Contents of Volume 122, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Subscription information (1997): Vols. 122–126 (15 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

L^2 and L^p estimates for oscillatory integrals and their extended domains

by

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Abstract. We prove the L^p boundedness of certain nonconvolutional oscillatory integral operators and give explicit description of their extended domains. The class of phase functions considered here includes the function $|x|^\alpha |y|^\beta$. Sharp boundedness results are obtained in terms of α, β, and rate of decay of the kernel at infinity.

0. Introduction. Our purpose in this paper is to study the (L^p, L^p) mapping properties of oscillatory integrals, or more commonly referred to as non-convolution operators of the form

$$Kf(x) = \int_0^\infty k(x,y)f(y)\,dy, \qquad x > 0, \tag{0.1}$$

the kernel k is of the form

$$k(x,y) = \varphi(x,y)\exp(ig(x,y)) \tag{0.2}$$

and g is a real-valued function. Conditions on g and φ are formulated in Section 1. It is clear that we can obtain similar type results for such operators defined on $(-\infty,\infty)$.

Such problems have a long, but somewhat uneven history; see for example [St] and the references there.

They come about in studying convergence questions for Fourier series, in solving boundary value problems for PDEs, for example as in [Ho], [Sj], [Wal], and in other settings as well, see [St], [Pan].

1991 *Mathematics Subject Classification*: 42A50, 42B20, 47G10.

Key words and phrases: L^p boundedness, oscillatory integrals, extended domains, Calderón–Zygmund kernels.

The first author is supported in part by a grant from the National Science Foundation.

The third author is supported in part by the General Research Fund, University of Kansas.

We also consider operators

$$Sf(x) = Kf(x^{a/b}), \quad x > 0, \tag{0.3}$$

where $b \geq a > 1$.

This is the beginning of a program to obtain the complete (L^p, L^p) mapping properties for the operator K (as well as its $(-\infty, \infty)$ version) with $1 < p < \infty$. A similar program was carried out in the convolution case (see [JS]).

In this paper, we obtain some of the mapping properties for these operators. In fact, we establish sharp results. For the $(2,2)$ results, see Theorem 2.1. For the (p,p) results, see Corollary 3.2.

In Section 5, we consider K in function spaces other than L^p. We describe explicitly the extended domain of K in the sense of [AS]. Not surprisingly, this turns out to be a compressed weighted amalgam space of l^2 and L^1. The description of the extended domain of K generalizes previous work of [Sz] and [LS].

It would be desirable to find in this context a description of the range of K more precise than that given by our estimates.

We sometimes write $\int f(u)\,du$ to mean $\int_0^\infty f(u)\,du$. C indexed if necessary will denote a positive constant depending only on the kernel k. It is understood that even in the same string of formulas, C in different places may stand for different such constants.

1. Preliminary estimates and conditions on k.

We now formulate the conditions on k in (0.2): for $b \geq a \geq 1$,

$$g(x,y) = \beta(x)^b \alpha(y)^a \tag{1.1}$$

where $\alpha, \beta : [0,\infty) \to [0,\infty)$ satisfy:

$$\begin{cases} \text{(a) For some } m, M > 0,\ m \leq \beta'(x),\ \alpha'(y) \leq M \text{ for all } x, y \geq 0, \\ \text{(b) } \alpha(0) = \beta(0) = 0, \text{ and} \\ \text{(c) } \beta' \text{ is absolutely continuous with } \beta''(x)/x^\varepsilon \in L^1(2,\infty) \\ \quad\ \text{for some } \varepsilon > 0. \end{cases} \tag{1.2}$$

The actual value of ε will be determined in the next section.

With this choice of g, we can consider φ satisfying the following conditions:

$$\begin{cases} \text{(a) } |\varphi(x,y)| \leq C|x-y|^{-\gamma}, \\ \text{(b) } |\partial_x \varphi(x,y)| \leq C|x-y|^{-\gamma-1}, \end{cases} \tag{1.3}$$

where $0 \leq \gamma \leq (b-a)/(2b)$ and $b \geq a \geq 1$ are as in (1.1).

In the case when $\gamma = 0$ we also need the following version of (1.2), (1.3) with the roles of x, y and α, β reversed:

$$\begin{cases} \alpha'' \in L^1(0,2), \\ |\partial_y \varphi(x,y)| \leq C|x-y|^{-1} \quad \text{for } |x-y| \geq 2. \end{cases} \tag{1.3$'$}$$

Our L^2 estimates in the next section will be based on the following proposition.

PROPOSITION 1.1. *For real ξ, $b \geq 1$, and $-1 < c \leq b-1$ we have*

$$\Big|\int_{t_1}^{t_2} t^c \exp(it^b\xi)\,dt\Big| \leq C|\xi|^{-(1+c)/b}, \quad 0 \leq t_1 < t_2, \tag{1.4}$$

$$\Big|\int_{t_1}^{t_2} t^c \exp(it^b\xi)\,dt\Big| \leq C|\xi|^{-1} t_1^{1+c-b}, \quad 0 < t_1 < t_2. \tag{1.5}$$

We also have the convex combinations $(1.4)^{1-\theta}(1.5)^\theta$:

$$\Big|\int_{t_1}^{t_2} t^c \exp(it^b\xi)\,dt\Big| \leq C|\xi|^{(1+c)(\theta-1)/b-\theta} t_1^{(c+1-b)\theta}, \quad 0 \leq \theta \leq 1. \tag{1.6}$$

Proof. Notice (using contour integration) that the integrals $\int t^c e^{it^b}\,dt$ are convergent for $b-1 > c-1$ and hence integrals over finite intervals are uniformly bounded. This is also the case when $c = b-1, b > 1$ and $b = 1, c = 0$. For (1.5) just write

$$t^c e^{it^b\xi} = (ibt^{b-c-1}\xi)^{-1}\partial_t e^{it^b\xi}$$

and integrate by parts.

In some of the estimates in the forthcoming sections, we shall use the following version of Schur's lemma (see [G]).

PROPOSITION 1.2. *If U, V are (σ-finite) measure spaces and $\kappa(u,v) \geq 0$ is measurable on $U \times V$, then the integral operator*

$$f \to \int_V \kappa(u,v) f(v)\,dv$$

is bounded from $L^p(V)$ into $L^p(U)$ if and only if there exist functions

$$\lambda : U \to (0,\infty), \quad \mu : V \to (0,\infty)$$

and constants $C_1, C_2 > 0$ so that

$$\int_U \kappa(u,v)\lambda(u)\,du \leq C_1 \mu(v)^{p/p'}$$

and

$$\int_V \kappa(u,v)\mu(v)\,dv \leq C_2 \lambda(u)^{p'/p}.$$

If $p = 2$ and the conditions are satisfied, then the norm of the operator is bounded by $\sqrt{C_1C_2}$.

The following special cases are useful in what follows.

COROLLARY 1.3i. *Let $\kappa : (0,\infty)\times(0,\infty)\to[0,\infty)$ be positive homogeneous of degree -1 and such that*
$$\int[\kappa(1,v)+\kappa(v,1)]v^{-\varepsilon}\,dv<\infty$$
for some $\varepsilon\in(0,1)$. Then the integral operator
$$f\to\int\kappa(u,v)f(v)\,dv$$
is bounded in $L^2(0,\infty)$.

Denote by $\mathbf{1}_E$ the characteristic function of the set E.

COROLLARY 1.3ii. *Let $\kappa(u,v)=(u+v)^{-\varrho}|u-v|^{-\sigma}\mathbf{1}_{[0,T]}(|u-v|)$ where $T>0$, $0<\sigma<1$, $\varrho\ge0$ and $\varrho+\sigma<1$. Then the operator with the kernel κ is bounded in $L^2(0,\infty)$.*

A result which we use to lift $(2,2)$ mappings to (p,p) mappings is well known in the context of Fourier series (see p. 125 of [Zyg, Vol. II]).

