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On some vector balancing problems

by

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Abstract. Let V be an origin-symmetric convex body in \mathbb{R}^n , $n \geq 2$, of Gaussian measure $\gamma_n(V) \geq 1/2$. It is proved that for every choice u_1, \dots, u_n of vectors in the Euclidean unit ball B_n , there exist signs $\varepsilon_j \in \{-1, 1\}$ with $\varepsilon_1 u_1 + \dots + \varepsilon_n u_n \in (c \log n)V$. The method used can be modified to give simple proofs of several related results of J. Spencer and E. D. Gluskin.

1. Introduction. Let \mathcal{C}_n denote the class of all origin-symmetric convex bodies in \mathbb{R}^n , $n \geq 2$. Following W. Banaszczyk [2], for each pair $U, V \in \mathcal{C}_n$ we define $\beta(U, V)$ as the smallest $r > 0$ satisfying the following condition: given $u_1, \dots, u_n \in U$, there exist signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that $\varepsilon_1 u_1 + \dots + \varepsilon_n u_n \in rV$.

Several “vector balancing” results, proved by various authors for quite different purposes, can be described as estimates on $\beta(U, V)$ for specific choices of U, V , or both of them:

(a) W. Banaszczyk [2] established a general lower bound for $\beta(U, V)$ in terms of the volumes of U, V : for some absolute constant $c > 0$, and for any $U, V \in \mathcal{C}_n$, one has

$$\beta(U, V) \geq c\sqrt{n}(|U|/|V|)^{1/n}.$$

(b) I. Bárány and V. S. Grinberg [5] show that $\beta(U, U) \leq 2n$ for every $U \in \mathcal{C}_n$.

(c) The vector form of a well-known result of J. Beck and T. Fiala [6] states that $\beta(B_1^n, Q_n) \leq 2$, where B_1^n is the unit ball of ℓ_1^n and Q_n is the unit cube in \mathbb{R}^n .

(d) J. Spencer [11] and E. D. Gluskin [7] have proved independently that $\beta(Q_n, Q_n) \leq c\sqrt{n}$, where $c > 0$ is an absolute constant.

(e) We write B_n for the Euclidean unit ball in \mathbb{R}^n . Suppose that $\mathcal{E} \in \mathcal{C}_n$ is an ellipsoid with principal semi-axes a_1, \dots, a_n . W. Banaszczyk [3] proves that $\beta(B_n, \mathcal{E}) = (a_1^{-2} + \dots + a_n^{-2})^{1/2}$.

A standard reference for many of these results, especially those of them motivated by combinatorial questions, is the book [13] of J. Spencer.

J. Komlós conjectures that the sequence $\beta(B_n, Q_n)$ is bounded. The best known result on this problem (J. Spencer [11]) states that $\beta(B_n, Q_n) = O(\log n)$ as $n \rightarrow \infty$.

Closely related to the Komlós conjecture is recent work of W. Banaszczyk and S. J. Szarek [4]. They define and study the quantity

$$(1.1) \quad \alpha(U, V) = \sup_L \frac{\mu(L, V)}{\lambda_n(L, U)},$$

where the supremum is taken over all lattices L in \mathbb{R}^n , $\mu(L, V)$ is the covering radius and $\lambda_n(L, U)$ is the n th successive minimum of L with respect to V and U respectively. If γ_n denotes the standard Gaussian measure on \mathbb{R}^n with density $(2\pi)^{-n/2} e^{-|x|^2/2}$, where $|x|$ is the Euclidean norm of x , the main result in [4] states that

$$(1.2) \quad \alpha(B_n, V) \leq \theta^{-1},$$

for every closed convex set V in \mathbb{R}^n with $\gamma_n(V) \geq 1/2$, where θ is an absolute constant defined by the equation $\gamma_1([- \theta/2, \theta/2]) = 1/2$. On the other hand, one can see that $\alpha(U, V) \leq \beta(U, V)$ for every $U, V \in \mathcal{C}_n$. This motivates the question if $\alpha(B_n, V)$ may be replaced by $\beta(B_n, V)$ in (1.2). More precisely, it is conjectured in [4] that for some function $f : (0, 1) \rightarrow \mathbb{R}^+$ and for all $V \in \mathcal{C}_n$ one has $\beta(B_n, V) \leq f(\gamma_n(V))$. This would imply in particular that $\beta(B_n, Q_n) = O(\sqrt{\log n})$ as $n \rightarrow \infty$.

