ is closed and $\dim[N(T)/(N(T)\cap R^\infty(T))]<\infty$.

From a number of equivalent properties of essentially semiregular operators we point out the following Kato decomposition (see [16, Theorem 3.1], [19, Theorem 2.1]).

PROPOSITION 12. *An operator $T\in\mathcal{L}(X)$ is essentially semiregular if and only if $R(T)$ is closed and there exist closed subspaces $X_1,X_2\subset X$ invariant with respect to T such that $X=X_1\oplus X_2$, $\dim X_1<\infty$, $T|_{X_1}$ is nilpotent and $T|_{X_2}$ is semiregular.*

If $T\in\mathcal{L}(X)$ is a lower semi-Browder operator then the space X_2 in the Kato decomposition is uniquely determined and $X_2=R^\infty(T)$. Thus $T|_{X_2}$ is onto. The analogous statement for n-tuples of commuting operator is not true.

EXAMPLE. Denote by H the Hilbert space with an orthonormal basis $\{e_{i,j}: i,j\in\mathbb{Z},\ i\geq 0 \text{ or } j\geq 0\}\cup\{e_{-1,-1}\}$. Define operators $T_1,T_2\in\mathcal{L}(X)$ by

$$T_1e_{i,j}=e_{i+1,j},\qquad T_2e_{i,j}=e_{i,j+1}.$$

We list some properties of the pair (T_1,T_2):

(a) T_1 and T_2 are commuting isometries so that $(T_1,T_2)\in\mathcal{B}_+^{(n)}(X)$.

(b) Set

$$Y=\bigvee\{e_{i,j}: i,j\in\mathbb{Z},\ i\geq 0 \text{ or } j\geq 0\}=\{e_{-1,-1}\}^\perp.$$

Then $T_iY\subset Y$ $(i=1,2)$, $T_1Y+T_2Y=Y$ and $\operatorname{codim} Y=1$. Thus $(T_1,T_2)\in\mathcal{B}_-^{(n)}(X)$.

(c) Denote by σ_{T} the Taylor spectrum. Then $(0,0)\in\sigma_{\mathrm{T}}(T_1,T_2)$. Indeed, $e_{-1,-1}\notin T_1H+T_2H$ so that $T_1H+T_2H\neq H$.

(d) $(0,0)$ is a limit point of the Taylor spectrum of (T_1,T_2). Indeed, if $(0,0)$ were an isolated point of $\sigma_{\mathrm{T}}(T_1,T_2)$ then, using the Taylor functional calculus, it would be possible to decompose H as $H=H_1\oplus H_2$ where $T_iH_j\subset H_j$ $(i,j=1,2)$, $\sigma_{\mathrm{T}}(T_1|_{H_1},T_2|_{H_1})=\{0,0\}$ and $\{0,0\}\notin\sigma_{\mathrm{T}}(T_1|_{H_2},T_2|_{H_2})$.

Since T_1 and T_2 are commuting isometries it would mean that the approximate point spectrum

$$\begin{aligned}&\sigma_\pi(T_1|_{H_1},T_2|_{H_1})\\&=\{(\lambda_1,\lambda_2)\in\mathbb{C}^2:\inf\{\|(T_1-\lambda_1)x\|+\|(T_2-\lambda_2)x\|: x\in H_1,\|x\|=1\}=0\}\end{aligned}$$

is empty. Thus $H_1=\{0\}$, a contradiction with the fact that

$$(0,0)\in\sigma_{\mathrm{T}}(T_1|_{H_1},T_2|_{H_1}).$$

(e) We have

$$(0,0)\in\sigma_{\mathrm{T}}(T_1,T_2)'\subset\sigma_{\mathrm{b}}(T_1,T_2)$$

and

$$(0,0)\notin\sigma_{\mathcal{B}}(T_1,T_2)=\sigma_{\mathcal{B}_+}(T_1,T_2)\cup\sigma_{\mathcal{B}_-}(T_1,T_2).$$

Thus the joint spectra $\sigma_{\mathcal{B}}$ and σ_{b} are different.

(f) In the same way as in (d) one can show that there is no (not necessarily orthogonal) decomposition $H=H_1\oplus H_2$ such that $T_iH_j\subset H_j$ $(i,j=1,2)$, $T_1|_{H_1}$ and $T_2|_{H_1}$ are nilpotent and $T_1H_2+T_2H_2=H_2$. Thus there is no analogy to the Kato decomposition of a single semi-Browder operator.

PROBLEM. Let $T=(T_1,\dots,T_n)$ be a commuting n-tuple of operators in a Banach space X. Denote by σ_δ the *defect spectrum* of T, i.e.,

$$\sigma_\delta(T)=\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n:(T_1-\lambda_1)X+\dots+(T_n-\lambda_n)X\neq X\}.$$

Using Theorem 1 one can obtain

$$\sigma_{\Phi_-}(T)\cup\sigma_\delta(T)'\subset\sigma_{\mathcal{B}}(T).$$

For $n=1$ the opposite inclusion also holds. It is an open problem whether $\sigma_{\Phi_-}(T)\cup\sigma_\delta(T)'=\sigma_{\mathcal{B}}(T)$ for $n\geq 2$.