PROPOSITION 1.4. *Let S be a sublinear operator in the space of functions on $(0,\infty)$ and suppose that the domain of S, $D(S)$, contains all step functions. If* (i) $\|Sf\|_\infty\le C\|f\|_1$ *and* (ii) $\|Sf\|_2\le C\|f\|_2$, $f\in D(S)$, *then*
$$\int|Sf(x)|^px^{p-2}\,dx\le C\int|f|^p\,dy\quad \text{if } 1<p\le2, \tag{1.7}$$
and in case S is linear (and S^* denotes its adjoint), we get
$$\int|S^*f|^q\,dx\le C\int|f|^qy^{q-2}\,dy\quad \text{if } 2\le q<\infty. \tag{1.8}$$

Proof. Consider $Tf(x)=xSf(x)$ with $d\nu(x)=\frac{dx}{x^2}$. Now T is a sublinear operator and
$$|Tf(x)|\le|x|\cdot\|Sf\|_\infty\le C|x|\cdot\|f\|_1$$
by (i). Therefore,
$$\{x:|Tf(x)|>\lambda\}\subset\{x:C|x|\cdot\|f\|_1>\lambda\}=E_\lambda,$$
thus
$$\nu\{x:|Tf(x)|>\lambda\}\le\int_{E_\lambda}d\nu\le\frac{C}{\lambda}\|f\|_1.$$
Also by (ii) we get
$$\int|Tf(x)|^2\,d\nu(x)\le C\int|f|^2\,dy.$$
Then (1.7) follows from the Marcinkiewicz interpolation theorem. Also, (1.8) follows by duality.

2. L^2 estimates. In this section, we shall discuss (L^2,L^2)-estimates for the operator Kf defined in (0.1) and (0.2) and the companion operator Sf defined in (0.3). We will prove the following three theorems:

THEOREM 2.1. *Suppose that $\gamma=(b-a)/(2b)$, that either $b\ge a>1$ or $b>a=1$, (1.2), (1.3) are satisfied and that α and φ satisfy (1.3′) when $a=b$. Then the operator K is bounded in $L^2(0,\infty)$.*

THEOREM 2.2. *In addition to (1.2), (1.3) assume that*
$$\begin{cases}|\varphi(x,y)|\le C\quad \text{for } |x-y|\le5,\\ \displaystyle\int_{\max\{|x-u|,|x-v|\}\le5}\left|\partial_x\frac{\varphi(x,u)\varphi(x,v)}{\beta'(x)}\right|dx\le C,\end{cases}\tag{2.1}$$
and let $c=(b(1-2\gamma)-a)/a$. Suppose that either $b\ge a>1$ or $b>a\ge1$ and $\gamma>0$, and that α and φ satisfy (1.3′) when $\gamma=0$. Then K is a bounded operator from $L^2(0,\infty)$ to $L^2(0,\infty;x^cdx)$, i.e., $\int_0^\infty|Kf(x)|^2x^c\,dx\le C\int_0^\infty|f(y)|^2\,dy$.

For $d>0$ define the operator $Sf(\xi)=Kf(\xi^d)$.

THEOREM 2.3. *If the assumptions of Theorem 2.2 are satisfied and if $d=a/(b(1-2\gamma))$, then S is a bounded operator in $L^2(0,\infty)$. If $\gamma=0$, then $d=a/b$.*

Remark 1. Theorem 2.3 is an immediate consequence of Theorem 2.2: letting $x=\xi^d$ we get
$$\int_0^\infty|Kf(\xi^d)|^2\,d\xi=\frac1d\int_0^\infty|Kf(x)|^2x^{(1-d)/d}\,dx\le C\int_0^\infty|f(y)|^2\,dy,$$
provided $(1-d)/d=c$.

Remark 2. If $a>1$ then (2.1) can be replaced by
$$\int_{\max\{|x-u|,|x-v|\}\le5}|\partial_x[\varphi(x,u)\varphi(x,v)\beta'(x)^{-1}]|dx\le C\max\{u^\varepsilon,v^\varepsilon,1\},\tag{2.1′}$$
where $\varepsilon\le(a-1)(1+c)/b$.

It is easy to translate (2.1) and (2.1′) into (sufficient) conditions on φ and β: (2.1) corresponds to the condition that $\beta''\in L^1$ and (2.1′) corresponds to $x^{-\varepsilon}\beta''\in L^1$.

Remark 3. Let χ be a smooth cutoff function, $\chi(t)=1$ for $0\le t\le4$, $\chi(t)=0$ for $t\ge5$ and let
$$\begin{aligned}k(x,y)&=\chi(|x-y|)k(x,y)+(1-\chi(|x-y|))k(x,y)\\&=k_1(x,y)+k_2(x,y).\end{aligned}\tag{2.2}$$

We will obtain the relevant estimates separately for the operators K_1 and K_2 corresponding to the kernels k_1 and k_2. We have $\int|k_1(x,y)|\,dy$, $\int|k_1(x,y)|\,dx\le C$ by (1.3) and Theorem 2.1 is immediate for K_1 (which is actually a bounded operator in L^p for $1 \le p \le \infty$). It is not so for Theorem 2.2 and here the condition (2.1) is needed. As concerns K_2, Theorem 2.1 is the special case of Theorem 2.2 with $\gamma = (b-a)/(2b)$.

The conclusion of this remark is that it is sufficient to prove Theorem 2.2.

Remark 4. The case when $a = 1$ and $\gamma = 0$ is excluded in Theorem 2.2 and will be dealt with in Section 4.

We now proceed with the proof of Theorem 2.2 (and according to Remark 3, of Theorem 2.1).

We decompose K_2 by letting $E_1 = [0,2]$, $E_2 = [2,\infty)$ and consider the operators $K_{ij} := K_2 : L^2(E_j) \to L^2(E_i, x^c dx)$, $i,j = 1,2$. Clearly $K_{11} = 0$ and we will show, in this order, that the operators K_1, K_{12}, K_{21}, K_{22} are bounded.

Because of (1.2) we have $m^c x^c \le \beta(x)^c \le M^c x^c$, with the corresponding inequalities between integrals with weights x^c and $\beta(x)^c$.

For $Hf(x) = \int_F h(x,y) f(y)\,dy$ we can write

$$\int_E \beta(x)^c |Hf(x)|^2\,dx \le \int_F \int_F A(u,v) f(u)\overline{f(v)}\,du\,dv, \tag{2.3}$$

where

$$A(u,v) = \int_E \beta(x)^c h(x,u)\overline{h(x,v)}\,dx. \tag{2.4}$$

The desired estimate $\int_E |Hf(x)|^2 \beta(x)^c\,dx \le \int_F |f(y)|^2\,dy$ is equivalent to $\|A\|_{L^2(F),L^2(F)} \le C$ where A is the operator with the kernel $A(u,v)$.

In all four cases we use variants of the same approach based on Proposition 1.2, on inequalities (1.4)–(1.6) and integration by parts to estimate $\|A\|$.

1) *Estimate of* K_1. We recall that φ satisfies (2.1) and vanishes for $|x-y| \ge 5$. The kernel in (2.4) can be written in the form

$$\begin{aligned} A(u,v) &= \int_0^\infty \beta(x)^c \varphi(x,u)\overline{\varphi(x,v)} \exp\{i\beta(x)^b[\alpha(u)^a - \alpha(v)^a]\}\,dx \\ &= \int_0^\infty \frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}\,d\int_0^{\beta(x)} t^c \exp\{it^b[\alpha(u)^a - \alpha(v)^a]\}\,dt\,dx. \end{aligned}$$

Also $A(u,v) = 0$ if $|u-v| \ge 10$. Integrate by parts denoting by $[x_1,x_2]$ the interval outside of which $\varphi(x,u)\varphi(x,v)$ vanishes ($x_1 \ge \max\{0, u-5, v-5\}$, $x_2 \le \min\{u+5, v+5\}$).