The purpose of this note is to discuss upper bounds for $\beta(B_n, V)$ when V is an arbitrary origin-symmetric convex body in \mathbb{R}^n . In §2 we give a simple proof of the following fact:

THEOREM 1. *Let $V \in \mathcal{C}_n$, with $\gamma_n(V) \geq 1/2$. Then $\beta(B_n, V) \leq 6 \log n$.*

Theorem 1 and standard estimates involving Sidák’s lemma ([10], see also [7]) allow a reasonable upper bound for $\beta(B_n, V)$ when V is a body with few faces (in the terminology of [1]). As an example, consider the case of the intersection of N strips defined by unit vectors in \mathbb{R}^n :

THEOREM 2. *Let $V = \{x \in \mathbb{R}^n : |\langle x, z_j \rangle| \leq 1, j = 1, \dots, N\}$, where $z_j \in B_n, j \leq N$, are N vectors spanning \mathbb{R}^n . Then*

$$\beta(B_n, V) \leq 9 \log n \sqrt{\log(3N)}.$$

When $V = Q_n$, the estimate given by Theorem 2 is worse than Spencer’s. In §3 we formulate a more precise version of Theorem 1, in which $\beta(B_n, V)$ is bounded in terms of the quantities $\inf_H \gamma_r(V \cap H)$, $r \leq n$, where the inf is taken over all r -dimensional subspaces of \mathbb{R}^n . Spencer’s log n -theorem is a consequence of this Theorem 3 and of the fact that $\inf_H \gamma_r(Q_n \cap H)$ is roughly “of the order” of 2^{-r} .

2. Proof of Theorems 1 and 2. For the proof of Theorem 1 we shall make use of two well-known facts:

(I) If $V \in \mathcal{C}_n$ and $x \in \mathbb{R}^n$, then $\gamma_n(x + V) \geq e^{-|x|^2/2} \gamma_n(V)$. This is a simple consequence of the symmetry of V and the convexity of the exponential function.

(II) Consider the entropy function $F(\alpha) = -\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)$, on $(0, 1/2)$. If \mathcal{A} is a subset of $\{-1, 1\}^r$ with cardinality $|\mathcal{A}| \geq 2^{rF((1-\theta)/2)}$, $0 < \theta < 1$, then we can find $\varepsilon', \varepsilon'' \in \mathcal{A}$ with $|\{j \leq r : \varepsilon'_j = \varepsilon''_j\}| \leq \theta r$ (D. Kleitman [9]).

LEMMA 2.1. *Let $V \in \mathcal{C}_n, n \geq 7$, with $\gamma_n(V) \geq 1/2$, and $u_1, \dots, u_r \in B_n, 7 \leq r \leq n$. There exists a subset σ of $\{1, \dots, r\}$ with $|\sigma| \geq r/2$, and signs $\varepsilon_j \in \{-1, 1\}, j \in \sigma$, such that*

$$\sum_{j \in \sigma} \varepsilon_j u_j \in 4V.$$

Proof. For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \{-1, 1\}^r$ we write $L(\varepsilon) = \varepsilon_1 u_1 + \dots + \varepsilon_r u_r$. By the parallelogram law,

$$(2.1) \quad \text{Ave}_\varepsilon |L(\varepsilon)|^2 = \sum_{j=1}^r |u_j|^2 \leq r.$$

Consider the sets $L(\varepsilon)/4 + V, \varepsilon \in \{-1, 1\}^r$. Using fact (I) and the arithmetic-geometric means inequality, we obtain

$$(2.2) \quad \int_{\mathbb{R}^n} \left[\sum_\varepsilon \chi_{L(\varepsilon)/4+V}(y) \right] \gamma_n(dy) \\ = \sum_\varepsilon \gamma_n \left(\frac{L(\varepsilon)}{4} + V \right) \\ \geq \left(\sum_\varepsilon e^{-|L(\varepsilon)|^2/32} \right) \gamma_n(V) \geq 2^r e^{-\text{Ave}_\varepsilon |L(\varepsilon)|^2/32} \gamma_n(V) \\ \geq 2^{(1-1/(32 \log 2) - 1/7)r} \geq 2^{F(1/4)r}.$$

