factors (see [4]) and for example can be applied in the case of Gaussian–Kronecker automorphisms (see [16]).

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Product $\mathbb{Z}^d$-actions on a Lebesgue space and their applications

by

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Abstract. We define a class of $\mathbb{Z}^d$-actions, $d \geq 2$, called product $\mathbb{Z}^d$-actions. For every such action we find a connection between its spectrum and the spectra of automorphisms generating this action. We prove that for any subset A of the positive integers such that $1 \in A$ there exists a weakly mixing $\mathbb{Z}^d$-action, $d \geq 2$, having A as the set of essential values of its multiplicity function. We also apply this class to construct an ergodic $\mathbb{Z}^d$-action with Lebesgue component of multiplicity $2^d k$, where k is an arbitrary positive integer.

1. Introduction. One of the most important open problems in ergodic theory is the following: does there exist a dynamical system with a given spectrum? This very difficult problem has been solved only for some types of spectra. It is not known in particular whether there exists a dynamical system with Lebesgue spectrum of a finite multiplicity.

Let $T : X \to X$ be an automorphism of a Lebesgue probability space $(X, \mathcal{B}, \mu)$. The spectrum of T is uniquely described by the maximal spectral type and the spectral multiplicity function. We denote the set of essential values of the spectral multiplicity function by $E(T)$. The problem of what subsets of $\mathbb{N}^+ \cup \{\infty\}$ (where $\mathbb{N}^+$ is the set of all positive integers) can be realized as $E(T)$ for an automorphism T is considered e.g. in [A], [BL], [CFS], [GKLL], [MN], [O], [Ro].

Recently Kwiatkowski and Lemańczyk ([KL]) have shown that, for a given set $A \subseteq \mathbb{N}^+$ with $1 \in A$, there exists a weakly mixing T such that $E(T) = A$. In addition, if A is finite then one can find a smooth such T. The goal of this paper is to extend this result to dynamical systems which are actions of the group $\mathbb{Z}^d$ of d-dimensional integers on a Lebesgue probability space. To do this, we introduce a special class of $\mathbb{Z}^d$-actions.

Let Φ be a $\mathbb{Z}^d$-action on a Lebesgue probability space $(X, \mathcal{B}, \mu)$, i.e. Φ is a homomorphism of $\mathbb{Z}^d$ into the group of all automorphisms of $(X, \mathcal{B}, \mu)$. The

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automorphism which corresponds to $g \in \mathbb{Z}^d$ is denoted by Φ^g. The $\mathbb{Z}^d$-action Φ yields the unitary representation $U = U_\Phi$ of $\mathbb{Z}^d$ on $L^2(X,\mu)$ given by

$$U^g f = f \circ \Phi^g, \quad g \in \mathbb{Z}^d,\ f \in L^2(X,\mu).$$

For each $f \in L^2(X,\mu)$ we define the *cyclic space* $Z_\Phi(f)$ as

$$Z_\Phi(f) = \overline{\operatorname{span}}\{U^g f : g \in \mathbb{Z}^d\},$$

and the *spectral measure* $\varrho_f^\Phi = \varrho_f$ on the d-dimensional torus $\mathbb{T}^d$ is defined by

$$\widehat{\varrho}_f[m_1,\dots,m_d] = (U^g f, f),$$

where $g = (m_1,\dots,m_d) \in \mathbb{Z}^d$ and $\widehat{\varrho}_f$ is the Fourier transform of ϱ_f, i.e.

$$\widehat{\varrho}_f[m_1,\dots,m_d] = \int_{\mathbb{T}^d} z_1^{m_1}\dots z_d^{m_d}\, \varrho_f(dz_1,\dots,dz_d).$$

Of course, the subspace $Z_\Phi(f)$ is U_Φ-invariant and it is known that U_Φ on $Z_\Phi(f)$ is spectrally equivalent to the unitary representation $V_\Phi = V_{\Phi,f}$ of $\mathbb{Z}^d$ on $L^2(\mathbb{T}^d,\varrho_f)$ defined by

$$(V_\Phi{}^g h)(z_1,\dots,z_d) = z_1^{m_1}\dots z_d^{m_d} h(z_1,\dots,z_d),$$

$g = (m_1,\dots,m_d)$, $h \in L^2(\mathbb{T}^d,\varrho_f)$.

