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Cohomology groups, multipliers and factors in ergodic theory

by

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Abstract. The problem of compact factors in ergodic theory and its relationship with the problem of extending a cocycle to a cocycle of a larger action are studied.

Introduction. Given an ergodic automorphism $\tau : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ of a Lebesgue space $(Y, \mathcal{C}, \nu)$ call any of its invariant σ-algebras a *factor*. Denote by

$$C(\tau) = \{S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu) : S\tau = \tau S,\ S \text{ invertible}\}$$

the *centralizer* of τ. Endowed with the weak topology in which

$$S_n \to S \quad \text{iff} \quad \mu(S_n^{\pm 1} A \triangle S^{\pm 1} A) \to 0 \quad \text{for each } A \in \mathcal{C},$$

it becomes a Polish group. If $\mathcal{H} \subset C(\tau)$ is a subgroup then it determines a factor $\mathcal{A}(\mathcal{H})$ given by

$$\mathcal{A}(\mathcal{H}) = \{A \in \mathcal{C} : SA = A \text{ for each } S \in \mathcal{H}\}.$$

On the other hand, a factor $\mathcal{A}$ determines a subgroup $\mathcal{H}(\mathcal{A}) \subset C(\tau)$ by

$$\mathcal{H}(\mathcal{A}) = \{S \in C(\tau) : SA = A \text{ for each } A \in \mathcal{A}\}.$$

From this point of view compact subgroups are of special interest as for them

$$\mathcal{H}(\mathcal{A}(\mathcal{H})) = \mathcal{H} \tag{1}$$

(see [5], [17]). Moreover, in this case τ can be represented as a compact group extension T_φ defined on the space $(X \times \mathcal{H}, \widetilde{\mu})$, where X stands for the quotient space corresponding to the factor $\mathcal{A}(\mathcal{H})$, $\widetilde{\mu}$ for the product measure of the corresponding image of ν with Haar measure $m_{\mathcal{H}}$ and T denotes the quotient action of τ; T_φ is defined by

$$T_\varphi(x, S) = (Tx, \varphi(x)S),$$

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where $\varphi : X \to \mathcal{H}$ is a cocycle (i.e. a measurable map, see next section for an explanation of the vocabulary).

Assume that we know all factors of T and we are interested what new factors appeared for T_φ (clearly T is a factor of $T_\varphi = \tau$). Under certain assumptions on T (e.g. 2-simplicity, see [9]) we show that any factor of T_φ will be determined by a compact subgroup in the centralizer of a certain natural factor of T_φ. Assume that $\mathcal{H}$ is additionally Abelian. Here, by a natural factor we mean a factor obtained by a compact subgroup of $\mathcal{H}$. It is clear that given a factor $\mathcal{E}$ of T_φ we can choose the largest compact subgroup $\mathcal{F} \subset \mathcal{H}$ such that $\mathcal{E}$ is still a factor of this natural factor (see [9], [11] for further details). Hence we conclude that $\mathcal{E}$ is a factor of $T_{\varphi\mathcal{F}} : X \times \mathcal{H}/\mathcal{F} \to X \times \mathcal{H}/\mathcal{F}$. So to simplify the notation we can assume that $\mathcal{F} = \{\mathrm{Id}\}$. Thus there exists a compact subgroup $\mathcal{H}' \subset C(T_\varphi)$ that determines $\mathcal{E}$. Due to our assumptions on T, each element $\widehat{W} \in \mathcal{H}'$ is of the form $\widehat{W} = W_{f,v}$, $W_{f,v}(x, S) = (W(x), f(x)v(S))$, where $W \in C(T)$, $f : X \to \mathcal{H}$ is measurable and $v : \mathcal{H} \to \mathcal{H}$ is a continuous group automorphism (see [11], [13]). Let $\sigma_S(x, R) = (x, RS)$; then clearly $\sigma_S \in C(T_\varphi)$. The group $\{\widehat{W}\sigma_S : \widehat{W} \in \mathcal{H}',\ S \in \mathcal{H}\}$ is then compact (e.g. [8], p. 55). However, if two liftings of the same W are in $\mathcal{H}'$ then they differ by a certain $S \in \mathcal{H}$ and hence all sets of $\mathcal{E}$ are invariant under σ_S. This, however, contradicts our maximal choice of the group $\mathcal{F}$. Hence our subgroup $\mathcal{H}'$ chooses only one lift (see Lemma 2.6 of [8]). Following [7], we call such a subgroup *diagonal*. Hence under all our assumptions all factors of our group extension are determined by compact diagonal subgroups.

We will now try to reverse the problem. Suppose that we are given a group extension $T_\varphi : X \times G \to X \times G$, where G is a compact Abelian group. Suppose moreover that a compact subgroup $\mathcal{R} \subset C(T)$ is given with the property that each element S of it has an extension to an element $\widehat{S}$ of $C(T_\varphi)$ (i.e. can be lifted to $C(T_\varphi)$). Is there a diagonal subgroup in $\{\widehat{S}\sigma_g : S \in \mathcal{R},\ g \in G\}$? A partial answer to this question has been given by J. Kwiatkowski [7] (in case of T an automorphism with discrete spectrum) in terms of some functional equations. In this note we give further clarification of this problem and exhibit its relationship with the problem of extending a cocycle to a cocycle defined for a larger group (see [2], [3]). Namely, we will show that (under some circumstances) the problem of the existence of compact factors is equivalent to the problem of extending a $\mathbb{Z}$-cocycle to a cocycle of a larger action.