It follows that for some subset \mathcal{A} of $\{-1, 1\}^r$ with cardinality $|\mathcal{A}| \geq 2^{F(1/4)r}$ we must have

$$(2.3) \quad \bigcap_{\varepsilon \in \mathcal{A}} \left(\frac{L(\varepsilon)}{4} + V \right) \neq \emptyset.$$

Using Kleitman’s result we find a pair $\varepsilon', \varepsilon'' \in \mathcal{A}$ for which

$$|\{j \leq r : \varepsilon'_j = \varepsilon''_j\}| \leq \frac{r}{2} \quad \text{and} \quad \frac{L(\varepsilon') - L(\varepsilon'')}{4} \in 2V.$$

Setting $\sigma = \{j \leq r : \varepsilon'_j \neq \varepsilon''_j\}$ and $\varepsilon_j = (\varepsilon'_j - \varepsilon''_j)/2$, $j \in \sigma$, we conclude the proof. ■

Remark 2.2. Instead of Kleitman's result, in the final step of the proof of Lemma 2.1, one may use the Sauer–Shelah lemma or an even simpler argument based on the computation of the cardinality of a neighborhood of a point in $\{-1, 1\}^r$. This would only affect the value of the constant in the statement.

Proof of Theorem 1. We first observe that if $\gamma_n(V) \geq 1/2$, and if ϱB_n is the largest origin-symmetric ball inscribed in V , then V is contained in a symmetric strip of width 2ϱ , hence $\gamma_1([- \varrho, \varrho]) \geq \gamma_n(V) \geq 1/2$. It follows that $\varrho > 1/2$, that is, $\frac{1}{2}B_n \subseteq V$.

Case 1. If $2 \leq n \leq 6$, then $\beta(B_n, V) \leq \beta(B_n, \frac{1}{2}B_n) = 2\sqrt{n} < 6 \log n$.

Case 2. If $n \geq 7$, we use an inductive argument. For the first step, set $r_0 = n$ and apply Lemma 2.1 to find $\sigma_1 \subseteq \{1, \dots, n\}$ with $|\sigma_1| \geq n/2$, and signs $\varepsilon_j \in \{-1, 1\}$, $j \in \sigma_1$, so that $\sum_{j \in \sigma_1} \varepsilon_j u_j \in 4V$. Then, define $\tau_1 = \sigma_1^c$ and $r_1 = |\tau_1| \leq r_0/2 = n/2$.

If $\tau_{k-1}, r_{k-1} = |\tau_{k-1}|$ have been defined for some $k \geq 2$ and if $r_{k-1} \geq 7$, then, using Lemma 2.1 again, we find $\sigma_k \subseteq \tau_{k-1}$ with $|\sigma_k| \geq r_{k-1}/2$ and $\varepsilon_j \in \{-1, 1\}$, $j \in \sigma_k$, with $\sum_{j \in \sigma_k} \varepsilon_j u_j \in 4V$. We define $\tau_k = \tau_{k-1} \setminus \sigma_k$, $r_k = |\tau_k| \leq r_{k-1}/2 \leq n/2^k$, and continue in the same way until, for some k_0 , $r_{k_0} < 7$. The number of steps needed does not exceed $\log_2(n/6) + 1$: if $r_k > 6$ then $2^k < n/6$. At this point we choose ε_j , $j \in \tau_{k_0}$, with $\sum_{j \in \tau_{k_0}} \varepsilon_j u_j \in \sqrt{r_{k_0}} B_n \subseteq 2\sqrt{6}V$. The choice of signs $\bigcup_{k=1}^{k_0} \{\varepsilon_j : j \in \sigma_k\} \cup \{\varepsilon_j : j \in \tau_{k_0}\}$ satisfies