PROPOSITION 13. *Let T be an essentially semiregular operator on a Banach space X. Then $R^\infty(T)$ is closed, $TR^\infty(T)=R^\infty(T)$ and the operator $\widetilde{T}: X/R^\infty(T)\to X/R^\infty(T)$ induced by T is upper semi-Browder.*

Proof. Set $M=R^\infty(T)$. Let $X=X_1\oplus X_2$ be the Kato decomposition of T (see Proposition 12) and set $T_i=T|_{X_i}$ $(i=1,2)$. Clearly $M=R^\infty(T_2)\subset X_2$. It is well known that M is closed and $TM=M$ (see e.g. [16, Lemma 1.4]). Let $k\geq 1$ and $x=x_1\oplus x_2\in X$ satisfy $T^kx\in M$. Then $T_2^kx_2\in M$ so that $x_2\in M$ (see [16, Lemma 1.4]). Thus $x\in X_1+M$ and $\dim N(\widetilde{T}^k)\leq\dim X_1$. Consequently, $\dim N^\infty(\widetilde{T})\leq\dim X_1<\infty$.

Let $\pi: X\to X/M$ be the canonical projection. As $M\subset R(T)$ and $R(\widetilde{T})=\{Tx+M: x\in X\}=\pi R(T)$, the range of $\widetilde{T}$ is closed. Thus $\widetilde{T}$ is upper semi-Browder.

THEOREM 14. *Let T be an operator on a Banach space X. Then the following conditions are equivalent:*

(a) *T is essentially semiregular.*

(b) *There exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Fredholm and the induced operator $\widetilde{T} : X/M \to X/M$ is upper semi-Fredholm.*

(c) *There exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\widetilde{T} : X/M \to X/M$ is upper semi-Browder.*

(d) *There exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is surjective and the induced operator $\widetilde{T} : X/M \to X/M$ is upper semi-Browder.*

(e) *There exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\widetilde{T} : X/M \to X/M$ is bounded below.*

Proof. By Proposition 13, (a)⇒(d). The implications (d)⇒(c)⇒(b) are straightforward.

(b)⇒(a). First we show that $R(T)$ is closed. Let $\pi : X \to X/M$ be the canonical projection. If $y \in R(T)$, $y = Tx$ for some $x \in X$, then $\pi y = Tx + M = \widetilde{T}(x + M) \in R(\widetilde{T})$, so that $R(T) \subset \pi^{-1}R(\widetilde{T})$. Let $y \in X$ and $\pi y \in R(\widetilde{T})$, i.e., $y + M = Tx + M$ for some $x \in X$. Then $y \in R(T) + M = R(T) + (F + TM) \subset R(T) + F$ for some finite-dimensional subspace F of M. Thus $\pi^{-1}(R(\widetilde{T})) \subset R(T) + F \subset \pi^{-1}(R(\widetilde{T})) + F$. Further $\pi^{-1}(R(\widetilde{T})) + F$ is closed since π is continuous, $R(\widetilde{T})$ is closed and F finite-dimensional. Hence $R(T) + F$ is closed, and so $R(T)$ is closed.

As $\pi N(T) \subset N(\widetilde{T})$ and $\dim N(\widetilde{T})$ is finite-dimensional, there exists a finite-dimensional subspace $G_1 \subset N(T)$ such that $N(T) \subset G_1 + N(T|_M)$. The operator $T|_M$ is lower semi-Fredholm and consequently essentially semiregular, i.e., there exists a finite-dimensional subspace G_2 of M such that $N(T|_M) \subset G_2 + R^\infty(T|_M)$. Thus

$$N(T) \subset G_1 + N(T|_M) \subset G_1 + G_2 + R^\infty(T|_M) \subset (G_1 + G_2) + R^\infty(T),$$

and T is essentially semiregular.

(a)⇒(e). Let $X = X_1 \oplus X_2$ be the Kato decomposition of T, i.e., $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $T|_{X_1}$ is nilpotent and $T_2 = T|_{X_2}$ is semiregular. Set $M = X_1 \oplus R^\infty(T_2) = X_1 \oplus R^\infty(T)$. Clearly, M is closed and since $TR^\infty(T) = R^\infty(T)$, we see that $T|_M$ is a lower semi-Browder operator.

Let $\widetilde{T} : X/M \to X/M$ be the operator induced by T. If $x = x_1 \oplus x_2$ satisfies $Tx \in M$ then $T_2x_2 \in R^\infty(T_2)$, so that $x_2 \in R^\infty(T_2)$ and $x \in M$. Hence $N(\widetilde{T}) = \{0\}$.

We show that $R(\widetilde{T})$ is closed. Let $x, x_k \in X$ $(k = 1, 2, \ldots)$ and let $Tx_k + M \to x + M$ in the topology of X/M. Then $x \in \overline{R(T) + M} = R(T) + M$ since $M \subset R(T) + X_1$. Consequently, $x + M \in R(\widetilde{T})$. Hence $R(\widetilde{T})$ is closed and $\widetilde{T}$ is bounded below.