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1. A scheme from cohomology theory. Assume that $\mathcal{S}$ is a group (so $Z(\mathcal{S})$ will be a natural ring coming from $\mathcal{S}$). Let A be an Abelian group, where we assume that A has a fixed structure of an $\mathcal{S}$-module. For each $n \ge 0$ denote by

$$F^n(\mathcal{S}, A)$$

the corresponding module of all functions from $\mathcal{S}^n$ to A. The elements of $F^n(\mathcal{S}, A)$ are called *n-cochains*. The nth derivative $\delta^n : F^n(\mathcal{S}, A) \to F^{n+1}(\mathcal{S}, A)$ is introduced by the formula

$$(2)\qquad \delta^n(\varphi)(s_1, \ldots, s_{n+1}) = s_1\varphi(s_2, \ldots, s_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(s_1, \ldots, s_{i-1}, s_is_{i+1}, s_{i+2}, \ldots, s_{n+1}) + (-1)^{n+1}\varphi(s_1, \ldots, s_n).$$

We then have $\delta^{n+1} \circ \delta^n = 0$. In standard cohomology language, a 1-*cocycle* is $f : \mathcal{S} \to A$ with

$$\delta^1 f = 0, \quad \text{i.e.} \quad f(s_1s_2) - s_1f(s_2) - f(s_1) = 0.$$

A 1-cocycle f is a 1-*coboundary* if there exists $a \in A$ such that

$$f(s) = sa - a.$$

Also, a 2-cochain φ is a 2-*cocycle* if

$$s_1\varphi(s_2, s_3) - \varphi(s_1s_2, s_3) + \varphi(s_1, s_2s_3) - \varphi(s_1, s_2) = 0,$$

and it will be a 2-*coboundary* provided that there exists a 1-cochain f such that

$$\varphi(s_1, s_2) = s_1f(s_2) + f(s_1) - f(s_1s_2).$$

In the measure-theoretic context, to all the above notions we will always add certain measurability conditions. The corresponding groups of n-cocycles and n-coboundaries will be denoted by $Z^n(\mathcal{S}, A)$ and $B^n(\mathcal{S}, A)$ respectively. In what follows 1-cocycles (1-coboundaries) will be called *cocycles* (*coboundaries*).

EXAMPLE 1. Suppose that $\mathcal{S}$ acts on $(X, \mathcal{B}, \mu)$ ($s \mapsto T_s$ with $T_{s_1s_2} = T_{s_1}T_{s_2}$) and let G be a compact Abelian group. Denote by $A = \mathcal{M}(X, G)$ the group (under pointwise multiplication) of all measurable functions from X into G. Then $\mathcal{S}$ acts on A as

$$sf := fT_s^{-1} = T_{s^{-1}}.$$

Consequently, 1-cocycles can be identified with measurable functions $f = f(s, x)$ of two variables with values in G satisfying

$$f(s_1s_2, x) = f(s_1, x) + f(s_2, T_{s_1}^{-1}x),$$

while 1-coboundaries are of the form

$$f(s, x) = gT_s^{-1}(x) - g(x)$$

for a $g \in \mathcal{M}(X, G)$.

Example 2. We will now complicate our picture by adding to the above representation of $\mathcal{S}$ another representation of $\mathcal{S}$ in $\mathrm{Aut}(G)$ (the group of continuous group automorphisms of G). We will denote this new representation by $s \mapsto v_s$ with $v_{s_1s_2} = v_{s_1}v_{s_2}$. The corresponding action of $\mathcal{S}$ on $A = \mathcal{M}(X, G)$ is given by

$$sf = v_s(fT_s^{-1}).$$

This definition is correct:

$$s(f_1 + f_2) = v_s((f_1 + f_2)T_s^{-1}) = v_s(f_1T_s^{-1} + f_2T_s^{-1}) = sf_1 + sf_2$$

and moreover,

$$(s_1s_2)f = v_{s_1s_2}fT_{s_1s_2}^{-1} = v_{s_1}(v_{s_2}(fT_{s_2}^{-1}))T_{s_1}^{-1} = s_1(s_2f).$$

We find that a 1-cocycle f is a measurable function satisfying

$$f(s_1s_2, x) = f(s_1, x) + v_{s_1}f(s_2, T_{s_1}^{-1}x),$$

and a 1-coboundary satisfies

$$f(s, x) = v_sg(T_s^{-1}x) - g(x)$$

for a $g \in A$.

Assume now that two groups $\mathcal{S}, \mathcal{R}$ act on $(X, \mathcal{B}, \mu)$ via $s \mapsto T_s$, $r \mapsto T_r$. We will assume that both act *freely* (i.e. for a.e. $x \in X$ the maps $s \mapsto T_sx$ and $r \mapsto T_rx$ are bijections) and that $\{T_s : s \in \mathcal{S}\} \cap \{T_r : r \in \mathcal{R}\} = \{\mathrm{Id}\}$. Suppose moreover that

$$T_sT_r = T_rT_s \quad \text{for all } s \in \mathcal{S},\ r \in \mathcal{R}.$$

Let $\mathcal{P}$ denote the group generated by T_sT_r, $r \in \mathcal{R}$, $s \in \mathcal{S}$ (i.e. $\mathcal{P}$ can be identified with $\mathcal{S} \times \mathcal{R}$). Assume that $\varphi : \mathcal{S} \times X \to G$ and $\psi : \mathcal{R} \times X \to G$ are cocycles. Moreover, assume that $v_s = \mathrm{Id}$ for all $s \in \mathcal{S}$ and let $r \mapsto v_r$ be a representation of $\mathcal{R}$ in $\mathrm{Aut}(G)$. Then we get easily the following.