$$(2.4) \quad \sum_{j=1}^n \varepsilon_j u_j \in \left[4 \left(\log_2 \left(\frac{n}{6} \right) + 1 \right) + 2\sqrt{6} \right] V \subseteq (6 \log n) V,$$

and this completes the proof. ■

Remark 2.3. Theorem 1 and a standard argument (see [11]) show that if $V \in \mathcal{C}_n$ with $\gamma_n(V) \geq 1/2$, and if $u_1, \dots, u_m \in B_n$, $m \geq n$, then there exist signs $\varepsilon_j \in \{-1, 1\}$ such that $\varepsilon_1 u_1 + \dots + \varepsilon_m u_m \in (12 \log n)V$.

There is nothing special about assuming that $\gamma_n(V) \geq 1/2$. If $\gamma_n(V) = \alpha \in (0, 1)$, one can find $c(\alpha)$ such that $\beta(B_n, V) \leq c(\alpha) \log n$. Moreover, the symmetry and the compactness of V are also not so important: if, say, V is any closed convex set in \mathbb{R}^n with $\gamma_n(V) \geq 3/4$, then

$$(2.5) \quad \beta(B_n, V) \leq \beta(B_n, V \cap (-V)) \leq c \log n.$$

Remark 2.4. It is easy to see that if the vectors $u_1, \dots, u_n \in B_n$ are orthogonal and if V is a centered cube in \mathbb{R}^n with $\gamma_n(V) \geq 1/2$, then there exist signs ε_j for which $\varepsilon_1 u_1 + \dots + \varepsilon_n u_n \in cV$. A careful examination of

the proof of (1.2) in [4] shows that the same is true for an arbitrary closed convex set V in \mathbb{R}^n with $\gamma_n(V) \geq 1/2$.

Remark 2.5. Let V be a symmetric convex set in \mathbb{R}^n , $n \geq 2$. If $x \in \mathbb{R}$, we write

$$V_x = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, \dots, x_{n-1}, x) \in V\}.$$

From the log-concavity of γ_{n-1} and the symmetry of V , one easily deduces that $h(x) = \gamma_{n-1}(V_x)$ is an even log-concave function on $\{x : \gamma_{n-1}(V_x) > 0\}$, hence it attains its maximum value at 0. It follows that

$$(2.6) \quad \gamma_n(V) = \int_{\mathbb{R}} h(x) \gamma_1(dx) \leq \gamma_{n-1}(V \cap e_n^\perp),$$

where $e_n^\perp = \{x \in \mathbb{R}^n : x_n = 0\}$. Induction and the rotational invariance of the Gaussian measure show that whenever H, H_1 are subspaces of \mathbb{R}^n with $H \subseteq H_1$, then $\gamma_{H_1}(V \cap H_1) \leq \gamma_H(V \cap H)$ (by γ_H we denote $\gamma_{\dim H}$ on H).

Another useful remark is that the set $I_s(h) = \{x \in \mathbb{R} : h(x) \geq s\}$ is a symmetric interval in \mathbb{R} (possibly the empty set) for every $s > 0$.

Using these observations one can give a proof of Sidák's lemma starting from the following lemma:

LEMMA 2.6. *Let K be a symmetric convex set in \mathbb{R}^n , $n \geq 1$, and $V_z = \{x \in \mathbb{R}^n : |\langle x, z \rangle| \leq 1\}$, $z \in \mathbb{R}^n$. Then*

$$\gamma_n(K \cap V_z) \geq \gamma_n(K) \gamma_n(V_z).$$

Proof. If $n = 1$ the inequality is trivially true since $K \cap V_z$ is either K or V_z , and γ_1 is a probability measure. Let $n \geq 2$. By the rotational invariance of γ_n we may assume that $z = \lambda^{-1} e_n$, $\lambda > 0$. Then

$$(2.7) \quad \begin{aligned} \gamma_n(K \cap V_z) &= \int_{-\lambda}^{\lambda} h(x) \gamma_1(dx) = \int_0^{\infty} \gamma_1(I_s(h) \cap [-\lambda, \lambda]) ds \\ &\geq \left(\int_0^{\infty} \gamma_1(I_s(h)) ds \right) \gamma_1([- \lambda, \lambda]) = \gamma_n(K) \gamma_n(V_z). \quad \blacksquare \end{aligned}$$