(e)⇒(b). Clear.

It is well known that if $T \in \mathcal{L}(X)$ is essentially semiregular and K is a compact operator commuting with T then $T + K$ is also essentially semiregular [5, Theorem 5.9]. Now we can prove a sharper result. Let us denote by

$$r_+(T) = \sup\{\varepsilon \ge 0 : T - \lambda I \in \Phi_+(X) \text{ for } |\lambda| < \varepsilon\}$$

and

$$r_-(T) = \sup\{\varepsilon \ge 0 : T - \lambda I \in \Phi_-(X) \text{ for } |\lambda| < \varepsilon\}$$

the *semi-Fredholm radii* of T. An operator $T \in \mathcal{L}(X)$ is upper (lower) semi-Fredholm if and only if $r_+(T) > 0$ $(r_-(T) > 0)$.

LEMMA 15. *Let A be an operator on a Banach space X and let M be a closed subspace of X such that $AM \subset M$. Then $r_e(A|_M) \le r_e(A)$ and $r_e(\widetilde{A}) \le r_e(A)$ where $\widetilde{A} : X/M \to X/M$ is the operator induced by A.*

Proof. Let $A \in \mathcal{L}(X)$ be a Fredholm operator and let $AM \subset M$. Then $R(A|_M)$ is closed (see [2, Lemma 4.3.1]) and $\dim N(A|_M) \le N(A) < \infty$. Thus $A|_M$ is upper semi-Fredholm. Further, $\operatorname{codim} R(\widetilde{A}) \le \operatorname{codim} R(A) < \infty$, and hence $\widetilde{A}$ is lower semi-Fredholm.

The rest follows from the fact that upper and lower semi-Fredholm spectra contain the boundary of the essential spectrum [7].

THEOREM 16. *Let $T, S \in \mathcal{L}(X)$, $TS = ST$ and let T be essentially semiregular. Let $\widehat{T} = T|_{R^\infty(T)}$ and let $\widetilde{T} : X/R^\infty(T) \to X/R^\infty(T)$ be the operator induced by T. If $r_e(S) < \min\{r_-(\widehat{T}), r_+(\widetilde{T})\}$ then $T + S$ is essentially semiregular.*

Proof. By Theorem 14, $\widehat{T} \in \Phi_-(X)$ and $\widetilde{T} \in \Phi_+(X)$. As $TS = ST$, we have $SR^\infty(T) \subset R^\infty(T)$ and we can define the operators $\widetilde{S} : X/R^\infty(T) \to X/R^\infty(T)$ and $\widehat{S} = S|_{R^\infty(T)}$. Clearly, $\widehat{T}\widehat{S} = \widehat{S}\widehat{T}$ and $\widetilde{T}\widetilde{S} = \widetilde{S}\widetilde{T}$. By Lemma 15, $r_e(\widehat{S}) \le r_e(S) < r_-(\widehat{T})$ and $r_e(\widetilde{S}) \le r_e(S) < r_+(\widetilde{T})$. As in [11, Theorem 1.9] we deduce that $\widehat{T} + \widehat{S}$ is lower semi-Fredholm and $\widetilde{T} + \widetilde{S}$ is upper semi-Fredholm. By Theorem 14, $T + S$ is essentially semiregular.

COROLLARY 17. *Let T be an essentially semiregular operator on a Banach space X, $S \in \mathcal{L}(X)$, $TS = ST$ and let S be a Riesz operator (i.e., $r_e(S) = 0$). Then $T + S$ is essentially semiregular.*

For $T \in \mathcal{L}(X)$ define

$$\sigma_\gamma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semiregular}\}$$

and

$$\sigma_{\gamma e}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not essentially semiregular}\}.$$

The spectrum $\sigma_\gamma(T)$ and its essential version $\sigma_{\gamma e}(T)$ were studied (under various names) by many authors (see e.g. [9]–[11], [13], [15], [16], [19] and [23]).

COROLLARY 18. *Let $T \in \mathcal{L}(X)$. Then*

$$\sigma_{\gamma e}(T) = \bigcap \sigma_\gamma(T + S)$$

where the intersection is taken over all Riesz operators in X commuting with T.

Proof. The inclusion $\supset$ follows from [19, Theorem 3.1]. The opposite inclusion follows from the previous corollary.

THEOREM 19. *Let X be an infinite-dimensional Banach space and $S \in \mathcal{L}(X)$. Then the following conditions are equivalent*:

(a) *$\sigma_{\gamma e}(T + S) = \sigma_{\gamma e}(T)$ for every $T \in \mathcal{L}(X)$ commuting with S.*
(b) *S is a Riesz operator.*

Proof. (b)$\Rightarrow$(a). See Corollary 17.
(a)$\Rightarrow$(b). Take $T = 0$. Then $\sigma_{\gamma e}(S) = \sigma_{\gamma e}(0) = \{0\}$. By [19, Corollary 3.4] or [16, Theorem 3.8], $\sigma_e(T) = \{0\}$ so that S is a Riesz operator.

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