Proposition 1. *The formula $F(T_sT_r, x) = \varphi(s, x) + \psi(r, T_s^{-1}x)$ defines a cocycle for $\mathcal{P}$ iff for every $s \in \mathcal{S}$ and $r \in \mathcal{R}$,*

$$\varphi(s, x) + \psi(r, T_s^{-1}x) = \psi(r, x) + v_r\varphi(s, T_r^{-1}x). \ \blacksquare$$

2. Cohomology, extensions of cocycles and lifting subgroups. We assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and let $\varphi : X \to G$ be a cocycle. According to Example 1 the $\mathbb{Z}$-cocycle generated by φ is given by

$$\varphi(n, x) = \begin{cases} \varphi(T^{-1}x) + \varphi(T^{-2}x) + \ldots + \varphi(T^{-n}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\varphi(x) - \varphi(Tx) - \ldots - \varphi(T^{-n-1}x) & \text{if } n < 0. \end{cases}$$

Denote by $T_\varphi : (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu})$ the corresponding group extension of T, where $T_\varphi(x, g) = (Tx, \varphi(x) + g)$ and $\widetilde{\mu}$ is the product measure of μ and Haar measure m_G on G. Let $L(\varphi, T) \subset C(T)$ denote the set of those elements from $C(T)$ that can be lifted to $C(T_\varphi)$. It is known (see [3]) that $L(\varphi, T)$ is a Borel subset of $C(T)$. We assume that φ is ergodic, i.e. that T_φ is ergodic. According to [9], [11], [13] each lift is then of the form $S_{f,v}$,

$$S_{f,v}(x, g) = (Sx, f(x) + v(g)),$$

where $v : G \to G$ (a continuous group automorphism) is unique while $f : X \to G$ (a measurable map) is determined up to a constant from G. In other words, the functional equation

$$\varphi S - v\varphi = fT - f \tag{3}$$

is satisfied. The group $\widetilde{C}(T_\varphi) := \{S_{f,v} : S \in L(\varphi, T)\}$ is a closed subgroup of $C(T_\varphi)$. Let $\pi : \widetilde{C}(T_\varphi) \to C(T)$, $\pi(S_{f,v}) = S$, be the natural projection. Assume now that $\mathcal{R} \subset L(\varphi, T)$ is a Borel subgroup. With each Borel selector $c : \mathcal{R} \to \widetilde{C}(T_\varphi)$ (they do exist, see e.g. [6]) for π we will associate a certain 2-cocycle (in an appropriate cohomology).

Let $c : \mathcal{R} \to \widetilde{C}(T_\varphi)$ be a Borel selector for π. So for each $S \in \mathcal{R}$ we have a measurable choice of $S_{f_S, v_S} \in \widetilde{C}(T_\varphi)$. Therefore, for each $S_1, S_2 \in \mathcal{R}$ we have

$$\varphi(S_1S_2) - v_{S_1S_2}\varphi = f_{S_1S_2}T - f_{S_1S_2}. \tag{4}$$

Moreover,

$$\varphi(S_1S_2) - v_{S_1}\varphi S_2 = f_{S_1}S_2T - f_{S_1}S_2$$

and

$$v_{S_1}\varphi(S_2) - v_{S_1S_2}\varphi = v_{S_1}f_{S_2}T - v_{S_1}f_{S_2}.$$

Adding the above two equalities we get

$$\varphi(S_1S_2) - v_{S_1S_2}\varphi = (f_{S_1}S_2 + v_{S_1}f_{S_2})T - (f_{S_1}S_2 + v_{S_1}f_{S_2}).$$

By comparing it with (4) and using the ergodicity of T we finally obtain

$$f_{S_1S_2} = f_{S_1}S_2 + v_{S_1}f_{S_2} + \overline{c}_{S_1,S_2} \tag{5}$$

for a constant $\overline{c}_{S_1,S_2} \in G$. Since the map $S \mapsto f_S$ is Borel (by our special choice) and $S \mapsto v_S$ is also Borel (see [3]), the function $\overline{c} : \mathcal{R} \times \mathcal{R} \to G$ is Borel. By a simple calculation we get the following.

Proposition 2. $v_{S_1}(\overline{c}_{S_2,S_3}) + \overline{c}_{S_1,S_2S_3} = \overline{c}_{S_1,S_2} + \overline{c}_{S_1S_2,S_3}$, *i.e.* $\overline{c} : \mathcal{R} \times \mathcal{R} \to G$ *belongs to* $Z^2(\mathcal{R}, G)$, *where* $\mathcal{R}$ *acts on* G *by* v_S*'s.*

Suppose now that the above 2-cocycle $\overline{c}$ is in fact a 2-coboundary, i.e.

$$\overline{c}_{S_1,S_2} = -d_{S_1S_2} + v_{S_1}d_{S_2} + d_{S_1}$$

for a measurable $d : \mathcal{R} \to G$. Set $\widetilde{f}_S = f_S + d_S$. Then

$$\widetilde{f}_{S_1S_2} = \widetilde{f}_{S_1}S_2 + v_{S_1}\widetilde{f}_{S_2}$$

and by further change $\widehat{f}_S = \widetilde{f}_S S^{-1}$ we finally obtain

$$\widehat{f}_{S_1S_2} = v_{S_1}\widehat{f}_{S_2}S_1^{-1} + \widehat{f}_{S_1},$$

which simply means that $\widehat{f}(S,x) = \widehat{f}_S(x)$ is a 1-cocycle (as in Example 2).

Let us put $f_{\mathrm{Id}} = 0$; we also have $v_{\mathrm{Id}} = \mathrm{Id}$. It follows that

$$\overline{c}_{S,\mathrm{Id}} = f_S - f_S \,\mathrm{Id} - v_S f_{\mathrm{Id}} = 0 = \overline{c}_{\mathrm{Id},S}.$$

Therefore, our 2-cocycle $\overline{c}$ has the following additional property:

$$\overline{c}_{S,\mathrm{Id}} = \overline{c}_{\mathrm{Id},S} = 0 \quad \text{for all } S \in \mathcal{R}. \tag{6}$$

We can summarize the remarks of this section in the following.

PROPOSITION 3. *Let $\mathcal{R} \subset L(\varphi, T)$ be a Borel subgroup such that no nontrivial power of T is in $\mathcal{R}$. The following conditions are equivalent.*

(1) There exists a Borel selector $c : \mathcal{R} \to \widetilde{C}(T_\varphi)$ such that the corresponding 2-cocycle $\overline{c}$ is in fact a 2-coboundary (equivalently, $(\widehat{f}_S)$ is a 1-cocycle).

(2) There exists a Borel selector $c : \mathcal{R} \to \widetilde{C}(T_\varphi)$ which is a group homomorphism (in other words, there is a measurable choice of a diagonal subgroup in the whole lifting of $\mathcal{R}$ in $\widetilde{C}(T_\varphi)$, i.e. of a subgroup which chooses exactly one element from $\pi^{-1}(S)$, $S \in \mathcal{R}$).