Let $\Phi(x) = \sqrt{2/\pi} \int_0^x e^{-t^2/2} dt$, $x \geq 0$. A simple inductive argument based on Lemma 2.6 provides a proof of

SIDÁK'S LEMMA. *Let $z_j \in \mathbb{R}^n$, $j \leq N$, and $V_j = \{x : |\langle x, z_j \rangle| \leq 1\}$. Then*

$$(2.8) \quad \gamma_n \left(\bigcap_{j \leq N} V_j \right) \geq \prod_{j=1}^N \gamma_n(V_j) = \prod_{j=1}^N \Phi(1/|z_j|). \quad \blacksquare$$

We include this proof of Lemma 2.6 because of its simplicity. The possibility of deducing Sidák's lemma in such an easy way became known to

several people more or less at the same time (see e.g. [14], where a much harder non-symmetric version of Lemma 2.6 is proved). The connection of Sidák's lemma with $\beta(B_n, V)$ is clear: given any $V \in \mathcal{C}_n$, we solve the equation $\gamma_n(\mu V) = 1/2$ in μ , and then apply Theorem 1 to obtain $\beta(B_n, V) \leq c\mu \log n$. If V is an intersection of strips, we can easily find an upper bound for μ using (2.8). As an example, let us see what happens if all the strips have width 2:

Proof of Theorem 2. Let $\mu \geq 1$. Using the standard estimate

$$\Phi(x) \geq 1 - e^{-x^2/2} \geq \exp(-2e^{-x^2/2}) \quad (x \geq 1),$$

we obtain

$$\gamma_n(\mu V) \geq [\Phi(\mu)]^N \geq \exp(-2Ne^{-\mu^2/2}).$$

It is clear that choosing $\mu = \sqrt{2 \log(2N/\log 2)}$, we get $\gamma_n(\mu V) \geq 1/2$. Therefore, Theorem 1 implies that

$$\beta(B_n, V) = \mu \beta(B_n, \mu V) \leq 6\sqrt{2} \log n \sqrt{\log(3N)}. \blacksquare$$

Finally, let us mention one slightly more delicate application of the method:

PROPOSITION 2.7. Let $z_j \in \mathbb{R}^n$, $j \leq N$, with $|z_j| \leq 1/\sqrt{\log(j+1)}$. If $V = \{x : |\langle x, z_j \rangle| \leq 1, j \leq N\}$, then

$$\beta(B_n, V) \leq c \log n,$$

where $c > 0$ is an absolute constant. \blacksquare

Note that the statement is independent of N .

3. A general upper bound for $\beta(B_n, V)$. When V is a parallelepiped in \mathbb{R}^n which contains B_n , the estimate given by Theorem 2 is $\beta(B_n, V) = O((\log n)^{3/2})$ as $n \rightarrow \infty$. Spencer's result for the cube can be recovered by a more precise version of Theorem 1 which we now describe:

DEFINITION. If $V \in \mathcal{C}_n$, $c \in (0, 1)$, and $r \in \{1, \dots, n\}$, we define

$$\varphi(V, c, r) = \min\{\varrho > 0 : \inf_H \gamma_r(\varrho V \cap H) \geq 2^{-cr}\},$$

where the inf is over all r -dimensional subspaces of \mathbb{R}^n . Note that $\varphi(V, c, r)$ is well defined since, for every H , $\gamma_r(\varrho V \cap H) \geq \gamma_n(\varrho V)$.