The modulus of the resulting first term,

$$\varphi(x_2,u)\overline{\varphi(x_2,v)}\beta'(x_2)^{-1} \int_0^{\beta(x_2)} t^c \exp\{it^b[\alpha(u)^a - \alpha(v)^a]\}\,dt,$$

is dominated by (using (1.4) and $|\varphi| \le C$)

$$|\alpha(u)^a - \alpha(v)^a|^{-(1+c)/b} \mathbf{1}_{[0,10]}(|u-v|). \tag{2.5}$$

The same estimate (using (2.1)) is valid also for the second term

$$\int_{x_1}^{x_2} \partial_x \frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)} \int_0^{\beta(x)} t^c \exp\{it^b[\alpha(u)^a - \alpha(v)^a]\}\,dt\,dx,$$

with the extra factor $\max\{1, u^\varepsilon, v^\varepsilon\}$ in case of the condition (2.1′).

We use the inequality

$$|\xi^a - \eta^a| \ge \tfrac{1}{2}|\xi - \eta|(\xi^{a-1} + \eta^{a-1}) \qquad \xi,\eta \ge 0,\ a \ge 1,$$

and (1.2) to get

$$|\alpha(u)^a - \alpha(v)^a|^{-(1+c)/b} \le C[(u^{a-1} + v^{a-1})|u-v|]^{-(1+c)/b}, \tag{2.6}$$

which together with (2.5) yields the corresponding bound of $A(u,v)$.

We now use Corollary 1.3ii with $\varrho = (a-1)(1+c)/b$, $\sigma = (1+c)/b$ (if $a > 1$ we could use $\varrho = (a-1)(1+c)/b - \varepsilon$ in the case (2.1′)) to conclude that $A(u,v)$ is the kernel of a bounded operator in $L^2(0,\infty)$.

This completes the estimate of K_1.

From now on, as explained above we assume that $\varphi(x,y) = 0$ for $|x-y| \le 4$. Also, by a suitable change of the variable x which should be done before splitting k as in (2.2), we may assume that $\beta(2) = 2$.

2) *Estimate of* K_{12}. According to (2.4) we need to find a suitable estimate for

$$\begin{aligned} A(u,v) &= \int_0^2 \beta(x)^c \varphi(x,u)\overline{\varphi(x,v)} \exp\{i\beta(x)^b[\alpha(u)^a - \alpha(v)^a]\}\,dx \\ &= \int_0^2 \frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}\,d\int_0^{\beta(x)} t^c \exp\{it^b[\alpha(u)^a - \alpha(v)^a]\}\,dt\,dx \end{aligned} \tag{2.7}$$

for $u,v \ge 2$.

Since $\varphi(x,y) = 0$ for $|x-y| \le 4$, we have, for $0 \le x \le 2 \le y$,

$$|\varphi(x,y)| \le Cy^{-\gamma}|1 - \max\{x/y : 0 \le x \le 2 < 4 \le y\}|^{-\gamma} = Cy^{-\gamma}\mathbf{1}_{[4,\infty)}$$

and similarly

$$|\partial_x \varphi(x,y)| \le Cy^{-\gamma-1}\mathbf{1}_{[4,\infty)}.$$

(1.2) implies that β'' is integrable over $[0,2]$.

Integration by parts in (2.7) produces two terms which can be estimated using (1.4) and (2.6) by

$$C|\varphi(2,u)\varphi(2,v)|\cdot|\alpha(u)^a-\alpha(v)^a|^{-(1+c)/b}$$

and

$$C\int_0^2\left|\partial_x\frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}\right|dx\,|\alpha(u)^a-\alpha(v)^a|^{-(1+c)/b}.$$

By the preceding remarks both terms can be estimated by

$$Cu^{-\gamma}v^{-\gamma}[(u^{a-1}+v^{a-1})|u-v|]^{-(1+c)/b}.$$

Since $2\gamma+a(1+c)/b=1$ (this implies $(1+c)/b<1$) it follows that the last bound satisfies the conditions of Corollary 1.3i with any $\varepsilon\in(\gamma,1-\gamma)\neq\emptyset$, and is the kernel of a bounded operator in $L^2(0,\infty)$.

This concludes the estimate of K_{12}.

3) *Estimate of* K_{21}. We write

$$\int_2^\infty\beta(x)^c|Kf(x)|^2\,dx=\lim_{T\to\infty}\int_2^T\beta(x)^c|Kf(x)|^2\,dx$$

and as in (2.4) we are led to estimating the kernel

$$A_T(u,v)=\int_2^T\beta(x)^ck(x,u)\overline{k(x,v)}\,dx \tag{2.8}$$

$$=-\int_2^T\frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}d\int_{\beta(x)}^{\beta(T)}t^c\exp\{t^b[\alpha(u)^a-\alpha(v)^a]\}\,dt\,dx,\qquad 0\le u,v\le 2.$$

The boundary term appearing in integration by parts in (2.8) contains the factor $\varphi(2,u)\overline{\varphi(2,v)}=0$ for $0\le u,v\le 2$ and drops out.

The second term is of the form

$$\int_2^T\partial_x\frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}\int_{\beta(x)}^{\beta(T)}t^c\exp\{t^b[\alpha(u)^a-\alpha(v)^a]\}\,dt\,dx.$$

Also, for $0\le y\le 2\le x$, $|\varphi(x,y)|\le Cx^{-\gamma}$ and $|\partial_x\varphi(x,y)|\le Cx^{-\gamma-1}$. Using these, (1.6) and (2.6) we get the estimate of the form

$$|A_T(u,v)|\le[(u^{a-1}+v^{a-1})|u-v|]^{-(1+c)(1-\theta)/b-\theta}$$
$$\times\int_2^y(x^{-\gamma-1}\beta'(x)^{-1}+x^{-2\gamma}\beta''(x))x^{\theta(c-b+1)}\,dx.$$

The last integral is finite as long as

$$\theta(b-1-c)+\gamma=\frac{a-1+2\gamma}{a}b\theta+2\gamma>\varepsilon$$

where ε is the parameter appearing in (1.2).

The factor $[(u^{a-1}+v^{a-1})|u-v|]^{-(1+c)(1-\theta)/b-\theta}$ is the kernel of a bounded operator in $L^2(0,2)$ provided the exponent $-(1+c)(1-\theta)/b-\theta$ is $>-1/a$ (this is a consequence of Proposition 1.2 with $\lambda=\mu=1$). If $\gamma>0$, this can be accomplished by choosing $\theta<2\gamma/(a-1+2\gamma)$.

We obtain a bound for $\|A_T\|$ independent of T.

When $\gamma=0$ we use the following duality argument. Instead of the operator K_{21} we consider its adjoint $K_{21}^*:L^2(2,\infty;x^c\,dx)\to L^2(0,2)$, given by the formula $K_{21}^*g(y)=\int_2^\infty k(x,y)g(x)x^c\,dx=\int_2^\infty k(x,y)\widetilde{g}(x)x^{c/2}\,dx$ where $\widetilde{g}(x)=x^{c/2}g(x)$.

The desired estimate is $\int_0^2|\int_2^\infty k(x,y)\widetilde{g}(x)x^{c/2}\,dx|^2\,dy\le C\int_2^\infty|\widetilde{g}(x)|^2\,dx$, which as before leads to consideration of the kernel

$$A^*(u,v)=(uv)^{c/2}\int_0^2\varphi(u,y)\overline{\varphi(v,y)}\exp\{i\alpha(x)^a[\beta(u)^b-\beta(v)^b]\}\,dy$$
$$=(uv)^{c/2}\int_0^2\frac{\varphi(u,y)\overline{\varphi(v,y)}}{\alpha'(y)}d\int_0^{\alpha(y)}\exp\{is^a[\beta(u)^b-\beta(v)^b]\}\,ds.$$

We integrate by parts and use (1.3′) to arrive at the estimate

$$|A^*(u,v)|\le C(uv)^{c/2}[(u^{b-1}+v^{b-1})|u-v|]^{-1/a}\qquad\text{for } u,v\ge 2.$$

In the present case $c=(b-a)/a$ and the last expression is a kernel satisfying conditions of Corollary 1.3i. We obtain a bound for $\|A^*\|$ and the estimate of K_{21} is complete.