Let

$$L_0^2(X,\mu) = \Big\{f \in L^2(X,\mu) : \int_X f\,d\mu = 0\Big\}.$$

The spectral theorem says that there exists a sequence $(f_n)_{n\in I} \subset L_0^2(X,\mu)$ where $I = [1,m] \cap \mathbb{N}^+$ for some $m \in \mathbb{N}^+$ or $I = \mathbb{N}^+$ such that

$$L_0^2(X,\mu) = \bigoplus_{n\in I} Z_\Phi(f_n)$$

and

$$\varrho_{f_1} \gg \varrho_{f_2} \gg \dots$$

Moreover, U_Φ on $L_0^2(X,\mu)$ is spectrally equivalent to the $\mathbb{Z}^d$-action Ψ on the space Ω, where

$$\Psi = \bigoplus_{n\in I} V_{\Phi,f_n} \quad \text{and} \quad \Omega = \bigoplus_{n\in I} L^2(\mathbb{T}^d,\varrho_{f_n}).$$

Set $\varrho_n = \varrho_{f_n}$, $n \in I$. The sequence $(\overline{\varrho}_n)$ of the types of (ϱ_n) is uniquely determined and it is in one-to-one correspondence with the set of spectral equivalence classes of $\mathbb{Z}^d$-actions. This sequence is called the *sequence of spectral types* of U_Φ and $\overline{\varrho}_1$ is the *maximal spectral type* of U_Φ. The sequence $(\overline{\varrho}_n)_{n\in I}$ is uniquely described by the pair $(\overline{\varrho}_1, m)$ where $m : \mathbb{T}^d \to \mathbb{N}^+ \cup \{\infty\}$ is a Borel function called the *spectral multiplicity function* of Φ.

A number $k \in \mathbb{N}^+ \cup \{\infty\}$ is said to be an *essential value* of m if

$$\varrho_1(\{z : m(z) = k\}) > 0.$$

A number $k \in \mathbb{N}^+$ is an essential value of m iff ϱ_k is not equivalent to ϱ_{k+1}, while $k = \infty$ is an essential value iff $\varrho_n \not\equiv 0$ for every $n = 1,2,\dots$ We denote the set of essential values of m by $E(\Phi)$.

In this paper we define a class of $\mathbb{Z}^d$-actions called product $\mathbb{Z}^d$-actions and we show that for every such action Φ, generated by automorphisms $T_1,\dots,T_d$, we have

$$E(\Phi) = \bigcup_{s=1}^{d} \bigcup_{1\le i_1<\dots<i_s}^{d} E(T_{i_1})\cdot\ldots\cdot E(T_{i_s}).$$

Next we describe the sequence of spectral types of Φ by the sequences of spectral types of $T_1,\dots,T_d$.

We apply these results to extend the results of [KL] and [L] (see also [A]). Namely, for a given set $A \subseteq \mathbb{N}^+$ such that $1 \in A$ we construct a weakly mixing product $\mathbb{Z}^d$-action Φ such that $E(\Phi) = A$. As another application, we show that for every $k \in \mathbb{N}^+$ there exists an ergodic product $\mathbb{Z}^d$-action Φ with Lebesgue component of multiplicity $2^d k$.

We present our results for $d = 2$. They may be easily extended to arbitrary $d \ge 2$.

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2. The product $\mathbb{Z}^d$-actions and their spectrum. Let T and S be automorphisms of Lebesgue probability spaces $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$, respectively. The action Φ of $\mathbb{Z}^2$, defined on the product space $(X\times Y, \mathcal{B}\otimes\mathcal{C}, \mu\times\nu)$ by the formula

$$\Phi^{(m,n)} = T^m \times S^n, \quad (m,n) \in \mathbb{Z}^2,$$

is called a *product $\mathbb{Z}^2$-action*.

It is clear how to extend this definition to arbitrary $\mathbb{Z}^d$-actions, $d \ge 2$. It is easy to show that Φ is free whenever T and S are aperiodic. Similarly, the ergodicity of T and S guarantees the ergodicity of Φ.

For $f \in L^2(X,\mu)$ we denote by $Z_T(f)$ and ϱ_f^T the cyclic space and the spectral measure generated by f, respectively. In the same way we define $Z_S(g)$ and ϱ_g^S for $g \in L^2(Y,\nu)$. We denote by m_T and m_S the multiplicity functions of the unitary operators U_T and U_S, respectively.