Our way to estimate $\beta(B_n, V)$ depends on an iteration scheme (similar to the one in the proof of Theorem 1), based on the following lemma:

LEMMA 3.1. Let $V \in \mathcal{C}_n$, $r \leq n$, and $u_1, \dots, u_r \in B_n$. We can find a subset σ of $\{1, \dots, r\}$ with $|\sigma| \geq r/2$, and signs $\varepsilon_j \in \{-1, 1\}$, $j \in \sigma$, for which

$$\sum_{j \in \sigma} \varepsilon_j u_j \in 4\varphi(V, 1/7, r)V.$$

Proof. We may clearly assume that u_1, \dots, u_r are linearly independent. Consider the subspace $H = \text{span}\{u_1, \dots, u_r\}$ and set $\varrho = \varphi(V, 1/7, r)$, $L(\varepsilon) = \varepsilon_1 u_1 + \dots + \varepsilon_r u_r$, $\varepsilon \in \{-1, 1\}^r$. We estimate

$$(3.1) \quad \sum_{\varepsilon} \gamma_H \left(\frac{L(\varepsilon)}{4} + (\varrho V \cap H) \right) \geq 2^{F(1/4)r},$$

and, exactly as in Lemma 2.1, we find $\sigma \subseteq \{1, \dots, r\}$, $|\sigma| \geq r/2$, and a sequence of signs $\varepsilon_j \in \{-1, 1\}$, $j \in \sigma$, with

$$(3.2) \quad \sum_{j \in \sigma} \varepsilon_j u_j \in 4\varrho V \cap H. \blacksquare$$

THEOREM 3. Let $V \in \mathcal{C}_n$, and $\psi_l = \lfloor n/2^l \rfloor$, $l = 0, 1, \dots$. Then

$$\beta(B_n, V) \leq 8 \sum_{l=0}^{\lfloor \log_2 n \rfloor} \varphi(V, 1/21, \psi_l).$$

Proof. Suppose that $u_1, \dots, u_n \in B_n$. We set $\tau_0 = \{1, \dots, n\}$, $r_0 = n$, $\sigma_0 = \emptyset$, and following the proof of Theorem 1 (with Lemma 3.1 playing now the role of Lemma 2.1), for $k \geq 1$ we choose σ_k, τ_k, r_k :

(i) $\sigma_k \subseteq \tau_{k-1}$, $|\sigma_k| \geq |\tau_{k-1}|/2$, and there exist $\varepsilon_j \in \{-1, 1\}$, $j \in \sigma_k$, with

$$\sum_{j \in \sigma_k} \varepsilon_j u_j \in 4\varphi(V, 1/7, r_{k-1})V.$$

(ii) $\tau_k = \tau_{k-1} \setminus \sigma_k$, $r_k = |\tau_k|$.

This procedure exhausts $\{1, \dots, n\}$ in a finite number of steps: for some $m \leq \lfloor \log_2 n \rfloor + 1$, we will have $r_m = 0$.

Each r_k , $k = 0, \dots, m-1$, lies in an interval of the form $(\psi_{l+1}, \psi_l]$, $l = 0, \dots, \lfloor \log_2 n \rfloor$, and at most two of them are in the same interval. If $\psi_{l+1} < r_k \leq \psi_l$, then it is easy to see that $\varphi(V, 1/7, r_k) \leq \varphi(V, 1/21, \psi_l)$: notice that if $\dim H = r_k$ and H_1 is any ψ_l -dimensional subspace of \mathbb{R}^n with $H \subseteq H_1$, then $\gamma_{H_1}(\varrho V \cap H_1) \geq 2^{-\psi_l/21}$ implies that $\gamma_H(\varrho V \cap H) \geq \gamma_{H_1}(\varrho V \cap H_1) \geq 2^{-(\psi_l/3)/7} \geq 2^{-r_k/7}$. It follows that

$$(3.3) \quad \sum_{k=0}^m \varphi(V, 1/7, r_k) \leq 2 \sum_{l=0}^{\lfloor \log_2 n \rfloor} \varphi(V, 1/21, \psi_l).$$

Therefore, the sequence of signs ε_j chosen in our m steps satisfies

$$\sum_{j=1}^n \varepsilon_j u_j \in \left[8 \sum_{l=0}^{\lfloor \log_2 n \rfloor} \varphi(V, 1/21, \psi_l) \right] V. \blacksquare$$

We shall apply Theorem 3 in the case where $V = Q_n$:

LEMMA 3.2. For some absolute constant $c > 0$, and for every $r \leq n$, one has $\varphi(Q_n, 1/21, r) \leq c$.