4) *Estimate of* K_{22}. We add now, to the procedure used in the preceding three cases, a dyadic partition of the quadrant $[2,\infty)\times[2,\infty)$:

$$\int_2^\infty\beta(x)^c\left|\int_2^\infty k(x,y)f(y)\,dy\right|^2$$
$$=\sum_{m=1}^\infty\int_{2^m}^{2^{m+1}}\left|\sum_{l=2}^\infty\int_2^\infty k(x,y)\chi_l(|x-y|)f(y)\,dy\right|^2\beta(x)^c\,dx$$
$$\le\sum_{m=1}^\infty\Big(\sum_{l=2}^\infty\Big(\int_{2^m}^{2^{m+1}}\beta(x)^c\left|\int_2^\infty k(x,y)f(y)\,dy\right|^2dx\Big)^{1/2}\Big)^2,$$

where $\chi_l=\mathbf{1}_{[2^l,2^{l+1}]}$ is the characteristic function of the interval $[2^l,2^{l+1}]$. Again we have used the fact that $k(x,y)=0$ for $|x-y|\le 4$.

The integrals inside the sum are of the form (2.4) with the kernels

$$A(u,v)=A_{ml}(u,v)=\int_{2^m}^{2^{m+1}}\beta(x)^c k(x,u)\overline{k(x,v)}\chi_l(|x-u|)\chi_l(|x-v|)\,dx.$$

Our objective now is to find estimates of the norms of the corresponding operators in $L^2(0,\infty)$ which would insure that

$$\sum_{m=1}^{\infty}\Big(\sum_{l=2}^{\infty}\|A_{ml}\|^{1/2}\Big)^2<\infty. \tag{2.9}$$

We write the kernels A_{ml} in the form

$$A_{ml}(u,v)=\int_{S_{ml}}\beta(x)^c k(x,u)\overline{k(x,v)}\,dx$$

where S_{ml} is the interval

$$S_{ml}=S_{ml}(u,v)=\{x\in[2^m,2^{m+1}]:|x-u|,|x-v|\in[2^l,2^{l+1}]\}$$

We observe that if $S_{ml}\neq\emptyset$, then

$$\text{(i) } |u-v|\le 2^{l+2},\qquad \text{(ii) } u,v\ge 2^{\max\{m,l\}-1}\quad\text{if } m\neq l,l+1. \tag{2.10}$$

Indeed, if $x\in S_{ml}$ then $|u-v|\le|u-x|+|x-v|\le 2^l+2^l=2^{l+1}$, which is (i). Also, with $y=u$ or v we have $y\ge x-|y-x|\ge 2^m-2^{l+1}\ge 2^{m-1}$ if $m\ge l+2$. Similarly $y\ge|y-x|-x\ge 2^l-2^{m+1}\ge 2^{l-1}$ if $l\ge m+2$. If $m=l-1$, then $y>x$, for the reverse inequality would imply that $2\le y\le x-2^l\le 0$, and it follows that $y\ge x+2^l>2^l$. This proves (ii).

Set $S_{ml}=[x_1,x_2]$ in order to write

$$A_{ml}=\int_{x_1}^{x_2}\frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}d\int_{\beta(x_1)}^{\beta(x)}t^c\exp\{it^b[\alpha(u)^a-\alpha(v)^a]\}\,dt\,dx$$

and integrate by parts.

The modulus of the boundary term is estimated using (1.2), (1.3), (1.6), (2.6) and (2.10) by

$$\begin{aligned}C|x_2-u|^{-\gamma}&|x_2-v|^{-\gamma}{x_1}^{(c-b+1)\theta}\\ &\times[(u^{a-1}+v^{a-1})|u-v|]^{-(1+c)(1-\theta)/b-\theta}\mathbf{1}_{[0,2^{l+2}]}\\ &\le C2^{-\varrho'_{ml}}|u-v|^{-\eta}\mathbf{1}_{[0,2^{l+2}]}(|u-v|),\end{aligned}$$

where

$$\varrho'_{ml}=2\gamma l+(b-c-1)\theta m+[(1+c)(1-\theta)/b+\theta](a-1)\max\{m,l\}$$

with the term $\max\{m,l\}$ omitted if $m=l$ or $m=l+1$.

The modulus of the second term (in the integration by parts formula) is estimated by

$$\int_{x_1}^{x_2}\bigg|\partial_x\frac{\varphi(x,u)\overline{\varphi(x,v)}}{\beta'(x)}\bigg|\,dx\,[(u^{a-1}+v^{a-1})|u-v|]^{-(1+c)(1-\theta)/b-\theta},$$

with the same θ as in the previous estimate.

We observe that

$$\int_{x_1}^{x_2}|\partial_x\varphi(x,u)\overline{\varphi(x,v)}|dx\le C2^{-l\gamma}\int_{x_1}^{x_2}|x-u|^{-\gamma-1}\,dx\le C2^{-\gamma}$$

for $\gamma>0$ and $\le C\big|\log\big|\frac{x_1-u}{x_2-u}\big|\big|\le C\log 2$ when $\gamma=0$, with the same estimate when u and v are interchanged.

The term

$$\int_{x_1}^{x_2}|\varphi(x,u)\overline{\varphi(x,v)}\beta''(x)|\,dx$$

is estimated using (1.2) by $C2^{-2\gamma l}2^{m\varepsilon}$. This together with the estimate of the boundary term results in the inequality

$$|A_{ml}(u,v)|\le C2^{-\varrho_{ml}}|u-v|^{-\eta}\mathbf{1}_{[0,2^{l+2}]}(|u-v|)$$

with $\eta=(1+c)(1-\theta)/b+\theta$ and $\varrho_{ml}=\varrho'_{ml}-m\varepsilon$.

By Corollary 1.3ii (with $\varrho=0$ and $\sigma=\eta$) the norm in $L^2(2,\infty)$ of the operator with the kernel

$$|u-v|^{-\eta}\mathbf{1}_{[0,2^{l+2}]}(|u-v|)$$

can be estimated by $C2^{l(1-\eta)}$, provided $0\le\eta<1$. Thus we arrive at the estimate

$$\|A_{ml}\|\le C2^{-\eta_{ml}}$$

where, using the definition of c,

$$\begin{aligned}\eta_{ml}&=\varrho_{ml}-(1-\eta)l\\ &=\Big[2\gamma-(1-\theta)\Big(1-\frac{1}{a}+\frac{2\gamma}{a}\Big)\Big]l+\Big[b-\frac{b}{a}(1-2\gamma)\theta-\varepsilon\Big]m\\ &\quad+(a-1)\Big[\frac{(1-2\gamma)(1-\theta)}{a}+\theta\Big]\max\{m,l\},\end{aligned}$$

with the same proviso as before concerning the last term. The point is now to choose θ (which may depend on m,l) so as to produce a lower bound for the last expression of the form $C\max\{m,l\}$ with $C>0$.

Consider first the case when $\gamma=0$. Then $a>1$ and for $m\neq l,l+1$ we can choose θ sufficiently near to 1 so that $(a-1)\theta>2(1-\theta(1-1/a))$. We then get the desired estimate provided $\varepsilon\le b-b/a$.