For $f \in L^2(X,\mu)$ and $g \in L^2(Y,\nu)$ we put

$$(f\otimes g)(x,y) = f(x)g(y), \quad x \in X,\ y \in Y.$$

LEMMA 1. *For every $f \in L^2(X,\mu)$ and $g \in L^2(Y,\nu)$ we have*
$$\varrho^{\Phi}_{f\otimes g} = \varrho^T_f \times \varrho^S_g.$$

Proof. It is enough to show that the Fourier transforms of the above measures coincide. Let $(m,n) \in \mathbb{Z}^2$. We have

$$\begin{aligned}\widehat{\varrho}^{\Phi}_{f\otimes g}[m,n] &= (U_{\Phi}^{(m,n)}(f\otimes g), f\otimes g) = (U_{T^m\times S^n}(f\otimes g), f\otimes g)\\ &= (U_T^m f\otimes U_S^n g, f\otimes g) = (U_T^m f, f)(U_S^n g, g)\\ &= \widehat{\varrho}^T_f[m]\cdot\widehat{\varrho}^S_g[n] = \widehat{\varrho^T_f\times\varrho^S_g}[m,n],\end{aligned}$$

which gives the desired equality. ∎

Let C_X, C_Y and $C_{X\times Y}$ be the subspaces of $L^2(X,\mu)$, $L^2(Y,\nu)$ and $L^2(X\times Y,\mu\times\nu)$ respectively, consisting of the constant functions. Then
$$L^2(X,\mu) = C_X \oplus L^2_0(X,\mu),\qquad L^2(Y,\nu) = C_Y\oplus L^2_0(Y,\nu)$$
and
$$L^2(X\times Y,\mu\times\nu) = C_{X\times Y}\oplus L^2_0(X\times Y,\mu\times\nu).$$

LEMMA 2. *If $f_i \in L^2_0(X,\mu)$, $i\in\mathcal{I}$, and $g_j\in L^2_0(Y,\nu)$, $j\in\mathcal{J}$, are such that*
$$L^2_0(X,\mu) = \bigoplus_{i\in\mathcal{I}} Z_T(f_i)\quad\text{and}\quad L^2_0(Y,\nu) = \bigoplus_{j\in\mathcal{J}} Z_S(g_j)\tag{1}$$
then
$$L^2_0(X\times Y,\mu\times\nu) = \bigoplus_{i\in\mathcal{I}}\bigoplus_{j\in\mathcal{J}} Z_\Phi(f_i\otimes g_j)\oplus\bigoplus_{i\in\mathcal{I}} Z_\Phi(f_i\otimes 1)\oplus\bigoplus_{j\in\mathcal{J}} Z_\Phi(1\otimes g_j).$$

Proof. The set of functions $f\otimes g$, where $f\in L^2(X,\mu)$ and $g\in L^2(Y,\nu)$, is linearly dense in $L^2(X\times Y,\mu\times\nu)$. Hence, the set of functions $f\otimes g$, $f\otimes 1$, $1\otimes g$, where $f\in L^2_0(X,\mu)$ and $g\in L^2_0(Y,\nu)$, is linearly dense in $L^2_0(X\times Y,\mu\times\nu)$.

Of course, $f\otimes 1\in\bigoplus_{i\in\mathcal{I}} Z_\Phi(f_i\otimes 1)$ and $1\otimes g\in\bigoplus_{j\in\mathcal{J}} Z_\Phi(1\otimes g_j)$ whenever $f\in L^2_0(X,\mu)$ and $g\in L^2_0(Y,\nu)$. So, it is enough to prove that
$$f\otimes g\in\bigoplus_{i\in\mathcal{I}}\bigoplus_{j\in\mathcal{J}} Z_\Phi(f_i\otimes g_j)$$
for $f\in L^2_0(X,\mu)$ and $g\in L^2_0(Y,\nu)$.