(3) The $\mathbb{Z}$-cocycle φ has an extension to a cocycle of the subgroup of $C(T)$ generated by T and $\mathcal{R}$.

Proof. It is not difficult to check that $(\widehat{f}_S)_{S\in\mathcal{R}}$ with $\widehat{f}_S := f_S S^{-1}$ satisfies the 1-cocycle equation if and only if $\{S_{f_S,v_S} : S \in \mathcal{R}\}$ is in fact a subgroup of $C(T_\varphi)$. The rest follows from Proposition 1. ■

Remark 1. Notice that if we have two diagonal subgroups over $\mathcal{R}$ then we obtain two 1-cocycles with values in $\mathcal{M}(X,G)$, and since the cocycles form a group, their difference will be a 1-cocycle for $\mathcal{R}$ taking values in the space of constant functions. This means that two diagonal groups determine a Borel map $m : \mathcal{R} \to G$ satisfying $m(S_1S_2) = m(S_1) + v_{S_1}m(S_2)$. In particular, if each $v_S = \mathrm{Id}$ then m is a continuous group homomorphism. In this latter case, if a diagonal subgroup exists, then all others are in 1-1 correspondence with continuous group homomorphisms of $\mathcal{R}$ into G.

3. Kwiatkowski's observation. In [7], J. Kwiatkowski shows that if T is an ergodic rotation and $\varphi : X \to G$ an ergodic cocycle, and if a compact subgroup $\mathcal{R} \subset C(T)$ lifts to $C(T_\varphi)$ in such a way that there exists a diagonal compact subgroup in the lifting then (using our language—see Proposition 2) the extended cocycle acting on the group generated by T and $\mathcal{R}$ must be a coboundary on $\mathcal{R}$. Observe that $\mathcal{R}$ acts by rotations so in particular this action is free.

We will now briefly argue that such a result is true in a wider context.

PROPOSITION 4. *Assume that $\mathcal{S}$ is a compact second countable group acting freely on $(X, \mathcal{B}, \mu)$. Then for any cohomology of $\mathcal{S}$ with coefficients in $A = \mathcal{M}(X,G)$ all cohomology groups are trivial.*

Proof. It is classical that in the case of an action of a compact group $\mathcal{S}$, the partition of X into trajectories of the action is measurable. Hence there exists a Borel set B (in general of measure zero) such that

$$X = \bigcup_{x\in B} \mathcal{S}x$$

and this union is disjoint. Suppose now that $F = F(s_1,\ldots,s_{n+1},x)$ is in $Z^{n+1}(\mathcal{S}, A)$, where $A = \mathcal{M}(X,G)$. In view of (2) we wish to show that

$$F(s_1,\ldots,s_{n+1},x) = \varphi(s_2,\ldots,s_{n+1},T_{s_1}^{-1}x)$$
$$+\sum_{i=1}^{n}(-1)^i\varphi(s_1,\ldots,s_{i-1},s_is_{i+1},s_{i+2},\ldots,s_{n+1},x)+(-1)^{n+1}\varphi(s_1,\ldots,s_n,x)$$

for a $\varphi \in F^n(\mathcal{S}, A)$. So, given $x \in B$ choose $\varphi_x : \mathcal{S}^n \to G$ so that the map $\varphi_\cdot(\cdot) : B \times \mathcal{S}^n \to G$ is Borel and then extend it to a cocycle on the whole trajectory using the above equality. ■

In particular, we have obtained

COROLLARY 1. *If $\mathcal{R}$ is compact, acts freely and there is a diagonal subgroup in the lifting (or equivalently, if there exists a selector c whose 2-cocycle $\overline{c}$ is a 2-coboundary) then the extended cocycle must be a coboundary on $\mathcal{R}$.* ■

Remark 2. The only fact which we used in the proof that for compact actions the cohomology groups are trivial was that the partition into trajectories was measurable. Since this is a particular case of the so-called actions of type I, the above result can easily be obtained from [15].

4. Multipliers and extensions of cocycles. All the material of this section is taken from [14]. Let $\mathcal{S}$ be a locally compact second countable group. By $\mathbb{T}$ we denote the circle, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A Borel function $c : \mathcal{S} \times \mathcal{S} \to \mathbb{T}$ is called a *multiplier* if it satisfies

$$c(s_1, s_2s_3)c(s_2,s_3) = c(s_1,s_2)c(s_1s_2,s_3), \qquad c(s,e) = c(e,s) = 1$$

for all $s, s_1, s_2, s_3 \in \mathcal{S}$, where e stands for the unit of G. We say that a multiplier c is *trivial* if there exists a Borel function $d : \mathcal{S} \to \mathbb{T}$ such that

$$c(s_1,s_2) = d(s_1)d(s_2)d(s_1s_2)^{-1}.$$

Remark 3. It follows from Proposition 2 and (6) that if $v_S = \mathrm{Id}$ for each $S \in \mathcal{R}$, then for any character $\chi \in \widehat{G}$ the function $\chi(\overline{c})$ is a multiplier.