The cases $m = l$ or $m = l + 1$ differ only by the factor 2 in the bound for $|A_{ml}|$ and we consider only the case when $m = l$ (again $\gamma = 0$). Then

$$\eta_{ll} = \left[-(1-\theta)\left(1-\frac{1}{a}\right)+\left(b-\frac{b}{a}\right)\theta-\varepsilon\right]l > \left[\frac{1}{2}\left(b-\frac{b}{a}\right)-\varepsilon\right]l$$

for $\theta < 1$ sufficiently near to 1 and $\varepsilon < \frac{1}{2}(b - b/a)$. We end up with the restriction that $\varepsilon < \frac{1}{2}(b - b/a)$, with which we arrive again at an estimate of the desired form.

If $\gamma > 0$, then we can choose $\theta < 1$ so that $2\gamma-(1-\theta)(1-1/a+2\gamma/a) \ge \gamma$. Then $b-(b/a)(1-2\gamma)\theta-\varepsilon > 2\gamma b-\varepsilon \ge \gamma b$ provided $\varepsilon \le \gamma b$. We drop the term $\max\{m, l\}$ which appears with a nonnegative coefficient. We thus arrive at a lower bound for η_{ml} which implies (2.9).

This completes the estimate of K_{22} and the proof of Theorems 2.1–2.3.

3. The necessary conditions and L^p estimates. In Theorem 3.1, we show that the operators defined in (0.1) map L^p into itself where $p = (b+a)/b$. Next we consider the special case where $\alpha(x) = \beta(x) = x$ and $\varphi(x, y) = |x-y|^{i\tau}$. In Corollary 3.2, we show that for $p = (b+a)/b$, (p,p) is an endpoint for this operator. We also show that for some relevant operators, $(2, 2)$ is a one-sided endpoint; this is done in Theorem 3.3.

We begin with the following result.

THEOREM 3.1. *Let $b \ge a > 1$. If $\varphi(x, y)$ satisfies* (1.3) *with $\gamma = 0$ and* (1.3′) *and α and β satisfy* (1.2) *then*

$$\|Kf\|_p \le C\|f\|_p \quad \textit{for } p = (b+a)/b.$$

Proof. It is clear that the operator

$$Sf(x) = \int \varphi(x^{a/b}, y)e^{i\beta(x^{a/b})^b\alpha(y)^a} f(y)\,dy$$

satisfies (i) of Proposition 1.4. By Theorem 2.3, S also satisfies (ii) of Proposition 1.4. But

$$\|Kf\|_p^p = C\int x^{a/b-1}|Sf(x)|^p\,dx \le C\|f\|_p^p,$$

where the last inequality follows from Proposition 1.4 if $p - 2 = a/b - 1$, i.e., $p = (b + a)/b$.

In Section 4, we prove that the operator

$$T_\tau f(x) = \int_{-\infty}^{\infty} \frac{e^{i|x|^b|y|}f(y)}{|x-y|^{i\tau}}\,dy$$

maps L^p into itself for $p = (b + 1)/b$. As an immediate consequence of Theorem 3.1 along with this result, we get

COROLLARY 3.2. *Let $b \ge a \ge 1$ and set*

$$T_\tau f(x) = \int_{-\infty}^{\infty} \frac{e^{i|x|^b|y|^a}f(y)}{|x-y|^{i\tau}}\,dy.$$

Then T_τ is a bounded operator in L^p if and only if

$$p = (b+a)/b.$$

If this is the case, then $\|T_\tau\|_{p,p}$ is bounded polynomially in τ.

Proof. The sufficiency follows from Theorem 3.1 in case $b \ge a > 1$ and $b = a = 1$. If $b > 1$ and $a = 1$ we employ Theorem 4.1 from Section 4.

In order to see the necessity, consider

$$\|Tf\|_q^q = \int_{-\infty}^{\infty} dx\left|\int_{-\infty}^{\infty} \frac{e^{i|x|^b|y|^a}f(y)}{|x-y|^r}\,dy\right|^q$$

$$= C\int_{-\infty}^{\infty} dx\,|x|^{1/b-1}\left|\int_{-\infty}^{\infty} \frac{e^{i|x||y|}f(|y|^{1/a})|y|^{1/a-1}}{||x|^{1/b}-|y|^{1/a}|^r}\,dy\right|^q.$$

Take $f_N(y) = 1$ if $N/2 \le y \le N$, and $f_N(y) = 0$ elsewhere. Thus $f_N(y^{1/a}) = 1$ if $(N/2)^a \le y \le N^a$. Suppose that T maps L^q into L^q. That implies

$$\|Tf_N\|_q \le C\|f_N\|_q. \tag{3.1}$$

Thus (3.1) implies that

$$\int_{1/(2N^a)\le x\le 1/N^a} dx\,x^{1/b-1}\left|\int \frac{e^{ixy}y^{1/a-1}f_N(y^{1/a})}{|x^{1/b}-y^{1/a}|^r}\,dy\right|^q \le C\int |f_N(y)|^q\,dy. \tag{3.2}$$

First suppose that $N \gg 1$. Then since $xy \le 1$, from (3.2) we get

$$\frac{1}{N^{a/b}}\cdot\frac{N^q}{N^{rq}} \le CN$$

or

$$N^{q(1-r)-a/b-1} \le C \quad \text{for } N \gg 1. \tag{3.3}$$

Letting $N \to \infty$ we see that (3.3) is violated if $q(1-r) - a/b - 1 > 0$ or

$$q > \frac{b+a}{b(1-r)}. \tag{3.4}$$

In our case $r = 0$, thus from (3.3) and (3.4) we conclude that T does not map L^q into itself if $q > (b + a)/b$.

Next suppose $N \ll 1$. This time from (3.2) we get, again using $xy \le 1$,

$$\frac{N^q\cdot N^{(a/b)rq}}{N^{a/b}} \le CN,$$

or

$$N^{q+(a/b)rq-a/b-1} \le C \quad \text{for } N \ll 1. \tag{3.5}$$

Letting $N \to 0^+$ we see that (3.5) is violated if $q+(a/b)rq-a/b-1<0$, or

$$q < \frac{b+a}{b+ar}. \tag{3.6}$$

By (3.3)–(3.6) with $r=0$, we conclude that T_0 does not map L^q into itself if $q \ne (b+a)/b$. This completes our proof.

The next result pertains to an operator corresponding to $\gamma = r = (b-a)/(2b)$ in (1.3).

COROLLARY 3.3. *Let $p \ge 2$. Let $b \ge a \ge 1$, $\tau \in \mathbb{R}$ and set*

$$T_\tau f(x) = \int \frac{e^{ix^b y^a} f(y)}{|x-y|^{r+i\tau}}\,dy,$$

where $r=(b-a)/(2b)$. Then T_τ is a bounded operator in L^2, but is not a bounded operator in L^p for any $p>2$.

Proof. The boundedness in L^2 follows from Theorem 2.1. For the remaining part, we employ (3.3) and (3.4) with $r=(b-a)/(2b)$. This completes our proof.

4. The case where $b>1$ and $a=1$. Here we study the L^p mapping properties of the operator

$$Tf(x) = \int \frac{e^{ix^b y} f(y)}{|x-y|^{i\tau}}\,dy, \quad \tau \in \mathbb{R},\ x>0, \tag{4.1}$$

for $p=(b+1)/b$.

This complements the result in Theorem 2.2 in the special case when $a=1$ and $\gamma=0$.

Rather than dealing directly with the operator T we show that the dual operator

$$T^* f(x) = \int \frac{e^{ixy^b} f(y)}{|x-y|^{i\tau}}\,dy \tag{4.2}$$

maps L^p into itself for $p=b+1$.