Let $0<\varepsilon<1$. It follows from (1) that there exist finite sets $I_\varepsilon\subset\mathcal{I}$, $J_\varepsilon\subset\mathcal{J}$ and functions $u_n\in Z_T(f_n)$, $v_m\in Z_S(g_m)$, $n\in I_\varepsilon$, $m\in J_\varepsilon$, such that

$$\Big\|f-\sum_{n\in I_\varepsilon} u_n\Big\| < \frac{\varepsilon}{2c}\quad\text{and}\quad\Big\|g-\sum_{m\in J_\varepsilon} v_m\Big\|<\frac{\varepsilon}{2c},$$
where $c = \|f\|+\|g\|+1$. Hence,
$$\begin{aligned}\Big\|f\otimes g-\sum_{n\in I_\varepsilon}\sum_{m\in J_\varepsilon} u_n\otimes v_m\Big\| &\le \Big\|f-\sum_{n\in I_\varepsilon}u_n\Big\|\cdot\|g\| + \Big\|g-\sum_{m\in J_\varepsilon}v_m\Big\|\cdot\Big\|\sum_{n\in I_\varepsilon}u_n\Big\|\\ &< \frac{\varepsilon}{2c}\Big(\|f\|+\|g\|+\frac{\varepsilon}{2c}\Big)<\varepsilon.\end{aligned}\tag{2}$$

Now, we show that
$$u_n\otimes v_m\in Z_\Phi(f_n\otimes g_m),\qquad n\in I_\varepsilon,\ m\in J_\varepsilon.\tag{3}$$

There exist finite sets $K_n, L_m\subset\mathbb{N}$ and complex numbers $a_{n,k}$ and $b_{m,l}$, $k\in K_n$, $l\in L_m$, with
$$\|u_n-u_n^\varepsilon\|<\frac{\varepsilon}{2a},\qquad \|v_m-v_m^\varepsilon\|<\frac{\varepsilon}{2a},$$
where
$$u_n^\varepsilon = \sum_{k\in K_n} a_{n,k}U_T^k f_n,\qquad v_m^\varepsilon=\sum_{l\in L_m} b_{m,l}U_S^l g_m$$
and $a>\|u_n\|+\|v_m\|+1$. Therefore, we obtain
$$\begin{aligned}\|u_n\otimes v_m-u_n^\varepsilon\otimes v_m^\varepsilon\| &\le \|u_n-u_n^\varepsilon\|\cdot\|v_m\|+\|v_m-v_m^\varepsilon\|\cdot\|u_n^\varepsilon\|\\ &\le\|u_n-u_n^\varepsilon\|\cdot\|v_m\|+\|v_m-v_m^\varepsilon\|\Big(\|u_n\|+\frac{\varepsilon}{2a}\Big)<\varepsilon.\end{aligned}$$

Moreover,
$$\begin{aligned}u_n^\varepsilon\otimes v_m^\varepsilon &= \sum_{k\in K_n}\sum_{l\in L_m} a_{n,k}b_{m,l}(U_T^k f_n\otimes U_S^l g_m)\\ &= \sum_{k\in K_n}\sum_{l\in L_m} a_{n,k}b_{m,l}U_\Phi^{(k,l)}(f_n\otimes g_m)\in Z_\Phi(f_n\otimes g_m).\end{aligned}$$

As a consequence, we get (3). From (2) and (3) we obtain
$$f\otimes g = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} u_i\otimes v_j\in\bigoplus_{i\in\mathcal{I}}\bigoplus_{j\in\mathcal{J}} Z_\Phi(f_i\otimes g_j),$$
which ends the proof of Lemma 2. ∎

Now, we are in a position to describe all spectral types of the $\mathbb{Z}^2$-action Φ. Assume that $(\overline{\mu}_i)_{i\in I}$, $(\overline{\nu}_j)_{j\in J}$ are the sequences of spectral types of T and S, respectively. We can present them in the following way.

Let (n_i) and (r_i) be two increasing sequences of positive integers from I and J, respectively. Consider the following partitions of I and J, respectively:

$$I_1 = \{1, \ldots, n_1\}, \quad I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \ \ldots,$$
$$J_1 = \{1, \ldots, r_1\}, \quad J_2 = \{r_1 + 1, \ldots, r_1 + r_2\}, \ \ldots$$

Assume that the functions $n \to \overline{\overline{\mu}}_n$ and $n \to \overline{\overline{\nu}}_n$ are constant on I_k and J_k, respectively, $k \geq 1$. Let

$$\overline{\overline{\mu}}_n = \overline{\mu}_k, \quad n \in I_k,$$
$$\overline{\overline{\nu}}_n = \overline{\nu}_k, \quad n \in J_k.$$