Given a multiplier c on $\mathcal{S}\times\mathcal{S}$ we consider the group $\mathcal{S}^c$ (called the *c-extension* of $\mathcal{S}$) which lives on $\mathcal{S}\times\mathbb{T}$ and whose multiplication is defined by

$$(s_1,z_1)(s_2,z_2)=(s_1s_2,z_1z_2c(s_1,s_2)).$$

On $\mathcal{S}^c$ we consider its Weil topology (the product measure of Haar measure $m_{\mathcal{S}}$ on $\mathcal{S}$ and Lebesgue measure on $\mathbb{T}$ is a right invariant measure on $\mathcal{S}^c$) which makes $\mathcal{S}^c$ a locally compact group. Consider $U:\mathcal{S}^c\to\mathcal{U}(L^2(\mathcal{S},m_{\mathcal{S}}))$ given by

$$U_{(s,z)}f(t)=zc(t,s)f(ts).$$

A direct computation shows that U is a representation of $\mathcal{S}^c$ in $L^2(\mathcal{S},m_{\mathcal{S}})$. It is also continuous (Lemma 3.2 of [14]). The map

$$\widetilde{U}:\mathcal{S}\to\mathcal{U}(L^2(\mathcal{S},m_{\mathcal{S}})),\qquad \widetilde{U}_sf(t)=c(t,s)f(ts),$$

is not a representation of $\mathcal{S}$ ($\widetilde{U}_{s_1}\widetilde{U}_{s_2}=c(s_1,s_2)\widetilde{U}_{s_1s_2}$) but if a subspace $V\subset L^2(\mathcal{S},m_{\mathcal{S}})$ is invariant under U then it is also invariant under $\widetilde{U}$. If $\mathcal{S}$ is additionally compact then so is $\mathcal{S}^c$ (Theorem 5.1 of [14]). Hence U has an invariant finite-dimensional (say of dimension k) subspace $V\subset L^2(\mathcal{S},m_{\mathcal{S}})$. Since this subspace is also invariant for $\widetilde{U}$, by taking the determinant of $\widetilde{U}_s$ in V we obtain Bargman's theorem (Theorem 5.1 of [14]):

(7) if $\mathcal{S}$ is compact and c is a multiplier then c^k is trivial.

There are groups for which each multiplier is trivial. We have (see Corollary 1 and Theorem 5.3 of [14]):

(8) every multiplier is trivial whenever $\mathcal{S}$ is either a compact, connected and simply connected or $G=\mathbb{R}$.

5. Factors in ergodic theory and cohomology of cocycles. Suppose that $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is ergodic and $\varphi:X\to G$ is an ergodic cocycle, where G is a compact Abelian group. Suppose that a compact subgroup $\mathcal{R}\subset L(\varphi,T)$ with a representation $S\mapsto v_S\in\mathrm{Aut}(G)$ are fixed. Put

$$\Gamma=\{\psi:X\to G:\psi S=v_S\psi \text{ for each } S\in\mathcal{R}\}.$$

It is clear that Γ is a subgroup of cocycles. Suppose that φ extends to a cocycle of the group generated by T and $\mathcal{R}$. Since $\mathcal{R}$ is compact the extended cocycle on $\mathcal{R}$ must be a coboundary (and this is all we are going to use). So we have a measurable choice of $(f_S)_{S\in\mathcal{R}}$ such that $(\widehat{f}_S)_{S\in\mathcal{R}}$ is an $\mathcal{R}$-coboundary. Hence there exists a measurable $g:X\to G$ such that

$$f_SS^{-1}=\widehat{f}_S=v_SgS^{-1}-g$$

and therefore $f_S=v_Sg-gS$ for each $S\in\mathcal{R}$. Now we can write

$$\varphi S-v_S\varphi=(v_Sg-gS)T-(v_Sg-gS)$$

for all $S\in\mathcal{R}$, so

$$(\varphi+gT-g)S=v_S(\varphi+gT-g),$$

and if we put $\psi:=\varphi+gT-g$ we see that φ and ψ are cohomologous and $\psi\in\Gamma$.

In particular, we have proved the following.

PROPOSITION 5. *Let $\mathcal{R}$ be a compact group in the centralizer of T which lifts to $C(T_\varphi)$. Then there is a diagonal subgroup in the lifting if and only if φ is cohomologous to a cocycle from Γ. If in addition $\mathcal{R}$ is connected then this is equivalent to saying that φ is cohomologous to an $\mathcal{R}$-invariant cocycle (or, what is the same, to a cocycle already measurable with respect to the factor $\mathcal{A}(\mathcal{R})$ of T determined by $\mathcal{R}$).*

Proof. One need only apply the Continuous Embedding Lemma from [3] which in the case of connected groups says that all v_S's must be equal to the identity. ∎

Remark 4. If T belongs to $\mathcal{R}\subset L(\varphi,T)$ which is compact and if a diagonal subgroup exists then the corresponding cocycle for $\mathcal{R}$ is a coboundary, but it only means that $\varphi+g_0$ is a T-coboundary as we make a selection over each $S\in\mathcal{R}$ so we choose $T_\varphi\sigma_{g_0}$ as an element of a diagonal subgroup.

EXAMPLE 3. Let $X=\mathbb{T}\times\mathbb{T}$, $G=\mathbb{T}$. Assume that T is an ergodic rotation by (α,β), so $T=T_\alpha\times T_\beta$. Let $\varphi:X\to G$ be defined as

$$\varphi(z_1,z_2)=z_2.$$

It is then well known that the corresponding extension is ergodic, while our cocycle is invariant under $\mathcal{R}=\{T_\gamma\times\mathrm{Id}:\gamma\in[0,1)\}$. Obviously, we can consider the factor $(T_\beta)_\varphi$ as φ depends only on the second coordinate. This factor corresponds to $\mathcal{R}$ which is a diagonal subgroup of $C(T_\varphi)$.

EXAMPLE 4. Let $X=\mathbb{T}^n$ ($n=1,2,\ldots,\infty$) and T be an ergodic rotation. Suppose that $\varphi:X\to G$ is ergodic and *all* elements of $C(T)$ lift. Thus if there is a diagonal subgroup in the centralizer of $C(T_\varphi)$ then φ has to be cohomologous to a constant. (Indeed, here $\mathcal{R}$ acts ergodically so by Proposition 5 if a diagonal subgroup exists then φ must be cohomologous to a cocycle already measurable with respect to $\mathcal{A}(\mathcal{R})$, which by ergodicity is trivial.)

EXAMPLE 5. The assumption about the existence of a diagonal subgroup is essential. Rotation by α on the circle is a 2-point extension of the rotation by 2α. All elements from the centralizer of the rotation by 2α lift, but the cocycle is *not* cohomologous to a constant (hence no diagonal subgroup exists). (If φ were cohomologous to a constant then either it would be a coboundary and then the rotation by α would not be ergodic or it would be cohomologous to -1, which means that -1 would be an eigenfunction of the rotation by α.)