By using a method due to Phong and Stein ([PS]) and weighted norm inequalities we shall prove the following:

THEOREM 4.1. *Let $b \ge 1$. There is a constant C so that*

$$\|T^* f\|_p \le C\|f\|_p \quad \textit{for } p=b+1. \tag{4.3}$$

Remarks. (i) Let η be a C^∞ function so that $\eta(x)=1$ for $|x| \ge 1$ and $\eta(x)=0$ for $|x| \le 1/2$. To prove (4.3), we show that the operator

$$f \to \int \frac{e^{ixy^b}\eta(x-y)}{|x-y|^{i\tau}} f(y)\,dy$$

is bounded in L^{b+1}.

(ii) Let $\varphi \in C_0^\infty$ be such that $\mathrm{supp}(\varphi) \subset [-1,1]$ and $\varphi(x) \equiv 1$ for $|x| \le 1/2$. For any $\lambda>1$, define

$$K_\lambda(x) = \frac{\eta(x)}{|x|^{i\tau}}\varphi\left(\frac{x}{\lambda}\right).$$

Let

$$T_\lambda f(x) = \int e^{ixy^b} K_\lambda(x-y) f(y)\,dy \quad \text{for } x \in \mathbb{R}. \tag{4.4}$$

Then (4.3) follows from the following

THEOREM 4.1′. *Let $b \ge 1$. There exists a constant C independent of λ such that*

$$\|T_\lambda f\|_p \le C\|f\|_p \quad \textit{for } p=b+1. \tag{4.5}$$

We denote by $\widehat{K}_\lambda(\xi)$ the Fourier transform of K_λ, i.e.,

$$\widehat{K}_\lambda(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} K_\lambda(x)\,dx.$$

Then we have

LEMMA 4.2. *There are constants C_1, C_2, C_3 independent of λ so that*

$$\begin{cases} \text{(a)}\ |\widehat{K}_\lambda(\xi)| \le C_1|\xi|^{-2} & \textit{for } \xi \in \mathbb{R}, \\ \text{(b)}\ |\widehat{K}_\lambda(\xi)| \le C_2|\xi|^{-1} & \textit{for } |\xi| \le 2, \textit{ and} \\ \text{(c)}\ \left|\dfrac{d}{d\xi}\widehat{K}_\lambda(\xi)\right| \le C_3|\xi|^{-2} & \textit{for } |\xi| \le 2. \end{cases} \tag{4.6}$$

Remark. (4.6)(a) implies that (4.6)(b) holds for $|\xi|>2$ as well, but we use (4.6)(b) and (c) for small $|\xi|$ only.

The proof of Lemma 4.2 uses standard arguments, such as integration by parts, etc. Let T_λ be given as in (4.2). Then we have

$$\begin{aligned} \widehat{T_\lambda f}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} \int e^{ixy^b} K_\lambda(x-y) f(y)\,dy\,dx \\ &= \int e^{iy(y^b-\xi)} f(y) \Big[\int e^{-i(\xi-y^b)(x-y)} K_\lambda(x-y)\,dx\Big]\,dy \\ &= \int e^{iy(y^b-\xi)} \widehat{K}_\lambda(\xi-y^b) f(y)\,dy \end{aligned}$$

$$= \int e^{iy(y^b-\xi)}\widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)f(y)\,dy$$
$$+ \int e^{iy(y^b-\xi)}\widehat{K}_\lambda(\xi - y^b)(1-\varphi)(\xi - y^b)f(y)\,dy$$
$$= P_\lambda f(\xi) + Q_\lambda f(\xi).$$

We also define S_λ and R_λ by

$$S_\lambda f(\xi) = \int \widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)f(y)dy, \quad \text{and}$$
$$R_\lambda f(\xi) = \int |\xi - y^b| \cdot |\widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)f(y)|\,dy.$$

PROPOSITION 4.3. *There is a constant $C \ge 0$ independent of λ so that*

$$\int (|Q_\lambda f(\xi)|^p + |R_\lambda f(\xi)|^p)|\xi|^{p-2}\,d\xi \le C\int |f(y)|^p\,dy \quad \textit{for } p = b+1. \tag{4.7}$$

Proof. Using (4.6)(a) we have

$$|Q_\lambda f(\xi)| \le \int \frac{1}{1+|\xi - y^b|^2}|f(y)|\,dy,$$

and from (4.6)(b) we have

$$|\xi - y^b| \cdot |\widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)| \le C|\varphi(\xi - y^b)|,$$

therefore we get

$$|R_\lambda f(\xi)| + |Q_\lambda f(\xi)| \le C\int \left(|\varphi(\xi - t)| + \frac{1}{1+|\xi - t|^2}\right)|g(t)|\,dt$$

where

$$g(t) = \begin{cases} (1/b)f(t^{1/b})t^{1/b-1} & \text{for } t > 0, \\ 0 & \text{elsewhere.}\end{cases} \tag{4.8}$$

Hence we get

$$|R_\lambda f(\xi)| + |Q_\lambda f(\xi)| \le CMg(\xi)$$

where Mg denotes the Hardy–Littlewood maximal function of g. Since $|x|^{b-1} \in A_p$, where A_p denotes the weight class of Muckenhoupt [M], we get

$$\int_{-\infty}^{\infty} (|R_\lambda f(\xi)| + |Q_\lambda f(\xi)|)^p|\xi|^{p-2}\,d\xi \le C\int (Mg(\xi))^p|\xi|^{p-2}\,d\xi$$
$$\le C\int |g|^p|t|^{p-2}\,dt = C\int |f|^p\,dt$$

for $p = b+1$. This completes our proof.

Next we show that similar estimates hold for the operator S_λ.

PROPOSITION 4.4. *There is a constant C independent of λ so that*

$$\int_{-\infty}^{\infty} |S_\lambda f(\xi)|^p|\xi|^{p-2}d\xi \le C\int |f(y)|^p\,dy \quad \textit{with } p = b+1.$$

Proof. Let $\Omega_\lambda(x) = \widehat{K}_\lambda(x)\varphi(x)$. Then we have

$$S_\lambda f(x) = \Omega_\lambda * g(x)$$

with $g(t)$ as in (4.8).

Next we prove that Ω_λ is a Calderón–Zygmund kernel, i.e. there is a constant C independent of λ so that

$$\begin{cases} \text{(a)} & |\Omega_\lambda(x)| \le C|x|^{-1}, \\ \text{(b)} & |\frac{d}{dx}\Omega_\lambda(x)| \le C|x|^{-2}, \quad \text{and} \\ \text{(c)} & \|\widehat{\Omega}_\lambda\|_\infty \le C. \end{cases} \tag{4.9}$$

Clearly (4.9)(a), (b) follow from Lemma 4.2. In order to see (4.9)(c), we note that for $z \in \mathbb{R}$,

$$|\widehat{\Omega}_\lambda(z)| = |(\widehat{K}_\lambda\varphi)^\vee(-z)| = |K_\lambda * \varphi^\vee(-z)| \le \|K_\lambda\|_\infty\|\varphi^\vee\|_1 \le C.$$

Therefore (4.9)(c) holds and since $|x|^{b-1}$ is an A_p weight with $p = b+1$, we have

$$\int |S_\lambda f(x)|^p|x|^{p-2}dx = \int |\Omega_\lambda * g|^p|x|^{p-2}\,dx$$
$$\le C\int |g(t)|^p|t|^{p-2}\,dt = C\int |f|^p\,dy$$

for $p = b+1$. This completes the proof.

Next we need to obtain weighted L^p estimates of $P_\lambda f(\xi)$.