We assume that $\overline{\overline{\mu}}_n \neq \overline{\overline{\mu}}_m$ if n and m belong to different I_k, and $\overline{\overline{\nu}}_n \neq \overline{\overline{\nu}}_m$ if n and m belong to different J_k, $k \geq 1$, i.e.

$$(4) \qquad \begin{cases} \underbrace{\overline{\mu}_1 = \ldots = \overline{\mu}_1}_{n_1} \gg \underbrace{\overline{\mu}_2 = \ldots = \overline{\mu}_2}_{n_2} \gg \ldots, \\ \underbrace{\overline{\nu}_1 = \ldots = \overline{\nu}_1}_{r_1} \gg \underbrace{\overline{\nu}_2 = \ldots = \overline{\nu}_2}_{r_2} \gg \ldots, \end{cases}$$

where $(\overline{\mu}_n)$ and $(\overline{\nu}_n)$ are the sequences of spectral types of T and S, respectively. Let δ be the measure on $\mathbb{T}^1$ defined by

$$\delta(A) = \begin{cases} 1 & \text{if } 1 \in A, \\ 0 & \text{if } 1 \notin A. \end{cases}$$

THEOREM 1. *If Φ is a product $\mathbb{Z}^2$-action generated by ergodic automorphisms T and S, then*

(i) *$E(\Phi) = E(T) \cdot E(S) \cup E(T) \cup E(S)$, where $E(T)$, $E(S)$ are the sets of essential values of m_T and m_S, respectively.*

(ii) *The maximal spectral type of U_Φ is $\overline{(\mu_1 \times \nu_1) + (\mu_1 \times \delta) + (\delta \times \nu_1)}$, where μ_1 and ν_1 are measures of the maximal spectral types of T and S, respectively.*

Proof. By the definition of the essential value, we have

$$E(T) = \{n_1, n_1 + n_2, \ldots\}, \quad E(S) = \{r_1, r_1 + r_2, \ldots\}.$$

Applying the Lebesgue decomposition theorem, we see that there exist pairwise orthogonal measures δ_k, $k \geq 1$, such that

$$\mu_i \equiv \sum_{k=i}^{|I|} \delta_k, \quad i \in I,$$

where $|I|$ is the cardinality of I. Similarly, there exist pairwise orthogonal measures σ_l, $l \geq 1$, such that

$$\nu_j \equiv \sum_{l=j}^{|J|} \sigma_l, \quad j \in J.$$

There exist functions $h_{pk}^{(i)}$, $i, p \in I$, $p \geq i$, $k = 1, \ldots, n_i$, and $g_{ql}^{(j)}$, $j, q \in J$, $q \geq j$, $l = 1, \ldots, r_j$, such that

$$L_0^2(X, \mu) = \bigoplus_{i \in I} \bigoplus_{p \geq i} \bigoplus_{k=1}^{n_i} Z_T(h_{pk}^{(i)}), \quad \varrho^T_{h_{pk}^{(i)}} = \delta_p,$$
$$L_0^2(Y, \nu) = \bigoplus_{j \in J} \bigoplus_{q \geq j} \bigoplus_{l=1}^{r_j} Z_S(g_{ql}^{(j)}), \quad \varrho^S_{g_{ql}^{(j)}} = \sigma_q.$$

Then Lemmas 1 and 2 give

$$(5) \qquad L_0^2(X \times Y, \mu \times \nu) = \bigoplus_{i \in I} \bigoplus_{j \in J} \bigoplus_{p \geq i} \bigoplus_{q \geq j} \bigoplus_{k=1}^{n_i} \bigoplus_{l=1}^{r_j} Z_\Phi(h_{pk}^{(i)} \otimes g_{ql}^{(j)})$$
$$\oplus \bigoplus_{i \in I} \bigoplus_{p \geq i} \bigoplus_{k=1}^{n_i} Z_\Phi(h_{pk}^{(i)} \otimes 1)$$
$$\oplus \bigoplus_{j \in J} \bigoplus_{q \geq j} \bigoplus_{l=1}^{r_j} Z_\Phi(1 \otimes g_{ql}^{(j)})$$

and

$$\varrho_{h_{pk}^{(i)} \otimes g_{ql}^{(j)}} = \delta_p \times \sigma_q, \quad \varrho_{h_{pk}^{(i)} \otimes 1} = \delta_p \times \delta, \quad \varrho_{1 \otimes g_{ql}^{(j)}} = \delta \times \sigma_q$$