Remark 5. In the example above we have a 2-point extension T_φ of T (T_φ represents the rotation by α while T the rotation by 2α). Here the centralizer of T_φ is Abelian (as T_φ has simple spectrum) so each 2-cocycle $\overline{c}$ associated with a selector $c:\mathcal{R}\to\mathbb{Z}_2$ (here $\mathcal{R}=C(T)$) is symmetric, i.e. $\overline{c}(S_1,S_2)=\overline{c}(S_2,S_1)$. Consequently, if we regard φ as a circle cocycle then $\mathcal{R}^{\overline{c}}$ is compact and Abelian. If we look at the proof of the Bargman theorem outlined in Section 4 we are forced to conclude that $k=1$ and hence the multiplier $\overline{c}$ (we can consider the 2-cocycle $\overline{c}$ as a circle 2-cocycle) must be trivial. This, however, means that φ regarded as a circle cocycle must be cohomologous to a constant (and this is not surprising since the equation

$$\varphi=\lambda fT/f$$

for $\lambda\in\mathbb{T}$ and a measurable $f:X\to\mathbb{T}$ expresses exactly the fact that λ is an eigenvalue of T_φ: in our example $e^{2\pi i\alpha}$ is an eigenvalue of T_φ).

EXAMPLE 6. Suppose that T is ergodic and $\mathcal{R}\subset C(T)$ is compact and cyclic of order k. Assume that G is a group where all roots of order k exist (for instance, G is divisible). If $\mathcal{R}$ lifts with all v_S's equal to the identity then a diagonal subgroup exists. Indeed, if S is a generator of $\mathcal{R}$ then consider a lifting S_f of S. If it has order k we are done; if not, which means $(S_f)^k=\sigma_{g_0}$, consider S_{f+h_0}. Its kth power is equal to $\sigma_{g_0+kh_0}$ so we only need $kh_0=-g_0$.

6. Diagonal subgroups in case of circle extensions. We will now consider only circle extensions T_φ, with $\varphi:X\to\mathbb{T}$. Throughout, we will assume that $\mathcal{R}\subset L(\varphi,T)$ acts freely and that the representation $\mathcal{R}\ni S\mapsto v_S$ is trivial. Recall that this latter condition is automatically satisfied if $\mathcal{R}$ is locally compact and connected (see [3]) or even for each $\mathcal{R}$ provided that T_φ has a simple spectrum. Under these assumptions for any measurable selector $c:\mathcal{R}\to\widetilde{C}(T_\varphi)$ the corresponding 2-cocycle $\overline{c}:\mathcal{R}\times\mathcal{R}\to\mathbb{T}$ is a multiplier. We will also assume that no nontrivial power of T is in $\mathcal{R}$. Directly from (7) and Proposition 3 we obtain the following.

PROPOSITION 6. *If $\mathcal{R}\subset L(\varphi,T)$ is compact then there exists $k\geq 1$ such that for T_{φ^k} a diagonal subgroup over $\mathcal{R}$ exists. Equivalently, φ^k can be extended to a cocycle of a group generated by $\mathcal{R}$ and T.* ∎

COROLLARY 2. *If, additionally, T_φ has a simple spectrum then a diagonal subgroup exists.*

Proof. One can repeat the reasoning from Remark 5 as $C(T_\varphi)$ is Abelian and hence the corresponding multipliers will be symmetric. ∎

In view of (8), in case of $\mathcal{R}=\mathbb{R}$ we have the following result.

PROPOSITION 7. *A cocycle $\varphi:X\to\mathbb{T}$ has an extension to a cocycle generated by T and $\mathcal{R}$ if and only if $\mathcal{R}\subset L(\varphi,T)$.* ∎

7. On extensions of $\mathbb{Z}$-cocycles. Assume that groups $\mathcal{G},\mathcal{S}$ act as measure-preserving transformations on a Lebesgue space $(Y,\mathcal{C},\nu)$ (the corresponding actions on Y will be denoted by gy and sy respectively). We assume that $\mathcal{G}$ is compact and metric, while $\mathcal{S}$ is locally compact, second countable and it acts ergodically (both actions need not be free). Moreover, we assume that $\mathcal{G}$ is in the centralizer of $\mathcal{S}$, i.e. $\mathcal{G}\subset C(\mathcal{S})$. Given $y\in Y$ denote by $\mathcal{G}_y$ the *stabilizer* of $\mathcal{G}$ at y,

$$\mathcal{G}_y:=\{g\in G: gy=y\}.$$

It is a closed subgroup of $\mathcal{G}$ and since $\mathcal{G}\subset C(\mathcal{S})$, $\mathcal{G}_{sy}=\mathcal{G}_y$. Therefore, if we denote by M the space of closed subgroups of $\mathcal{G}$ endowed with the Hausdorff metric then M is clearly separable, and since the map

$$Y\ni y\mapsto \mathcal{G}_y\in M$$

is $\mathcal{S}$-invariant, it must be constant as the action of $\mathcal{S}$ is ergodic. This means that with no loss of generality, we can talk about the stabilizer, $\mathrm{stab}(G)$, of the action of $\mathcal{G}$. Denote by $X=Y/\mathcal{G}$ the space of orbits of $\mathcal{G}$. Clearly, $\mathcal{S}$ still acts on this new measure space $(X,\mathcal{B},\mu)$; we will denote this action by $\overline{\mathcal{S}}$ and write $\overline{s}x$ for the action of $s\in\mathcal{S}$ at $x\in X$. Finally, let $\xi,\eta:X\to Y$ be two measurable selectors for the map $y\mapsto\{gy:g\in\mathcal{G}\}=x$.

The following theorem is "folklore", so we will only sketch a proof.