PROPOSITION 4.5. *There exists a constant C independent of λ such that*

$$\int_{-\infty}^{\infty} |P_\lambda f(\xi)|^p|\xi|^{p-2}\,d\xi \le C\int |f(y)|^p\,dy \quad \textit{with } p = b+1.$$

Proof. Let $I_j = [j^{1/b}, (j+1)^{1/b}]$ and $f_j(y) = f(y)\chi_{I_j}(y)$. We have

$$P_\lambda f(\xi) = \sum_{j=0}^{\infty}\int e^{iy(y^b-\xi)}\widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)f_j(y)\,dy$$
$$= \sum_{j=0}^{\infty}[F_j(\xi) + G_j(\xi)]$$

where

$$F_j(\xi) = \int [e^{i(y-j^{1/b})(y^b-\xi)} - 1]e^{i(y^b-\xi)j^{1/b}}\widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)f_j(y)\,dy,$$

and

$$G_j(\xi) = e^{-i\xi\cdot j^{1/b}}\int \widehat{K}_\lambda(\xi - y^b)\varphi(\xi - y^b)e^{iy^b\cdot j^{1/b}}f_j(y)\,dy.$$

We observe that if $y \in I_j$ and $|\xi - y^b| \le 1$, then $j-1 \le \xi \le j+2$. Therefore, both

$$\operatorname{supp} F_j, \operatorname{supp} G_j \subset [j-1, j+2]. \tag{4.10}$$

Also note that

$$\begin{aligned}|F_j(\xi)| &\le \int |y-j^{1/b}|\cdot|y^b-\xi|\cdot|\widehat{K}_\lambda(\xi-y^b)\varphi(\xi-y^b)|\cdot|f_j(y)|\,dy\\ &\le \int |y^b-\xi|\cdot|\widehat{K}_\lambda(\xi-y^b)\varphi(\xi-y^b)|\cdot|f_j(y)|\,dy\\ &\qquad\qquad (\text{since } |y-j^{1/b}|\le 1)\\ &= R_\lambda f_j(\xi).\end{aligned}$$

Also,

$$|G_j(y)| = \Big|\int \widehat{K}_\lambda(\xi-y^b)\varphi(\xi-y^b)e^{iy^b\cdot j^{1/b}} f_j(y)\,dy\Big| = |S_\lambda \widetilde{f}_j(\xi)|$$

where $\widetilde{f}_j(y) = e^{iy^b\cdot j^{1/b}} f_j(y)$.

By (4.10) and since $\{[j-1,j+2]\}_{j=0}^{\infty}$ have the finite overlapping property, we get

$$\begin{aligned}\int |P_\lambda(\xi)|^p|\xi|^{p-2}\,d\xi &= \int \Big|\sum_{j=0}^{\infty}(F_j(\xi)+G_j(\xi))\Big|^p|\xi|^{p-2}\,d\xi\\ &\le C\sum_{j=0}^{\infty}\int(|R_\lambda f_j(\xi)|^p+|S_\lambda \widetilde{f}_j(\xi)|^p)|\xi|^{p-2}\,d\xi\\ &\le C\sum_{j=0}^{\infty}\Big(\int|f_j(y)|^p\,dy+\int|\widetilde{f}_j(y)|^p\,dy\Big)\end{aligned}$$

by Propositions 4.3 and 4.4, and this now completes our proof for $p=b+1$.

Finally, we are in a position to prove Theorem 4.1′.

Proof of Theorem 4.1′. Let $p=b+1$, $b\ge 1$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}|T_\lambda f(x)|^p\,dx &= \int_{-\infty}^{\infty}|(\widehat{T_\lambda f})^\vee(x)|^p\,dx \le C\int_{-\infty}^{\infty}|\widehat{T_\lambda f}(\xi)|^p|\xi|^{p-2}\,d\xi\\ &= C\int_{-\infty}^{\infty}|P_\lambda f(\xi)+Q_\lambda f(\xi)|^p|\xi|^{p-2}\,d\xi\\ &\le C\Big[\int_{-\infty}^{\infty}|P_\lambda f(\xi)|^p|\xi|^{p-2}\,d\xi+\int_{-\infty}^{\infty}|Q_\lambda f(\xi)|^p|\xi|^{p-2}\,d\xi\Big]\\ &\le C\int|f(y)|^p\,dy\end{aligned}$$

by Propositions 4.3 and 4.5. This completes our proof.

By Fatou's lemma we get

$$\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty}e^{ix|y|^b}\frac{\eta(x-y)}{|x-y|^{i\tau}}f(y)\,dy\Big|^p\,dx \le C\liminf_{\lambda\to\infty}\|T_\lambda f\|_p^p \le C\|f\|_p^p,$$

by Theorem 4.1′, and thus we get our proof of Theorem 4.1.

By duality we see that T maps boundedly $L^{(b+1)/b}$ into itself.

5. Extended domains. In this section we describe the extended domains of some of the integral operators considered in the preceding sections.

The general construction, properties of extended domains of integral operators and references to the subject can be found in [AS] and [LS]. We list here some of the relevant facts.

For a measure space (X,dx) (usually σ-finite) we denote by $L^0(X)$ the space of all scalar-valued, measurable, a.e. finite functions on X. $L^0(X)$ is furnished with the topology of convergence in measure on all subsets of X of finite measure. A subset $A\subset L^0(X)$ is *solid* if with every function f, A contains the order interval consisting of functions a.e. dominated in absolute value by f. A topological vector subspace of L^0 is *solid* if it is solid as a subset of L^0 and if its topology can be defined by a base of neighborhoods of 0 which are solid.

We now consider two measure spaces, (X,dx), (Y,dy), a function $k\in L^0(X\times Y)$ and the integral operator $Kf(x)=\int_Y k(x,y)f(y)\,dy$ with domain $D_K=\{f\in L^0(Y):\int_Y|k(x,y)f(y)|\,dy<\infty \text{ a.e.}\}$ acting from $L^0(Y)$ into $L^0(X)$. The extended domain of K is a solid metric complete vector subspace $\widetilde{D}_K$ of $L^0(Y)$, with the following properties:

(i) *D_K is dense in $\widetilde{D}_K$ and $K:\widetilde{D}_K\supset D_K\to L^0(X)$ is continuous. Hence K can be extended by continuity to $\widetilde{D}_K$; this extension is denoted by $\widetilde{K}$.*

(ii) (Maximality property) *If $V\subset L^0(Y)$ is a solid topological vector space such that $K:V\supset D_K\cap V\to L^0(X)$ is continuous, then the closure $\overline{D_K\cap V}^V$ is contained in $\widetilde{D}_K$ and the extension by continuity of K to this closure is the restriction to it of $\widetilde{K}$.*

The space $\widetilde{D}_K$ is described by the following condition:

(iii) *$f\in\widetilde{D}_K$ if and only if for every sequence $\{g_n\}$ of functions in D_K with disjoint supports and such that $|g_n|\le|f|$ a.e. we have $\sum_n|Kg_n(x)|^2<\infty$.*

It is not known how the last condition can be used to define the topology on $\widetilde{D}_K$ (an F-norm defining this topology can be found in the references above but will not play any role in what follows).

The following special cases will be useful:

(iv) *If $k\ge 0$, then $\widetilde{D}_K=D_K$.*

(v) *If X, Y are compact, $k(x,y)\ne 0$ for all $x,y\in X\times Y$ and if k is continuous, then $\widetilde{D}_K=D_K$.*

(vi) *If X and Y are locally compact and if k is as above, then $\widetilde{D}_K\subseteq D_{K,\mathrm{loc}}$.*

The following are some obvious consequences of (iii).

(vii) *Let* $k_1(z,y) = \gamma(z)k(\alpha(z),y)$ *where* $\gamma(z) \neq 0$ *for a.e.* $z \in Z$ *is in* $L^0(Z)$, *and* $\alpha : Z \to X$ *is a measurable bijection preserving sets of measure* 0. *Then* $\widetilde{D}_{K_1} = \widetilde{D}_K$.

(viii) *Let* $\beta : Z \to Y$ *be a measurable bijection preserving sets of measure* 0, *let* β' *denote the Radon–Nikodym derivative of* $d\beta$ *with respect to* dx, *and let* $k_\beta(x,z) = k(x,\beta(z))$. *Then* $f \in \widetilde{D}_K$ *if and only if* $\beta' f \circ \beta \in \widetilde{D}_{K_\beta}$.