for $k = 1, \ldots, n_i$ and $l = 1, \ldots, r_j$. We notice that (5) can be written as

$$L_0^2(X \times Y, \mu \times \nu) = \bigoplus_{p \in I} \bigoplus_{q \in J} \bigoplus_{i=1}^{p} \bigoplus_{j=1}^{q} \bigoplus_{k=1}^{n_i} \bigoplus_{l=1}^{r_j} Z_\Phi(h_{pk}^{(i)} \otimes g_{ql}^{(j)}) \oplus \bigoplus_{p \in I} H_p \oplus \bigoplus_{q \in J} K_q,$$

where

$$H_p = \bigoplus_{i=1}^{p} \bigoplus_{k=1}^{n_i} Z_\Phi(h_{pk}^{(i)} \otimes 1), \quad K_q = \bigoplus_{j=1}^{q} \bigoplus_{l=1}^{r_j} Z_\Phi(1 \otimes g_{ql}^{(j)}).$$

We also notice that the measures $\delta_p \times \sigma_q$, $\delta \times \sigma_q$, $\delta_p \times \delta$, $p \in I$, $q \in J$, are pairwise orthogonal. We set

$$G_{pq} = \bigoplus_{i=1}^{p} \bigoplus_{j=1}^{q} \bigoplus_{k=1}^{n_i} \bigoplus_{l=1}^{r_j} Z_\Phi(h_{pk}^{(i)} \otimes g_{ql}^{(j)}).$$

Hence,

$$L_0^2(X \times Y, \mu \times \nu) = \bigoplus_{p \in I} \bigoplus_{q \in J} G_{pq} \oplus \bigoplus_{p \in I} H_p \oplus \bigoplus_{q \in J} K_q.$$

Let us remark that

• G_{pq} is the Hilbert product of $(n_1+\ldots+n_p)\cdot(r_1+\ldots+r_q)$ cyclic spaces $Z_\Phi(h_{pk}^{(i)}\otimes g_{ql}^{(j)})$ and the spectral measure of each of them is equivalent to $\delta_p\times\sigma_q$,

• H_p is the Hilbert product of $n_1+\ldots+n_p$ cyclic spaces $Z_\Phi(h_{pk}^{(i)}\otimes 1)$ and the spectral measure of each of them is equivalent to $\delta_p\times\delta$,

• K_q is the Hilbert product of $r_1+\ldots+r_q$ cyclic spaces $Z_\Phi(1\otimes g_{ql}^{(j)})$ and the spectral measure of each of them is equivalent to $\delta\times\sigma_q$.

Now, consider the set of numbers of one of the forms $(n_1+\ldots+n_p)(r_1+\ldots+r_q)$, $n_1+\ldots+n_p$, $r_1+\ldots+r_q$, $p,q\ge 1$. Let $(u_s)_{s\in K}$ be the sequence formed from these numbers in such a way that $u_s<u_{s+1}$, $s\in K$. Set

$$D=(I\times J)\cup(I\times\{0\})\cup(\{0\}\times J)$$

and let

$$\begin{aligned} v(p,q)&=(n_1+\ldots+n_p)(r_1+\ldots+r_q) \quad \text{for } p\in I,\ q\in J,\\ v(p,0)&=n_1+\ldots+n_p \quad \text{for } p\in I,\\ v(0,q)&=r_1+\ldots+r_q \quad \text{for } q\in J.\end{aligned}$$

We put

$$N_s=\{(p',q')\in D: v(p',q')\ge u_s\} \quad \text{for } s\in K. \tag{6}$$

Clearly, $N_1\supset N_2\supset\ldots$ For $s\in K$, we define

$$\gamma_s=\sum_{(p',q')\in N_s}(\delta_{p'}\times\sigma_{q'}), \quad \text{where } \sigma_0=\delta_0=\delta. \tag{7}$$

Then

$$\underbrace{\overline{\gamma}_1=\ldots=\overline{\gamma}_1}_{u_1}\gg\underbrace{\overline{\gamma}_2=\ldots=\overline{\gamma}_2}_{u_2-u_1}\gg\ldots \tag{8}$$

is the sequence of spectral types of the action Φ on $L_0^2(X\times Y,\mu\times\nu)$. This means that $E(\Phi)=\{u_1,u_2,\ldots\}$, which concludes the proof. ■

Remark. The measures $\gamma_1,\gamma_2,\ldots$, defined by (7) depend on the measures $\delta_{p'}\times\sigma_{q'}$. However, $\delta_{p'}$ and $\sigma_{q'}$ are not given a priori in (4). There exists a description of (8) using the measures $\mu_i\times\nu_j$, $\mu_i\times\delta$, $\delta\times\nu_j$, $i,j\ge1$, but it is not necessary for our purpose and we omit it.