THEOREM 1. *The formula*

$$\Theta_\xi(s,x)=g\,\mathrm{stab}(\mathcal{G})\Leftrightarrow g(\overline{s}x)=s\xi(x)$$

defines a cocycle with values in $\mathcal{G}/\mathrm{stab}(\mathcal{G})$ such that the corresponding group extension of $\overline{\mathcal{S}}$ via Θ is isomorphic to the original action of $\mathcal{S}$. Moreover, Θ_ξ and Θ_η are cohomologous.

Proof. By a direct computation, $\Theta_\xi(s_1s_2,x)=\Theta_\xi(s_1,x)\Theta_\xi(s_2,s_1x)$. If we define

$$f(x)=g\,\mathrm{stab}(\mathcal{G})\Leftrightarrow g\xi(x)=\eta(x)$$

then $\Theta_\xi(s,x)=f(x)^{-1}\Theta_\eta(s,x)f(sx)$, so the two cocycles Θ_ξ and Θ_η are cohomologous. Finally, observe that the map

$$(x,g\,\mathrm{stab}(\mathcal{G}))\mapsto g\xi(x)$$

establishes an isomorphism of the extension of $\overline{\mathcal{S}}$ by Θ_ξ with $\mathcal{S}$. ∎

Let us now come back to our situation of an ergodic $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$. We assume that $\varphi:X\to G$ is an ergodic cocycle, where G is a compact Abelian group. Let $\mathcal{R}\subset L(\varphi,T)$ be a compact subgroup. We assume that $\mathcal{R}$ acts freely on $(X,\mathcal{B},\mu)$ and that no nontrivial power of T is in $\mathcal{R}$. In other words, the joint action of T and $\mathcal{R}$ can naturally be identified with an action of $\mathbb{Z}\times\mathcal{R}$ and furthermore, this action is free. Let $\widetilde{\mathcal{R}}$ denote the group of all liftings to $C(T_\varphi)$ of all elements of $\mathcal{R}$. Hence

$$\widetilde{S}\in\widetilde{\mathcal{R}}\Leftrightarrow\widetilde{S}=S_{f,v}\text{ and }\varphi S-v\varphi=fT-f.$$

Define by $\widetilde{G}=\{\sigma_g:g\in G\}$, where $\sigma_g(x,h)=(x,h+g)$. Hence $\widetilde{G}$ consists of all liftings of Id_X and as a subgroup of $C(T_\varphi)$ is naturally isomorphic to G. Moreover,

$$S_{f,v}\sigma_g=\sigma_{v(g)}S_{f,v}. \tag{9}$$

From now on we will assume that

$$\widetilde{G}\subset C(\widetilde{R}). \tag{10}$$

Equivalently (by (9)), the representation $\mathcal{R}\ni S\mapsto v_S$ is trivial. We will write S_f instead of S_{f,Id_G}.

Let $c:\mathcal{R}\to\widetilde{\mathcal{R}}$ be a measurable selector for π ($\pi(S_f)=S$) and let $\overline{c}:\mathcal{R}\times\mathcal{R}\to G$ be the associated 2-cocycle, i.e.

$$\overline{c}_{S_2,S_3}+\overline{c}_{S_1,S_2S_3}=\overline{c}_{S_1,S_2}+\overline{c}_{S_1S_2,S_3},\quad \overline{c}_{S,\mathrm{Id}}=\overline{c}_{\mathrm{Id},S}=0$$

for all $S,S_1,S_2,S_3\in\mathcal{R}$. Consider the Mackey $\overline{c}$-extension $\mathcal{R}^{\overline{c}}$ of $\mathcal{R}$, that is, $\mathcal{R}^{\overline{c}}=\mathcal{R}\times G$ and

$$(S_1,g_1)\circ(S_2,g_2)=(S_1S_2,g_1+g_2+\overline{c}(S_1,S_2)),$$
$$(S,g)^{-1}=(S^{-1},-g-\overline{c}(S,S^{-1})).$$

Since the product of Haar measures on $\mathcal{R}$ and G is a finite rotation invariant measure, $\mathcal{R}^{\overline{c}}$ with its Weil topology becomes a compact group. Note that G becomes a closed normal subgroup of $\mathcal{R}^{\overline{c}}$ and $\mathcal{R}^{\overline{c}}/G$ is exactly $\mathcal{R}$ hence $\mathcal{R}^{\overline{c}}$ can be viewed as a compact extension of $\mathcal{R}$.

Remark 6. Note that a 2-cocycle $\overline{c}:\mathcal{R}\times\mathcal{R}\to G$ is a 2-coboundary $\Leftrightarrow$ the Mackey extension $\mathcal{R}^{\overline{c}}$ of $\mathcal{R}$ contains a diagonal subgroup $\{(S,g_S):S\in\mathcal{R}\}$.

As noticed in [3], $\widetilde{\mathcal{R}}\subset C(T_\varphi)$ is also a compact group in the weak topology of the centralizer.

PROPOSITION 8. *The map $J:\widetilde{\mathcal{R}}\to\mathcal{R}^{\overline{c}}$ defined by*

$$J(S_f)=(S,g_0)\Leftrightarrow S_{g_0+f}=c(S)(=S_{f_S})$$

is an isomorphism of topological groups. In particular, if $c_1,c_2:\mathcal{R}\to\widetilde{\mathcal{R}}$ are two Borel selectors for π then the corresponding Mackey extensions $\mathcal{R}^{\overline{c}_1},\mathcal{R}^{\overline{c}_2}$ are isomorphic.

Proof. The fact that J is 1-1 and onto is clear. Moreover, if $S_f,S'_{f'}\in\mathcal{R}$ and $J(S_f)=(S,g_0)$, $J(S'_{f'})=(S',g'_0)$ then $S_{f_S}=S_{g_0+f}$ and $S'_{f_{S'}}=S'_{g'_0+f'}$.