PROOF. Let H be an r -dimensional subspace of \mathbb{R}^n . Let also $\{w_1, \dots, w_r\}$ be an orthonormal basis of H , and W be the $n \times r$ matrix with columns w_j , $j \leq r$. Then, for every $c > 0$,

$$\gamma_r(cQ_n \cap H) = \gamma_r(\{x \in \mathbb{R}^r : |\langle x, W^*e_i \rangle| \leq c, i = 1, \dots, n\}),$$

where $\{e_i\}_{i \leq n}$ is the standard orthonormal basis of \mathbb{R}^n .

CLAIM. If $t_1, \dots, t_n > 0$, then

$$(3.4) \quad \prod_{i=1}^n \Phi\left(\frac{1}{t_i}\right) \geq 2^{-\delta \sum_{i=1}^n t_i^2},$$

where $\delta > 0$ is an absolute constant.

PROOF OF THE CLAIM. We may assume that $t_1 \leq \dots \leq t_s \leq 1 < t_{s+1} \leq \dots \leq t_n$. We set $S = \sum_{i=1}^n t_i^2$, and $A_j = \{i \leq n : 2^{j-1} < t_i \leq 2^j\}$, $j = 1, 2, \dots$. Note that $|A_j| \leq S/2^{2j-2}$.

We have the estimates:

$$(3.5) \quad \prod_{i=1}^s \Phi\left(\frac{1}{t_i}\right) \geq \exp\left(-2 \sum_{i=1}^s e^{-1/(2t_i^2)}\right) \geq \exp\left(-4 \sum_{i=1}^s t_i^2\right) \geq e^{-4S},$$

$$(3.6) \quad \prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq \left(\sqrt{2/(\pi e)} \frac{1}{2^j}\right)^{|A_j|} \geq [(\sqrt{2/(\pi e)})^{1/2^{2j-2}} 2^{-j/2^{2j-2}}]^S$$

and hence

$$(3.7) \quad \prod_j \prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq [(\sqrt{\pi e/2})^{-\sum_{j=1}^{\infty} 1/2^{2j-2}} 2^{-\sum_{j=1}^{\infty} j/2^{2j-2}}]^S.$$

From (3.5) and (3.7) it follows that

$$\prod_{i=1}^n \Phi\left(\frac{1}{t_i}\right) = \prod_{i=1}^s \Phi\left(\frac{1}{t_i}\right) \prod_j \prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq 2^{-\delta S},$$

for some absolute constant $\delta > 0$. The claim is proved.

By Sidák's lemma we have $\gamma_r(cQ_n \cap H) \geq \prod_{i=1}^n \Phi(c/|W^*e_i|)$, and since $\sum_{i=1}^n |W^*e_i|^2 = \sum_{j=1}^r |w_j|^2 = r$, our claim provides the inequality

$$(3.8) \quad \gamma_r(cQ_n \cap H) \geq 2^{-\delta r/c^2} \geq 2^{-r/21}$$

if $c = c(\delta) > 0$ has been chosen large enough (independent of n and r). ■

As a consequence of Theorem 3 and Lemma 3.2 one has Spencer's estimate on the Komlós conjecture:

COROLLARY 3.3. $\beta(B_n, Q_n) = O(\log n)$ as $n \rightarrow \infty$.

REMARK 3.4. J. Spencer [11] and E. D. Gluskin [7] have proved that $\beta(Q_n, Q_n) = O(\sqrt{n})$ as $n \rightarrow \infty$, which is clearly optimal. The basic step towards this theorem is to prove the following:

CLAIM. If $u_1, \dots, u_r \in Q_n$, $r \leq n$, then there exist a subset σ of $\{1, \dots, r\}$ with cardinality $|\sigma| \geq \theta r$ and a choice of signs ε_j , $j \in \sigma$, such that

$$(3.9) \quad \sum_{j \in \sigma} \varepsilon_j u_j \in c\sqrt{r} \sqrt{\log(2n/r)} Q_n,$$

where $\theta \in (0, 1)$ and $c > 0$ are absolute constants.