COROLLARY 1. *For every set $A\subseteq\mathbb{N}^+$ with $1\in A$, there exists a weakly mixing $\mathbb{Z}^d$-action Φ such that $E(\Phi)=A$.*

Proof. For a given set $A\subseteq\mathbb{N}^+$ containing 1, a weakly mixing dynamical system $(X,\mathcal{B},\mu,T)$ with

$$E(T)=A \tag{9}$$

has been constructed in [KL]. Let $(Y,\mathcal{C},\nu,S)$ be a weakly mixing dynamical system with simple spectrum, i.e.

$$E(S)=\{1\}. \tag{11}$$

Let Φ be the product $\mathbb{Z}^2$-action defined by T and S. Observe that Φ is weakly mixing because so are T and S. It follows from Theorem 1 and (9), (10) that

$$E(\Phi)=E(T)=A. \ ■$$

COROLLARY 2. *For every $k\in\mathbb{N}^+$ there exists an ergodic $\mathbb{Z}^2$-action Φ with Lebesgue component of multiplicity 2^2k.*

Proof. It follows from [L] (see also [A]) that for every $k\in\mathbb{N}^+$ there exists an ergodic dynamical system $(X_k,\mathcal{B}_k,\mu_k,T_k)$ having the following sequence of spectral types:

$$\underbrace{\overline{\alpha_1^{(k)}+\lambda}}_{n_1=1}\gg\underbrace{\overline{\lambda}=\ldots=\overline{\lambda}}_{n_2=2k-1}, \tag{11}$$

where $\alpha_1^{(k)}\perp\lambda$ and λ is the Lebesgue measure.

Let $k\in\mathbb{N}^+$. Let Φ be the product $\mathbb{Z}^2$-action on $X_k\times X_1$ defined by T_k and T_1. Now, we apply (6) and (7). We obtain

$$\begin{aligned} &I=\{1,2\},\quad J=\{1,2\},\\ &N_1=(I\times J)\cup(I\times\{0\})\cup(\{0\}\times J)=D,\\ &N_2=\{(1,2),(2,1),(2,2),(2,0),(0,2)\},\\ &N_3=\begin{cases}\{(2,1),(2,2),(2,0)\} & \text{if } k>1,\\ \{(2,2)\} & \text{if } k=1,\end{cases}\\ &N_4=\begin{cases}\{(2,2)\} & \text{if } k>1,\\ \emptyset & \text{if } k=1.\end{cases}\end{aligned}$$

Then

$$u_1=1,\quad u_2=2,\quad u_3=2k,\quad u_4=2^2k \quad \text{for } k>1$$

and

$$u_1=1,\quad u_2=2,\quad u_3=2^2 \quad \text{for } k=1.$$

Hence,

$$\begin{aligned} \gamma_1&=(\alpha_1+\lambda)\times(\alpha_2+\lambda)+(\alpha_1+\lambda)\times\delta+\delta\times(\alpha_2+\lambda),\\ \gamma_2&=(\alpha_1+\lambda)\times\lambda+\lambda\times(\alpha_2+\lambda)+(\delta\times\lambda)+(\lambda\times\delta),\\ \gamma_3&=\begin{cases}(\lambda\times\alpha_2)+(\lambda\times\lambda)+(\lambda\times\delta) & \text{if } k>1,\\ \lambda\times\lambda & \text{if } k=1,\end{cases}\\ \gamma_4&=\begin{cases}\lambda\times\lambda & \text{if } k>1,\\ 0 & \text{if } k=1,\end{cases}\end{aligned}$$

where $\alpha_1 = \alpha_1^{(k)}$, $\alpha_2 = \alpha_1^{(1)}$. Therefore the $\mathbb{Z}^2$-action Φ has a Lebesgue component of multiplicity 2^2k. Similarly, the maximal spectral multiplicity of Φ equals 2^2k. ■

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