Now

$$\begin{aligned}J(S'_{f'}\circ S_f)&=J(S'_{f_{S'}-g'_0}\circ S_{f_S-g_0})\\&=J((S'S)_{f_{S'}S+f_S-(g'_0+g_0)})=J((S'S)_{f_{S'S}-c(S',S)-(g'_0+g_0)})\\&=(S'S,g_0+g'_0c(S',S))=J(S'_{f'})\circ J(S_f).\end{aligned}$$

It remains (see e.g. [10]) to show that J is a Borel map. This, however, is clear; the Borel structure on $\mathcal{R}^{\overline{c}}$ is exactly the product Borel structure of $\mathcal{R}\times G$ (see [14]), $J(S_f)=(S,c(S)(S_f)^{-1})$ and the maps

$$S_f\mapsto S\mapsto c(S),\qquad S_f\mapsto(S_f)^{-1}$$

are Borel. ∎

In view of the above proposition, on $X\times G$ we have the joint free action of T_φ and $\widetilde{\mathcal{R}}$ which naturally can be identified with a free action of $\mathcal{S}=\mathbb{Z}\times\mathcal{R}^{\overline{c}}$. Note that $\widetilde{G}$ is in the centralizer of the action of $\mathcal{S}$ and hence the factor determined by $\widetilde{G}$ (that is, $\mathcal{B}$) is a factor for $\mathcal{S}$. (Note, however, that the factor-action of $\mathcal{S}$ on $(X,\mathcal{B},\mu)$ is not free.) Collecting now the results contained in Proposition 3, Theorem 1 and Remark 6 we obtain our main result on extensions of $\mathbb{Z}$-cocycles.

THEOREM 2. *Let $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism of a Lebesgue space. Assume that $\varphi:X\to G$ is an ergodic cocycle, where G is a compact metric Abelian group. Assume moreover that $\mathcal{R}\subset L(\varphi,T)$ is compact, acts freely on $(X,\mathcal{B},\mu)$, $T^n\in\mathcal{R}$ iff $n=0$ and that the representation*

$$\mathcal{R}\ni S\mapsto v_S\in\mathrm{Aut}(G)$$

is trivial. Then there exists a 2-cocycle $\overline{c}:\mathcal{R}\times\mathcal{R}\to G$ such that for the compact Mackey extension $\mathcal{R}^{\overline{c}}=\{(S,g):S\in\mathcal{R},\ g\in G\}$ of $\mathcal{R}$ there exists an extension of the $\mathbb{Z}$-cocycle φ to a cocycle of an action of $\mathbb{Z}\times\mathcal{R}^{\overline{c}}$. Moreover, φ admits an extension to the action of $\mathbb{Z}\times\mathcal{R}$ iff $\mathcal{R}^{\overline{c}}$ contains a diagonal subgroup $\{(S,g_S):S\in\mathcal{R}\}$. ∎

Remark 7. (1) Although the action of $\mathcal{R}^{\overline{c}}$ on $(X,\mathcal{B},\mu)$ is an action of a compact group, the corresponding cocycle need not be a coboundary since the action is not free (see Example 5 for a concrete realization of this case).

(2) The only subactions of $\mathcal{R}^{\overline{c}}$ which are free on $(X,\mathcal{B},\mu)$ are subgroups of the form $\{(S,g_S):S\in\mathcal{R}'\}$, where $\mathcal{R}'$ is a subgroup of $\mathcal{R}$.

(3) Note that the extension of the $\mathbb{Z}$-cocycle φ to a cocycle of $\mathbb{Z}\times\mathcal{R}^{\overline{c}}$ is always possible as $\widetilde{\mathcal{R}}$ is compact and it corresponds to a factor of T_φ, while extensions of φ to cocycles of other groups (as $\mathbb{Z}\times\mathcal{R}$) depend on whether or not there exists an appropriate factor of T_φ.

(4) The situation where many groups of the centralizer can be lifted to a compact group extension appears in the recently developed theory of natural

factors (see [4]) and for example can be applied in the case of Gaussian-Kronecker automorphisms (see [16]).

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Product $\mathbb{Z}^d$-actions on a Lebesgue space and their applications

by

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Abstract. We define a class of $\mathbb{Z}^d$-actions, $d \geq 2$, called product $\mathbb{Z}^d$-actions. For every such action we find a connection between its spectrum and the spectra of automorphisms generating this action. We prove that for any subset A of the positive integers such that $1 \in A$ there exists a weakly mixing $\mathbb{Z}^d$-action, $d \geq 2$, having A as the set of essential values of its multiplicity function. We also apply this class to construct an ergodic $\mathbb{Z}^d$-action with Lebesgue component of multiplicity $2^d k$, where k is an arbitrary positive integer.

1. Introduction. One of the most important open problems in ergodic theory is the following: does there exist a dynamical system with a given spectrum? This very difficult problem has been solved only for some types of spectra. It is not known in particular whether there exists a dynamical system with Lebesgue spectrum of a finite multiplicity.

Let $T : X \to X$ be an automorphism of a Lebesgue probability space $(X, \mathcal{B}, \mu)$. The spectrum of T is uniquely described by the maximal spectral type and the spectral multiplicity function. We denote the set of essential values of the spectral multiplicity function by $E(T)$. The problem of what subsets of $\mathbb{N}^+ \cup \{\infty\}$ (where $\mathbb{N}^+$ is the set of all positive integers) can be realized as $E(T)$ for an automorphism T is considered e.g. in [A], [BL], [CFS], [GKLL], [MN], [O], [Ro].

Recently Kwiatkowski and Lemańczyk ([KL]) have shown that, for a given set $A \subseteq \mathbb{N}^+$ with $1 \in A$, there exists a weakly mixing T such that $E(T) = A$. In addition, if A is finite then one can find a smooth such T. The goal of this paper is to extend this result to dynamical systems which are actions of the group $\mathbb{Z}^d$ of d-dimensional integers on a Lebesgue probability space. To do this, we introduce a special class of $\mathbb{Z}^d$-actions.

Let Φ be a $\mathbb{Z}^d$-action on a Lebesgue probability space $(X, \mathcal{B}, \mu)$, i.e. Φ is a homomorphism of $\mathbb{Z}^d$ into the group of all automorphisms of $(X, \mathcal{B}, \mu)$. The

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