A modification of the proof of Lemma 2.1 gives a simple proof of this fact: define $K = \{x \in \mathbb{R}^r : |\langle x, W^*e_i \rangle| \leq 1, i \leq n\}$, where W is the $n \times r$ matrix with columns u_j , $j \leq r$. Note that $|W^*e_i| \leq \sqrt{r}$, $i = 1, \dots, r$. Choosing an absolute constant $c > 0$ large enough and using Sidák's lemma one has the inequality

$$\sum_{\varepsilon} \gamma_r(\varepsilon + c\sqrt{r} \sqrt{\log(2n/r)} K) \geq 2^{\theta r},$$

where $\theta = \theta(c) \in (0, 1)$ is some other absolute constant. The rest is as in Lemma 2.1: we find $\varepsilon \in \{-1, 0, 1\}^r$ with $|\{j : \varepsilon_j \neq 0\}| \geq \theta r$, and $\varepsilon \in c\sqrt{r} \sqrt{\log(2n/r)} K$.

This is equivalent to the claim, and an inductive argument analogous to the one in [7], [11] leads to the Spencer–Gluskin theorem. In this case, our method may be viewed as a (simplified) variation of Gluskin's method where Sidák's lemma was used for volume estimates and then combined with Minkowski's theorem from the geometry of numbers.

Another modification of Lemma 2.1, now combined with the binary block decomposition used by B. S. Kashin in [8], can give the following stronger result of J. Spencer [12]:

If $u_1, \dots, u_n \in Q_n$, then there exist signs $\varepsilon_j \in \{-1, 1\}$ for which

$$\max_{t \leq n} \left\| \sum_{j=1}^t \varepsilon_j u_j \right\|_{\infty} \leq c\sqrt{n},$$

where $c > 0$ is an absolute constant.

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BMO $_{\psi}$ -spaces and applications to extrapolation theory

by

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Abstract. We investigate a scale of BMO $_{\psi}$ -spaces defined with the help of certain Lorentz norms. The results are applied to extrapolation techniques concerning operators defined on adapted sequences. Our extrapolation works simultaneously with two operators, starts with BMO $_{\psi}$ - L_{∞} -estimates, and arrives at L_p - L_p -estimates, or more generally, at estimates between K-functionals from interpolation theory.

Introduction. Extrapolation techniques are an important tool to compare L_p -norms of operators defined on martingales. Basic results were proved by D. L. Burkholder, B. J. Davis, and R. F. Gundy ([11], [10]).

Let us consider a basic example. Assume $f = (d_k)_{k=0}^n \subset L_1(\Omega, \mathcal{F}, \mathbb{P})$ to be a sequence of martingale differences with respect to some filtration $(\mathcal{F}_k)_{k=0}^n$ such that $d_0 = 0$ and $|d_k|$ is \mathcal{F}_{k-1} -measurable for $k = 1, \dots, n$. An extension of the Azuma inequality proved by P. Hitczenko [16] (Lemma 4.3) (see also [12], [25], the comments in [16], and the proof of Proposition 1 of this paper) says that for $\lambda > 0$,

$$(1) \quad \mathbb{P}\left(\left|\sum_{k=1}^n d_k\right| > \lambda \|S_2 f\|_{\infty}\right) \leq 2e^{-\lambda^2/2}$$

where $S_2 f := (\sum_{k=1}^n |d_k|^2)^{1/2}$ is the usual square function operator. The above inequality is of importance for several reasons. For example, in [16] (Corollary 4.2) this inequality is used to prove the Burkholder–Davis–Gundy type inequality

$$(2) \quad \left\| \sup_{1 \leq k \leq n} \left| \sum_{i=1}^k d_i \right| \right\|_p \leq c\sqrt{p} \left\| \left(\sum_{i=1}^n |d_i|^2 \right)^{1/2} \right\|_p \quad \text{for } 1 \leq p < \infty,$$

which extends the corresponding one for dyadic martingales ([12], [6], [25]). In order to deduce (2) from (1) one has to modify (1) in two steps. First we observe that for $B \in \mathcal{F}_k$ we get a martingale difference sequence $(d_i)_{i=k+1}^n \subset L_1(B, \mathbb{P}_B)$, where \mathbb{P}_B is the normalized restriction of \mathbb{P} to B . Applying (1) to