Properties (ii) and (iii) can be used to justify the following argument to determine $\widetilde{D}_K$. We use the necessity part of (iii) to find an upper bound, say V, for $\widetilde{D}_K$: $\widetilde{D}_K \subset V$. If V has some natural solid topology such that K is continuous as an operator from V to L^0 then (ii) can be used to establish the reverse inclusion and thus to show that $\widetilde{D}_K = V$.

We now consider the concrete case where $X = Y = \mathbb{R}_+$; furthermore, we look at a model case and consider various examples which can be reduced to the model case using one or both of the remarks (vii) and (viii).

In the model case, using notation from the preceding sections, write the kernel in question in the form

$$k(x,y) = \varphi(x,y)\exp(ixy) = |\varphi(x,y)|\exp[i(xy + \psi(x,y))],$$

where the functions φ, ψ are subject to the following conditions:

(5.1) ψ is real, continuous and bounded relative to the term xy in the following sense: $\max_{n \le y \le n+1} |\psi(x,y) - \psi(x,n)| \to 0$ as $n \to \infty$ uniformly in x in any bounded interval.

(5.2) $\varphi(x,y) \neq 0$ for all x,y, is continuous and there exist positive functions $m(x) \le M(x)$, an increasing function $\omega(x)$ and a continuous positive function $\lambda(y)$ on $[0,\infty)$ such that a) $\varphi(x,y) \ge m(x)\lambda(y)$ and b) $|\partial_x^k \varphi(x,y)| \le M(x)\lambda(y)$ for $k = 0,1,2$, and $y \ge \omega(x)$.

(5.3) For every $T > 0$ the function $\mathbb{R}_+ \ni y \to \varphi(y,\cdot) \in L^1(0,T)$ is bounded.

We can now, according to the general scheme outlined above, describe an upper bound for $\widetilde{D}_K$.

PROPOSITION 5.1. *Suppose that the kernel* k *as above satisfies* (5.1) *and* (5.2)(a). *Then* $\widetilde{D}_K \subset \{f : \lambda f \in l^2(L^1(\mathbb{R}_+))\}$, *where*

$$l^2(L^1(\mathbb{R}_+)) = \Big\{ g \in L^1_{\mathrm{loc}} : \sum_{n=0}^{\infty} \Big[\int_n^{n+1} |g(y)|\,dy \Big]^2 < \infty \Big\}$$

is the usual amalgam space with the obvious norm.

Proof. (5.2)(a) implies that $D_K = L^1(\mathbb{R}_+, \lambda)$, the L^1-space with weight λ, in particular, by (vi), $\widetilde{D}_K \subset L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Suppose now that $f \in \widetilde{D}_K$. Since $\widetilde{D}_K$ is solid we may replace f by $|f|$ and thus assume that $f \ge 0$. We use the necessity of the condition in (iii) with conveniently chosen sequence $\{g_n\}$: we take an integer N sufficiently large and let $g_n = \chi_{[N+n-1,N+n]} f$, $n = 1,2,\ldots$ Then $\{g_n\} \subset D_K$ is a candidate to be used in (iii) to the effect that $\sum_{n=1}^{\infty} |Kg_n(x)|^2 < \infty$ a.e. We restrict x to the interval $[0,a]$ and write

$$\begin{aligned} |Kg_n(x)| &= \Big| \int_{N+n-1}^{N+n} \exp[ixy + i\psi(x,y)]|\varphi(x,y)|f(y)\,dy \Big| \\ &= \Big| \exp[ix(N+n-1) + \psi(x,N+n-1)] \\ &\quad \times \int_0^1 \exp\{ixt + i[\psi(x,N+n-1+t) - \psi(x,N+n-1)]\} \\ &\quad \times |\varphi(x,N+n-1+t)|f(N+n-1+t)\,dt \Big|. \end{aligned}$$

We now choose N so that for x in the interval $[0,a]$ for $t \in [0,1]$ and for all $n \ge 1$ we have

$$xt + \psi(x,N+n-1+t) - \psi(x,N+n-1) \in [-\pi/4, \pi/4].$$

Then

$$\begin{aligned} \Big| \int_{N+n-1}^{N+n} k(x,y)f(y)\,dy \Big| &\ge \Big| \int_{N+n-1}^{N+n} \mathrm{Re}[k(x,y)]f(y)\,dy \Big| \\ &\ge \frac{1}{\sqrt{2}} \int_{N+n-1}^{N+n} m(x)\lambda(y)f(y)\,dy \end{aligned}$$

and the conclusion follows, since $\int_0^N f(y)\,dy$ is finite (because $\widetilde{D}_K \subset L^1_{\mathrm{loc}}$).

The upper bound for $\widetilde{D}_K$ defined in Proposition 5.1 we denote by V, $V = \{f : \lambda f \in l^2(L^1(\mathbb{R}_+))\}$. We next prove:

PROPOSITION 5.2. *If* K *satisfies* (5.1), (5.2)(b) *and* (5.3)(b), *then* $V \subset \widetilde{D}_K$.

Proof. We use (ii), the maximality property of the extended domain. We notice first that D_K is dense in V and prove that $K : V \to L^0$ is continuous.

The continuity of K on V is established by observing that for every $S > 0$ we can represent V as the direct sum of $L^1([0,S])$ and $l^2(L^1([S,\infty);\lambda))$. Since

$$\int_0^T \int_0^S |k(x,y)f(y)|\,dy\,dx \le \|f\|_{L^1(0,S)} \sup_{[0,S]} \|k(\cdot,y)\|_{L^1(0,T)},$$

it follows that for every S, K is bounded as an operator from the first summand into $L^1_{\mathrm{loc}} \subset L^0$.

To take care of the second summand we choose $\varepsilon > 0$ and a smooth function $e(x) \geq 0$ equal to 1 in the interval $[\varepsilon, X)$, and equal to 0 for x near 0 and for $x > X+1$, say. We are going to get an estimate of the form

$$\int_0^\infty e(x)\Big|\int_S^\infty k(x,y)f(y)\,dy\Big|^2\,dx \leq C\|f\|_{l^2(L^1(S,\infty;\lambda))}$$

where $S \geq \omega(X+1)$.

The estimate is obtained by a procedure similar to the one used in Section 2: we write the integral in question in the form $\int_S^\infty\int_S^\infty A(u,v)f(u)\overline{f(v)}\,du\,dv$ where $A(u,v) = \int_0^\infty e(x)k(x,u)\overline{k(x,v)}\,dx$, and estimate the resulting expression by

$$\sum_{m,n=0}^{\infty} a_{mn}\int_{S+m}^{S+m+1}\lambda(u)|f(u)|\,du\int_{S+n}^{S+n+1}\lambda(v)|f(v)|\,dv,$$

where

$$a_{mn} = \sup\{|A(u,v)\lambda(u)^{-1}\lambda(v)^{-1} : u \in [S+m, S+m+1],\ v \in [S+n, S+n+1]\}.$$

The argument is then completed by showing that the matrix a_{mn} defines a bounded operator in l^2. Indeed, we will show that $a_{mn} \leq C/(1+|m-n|^2)$, then applying Schur's lemma (Proposition 1.2) with $\mu(m) = \nu(n) = 1$.

First of all, since $e(x)$ has bounded support, we have, using (5.3),

$$|A(u,v)| \leq \Big(\int e(x)|k(x,u)\lambda(u)^{-1}|^2\,dx\Big)^{1/2}\Big(\int e(x)|k(x,v)\lambda(v)^{-1}|^2\,dx\Big)^{1/2} \leq C\lambda(u)\lambda(v)$$

for all $u, v \in [S,\infty)$.

On the other hand, integrating by parts twice and using (5.2) we get

$$\Big|\int e(x)k(x,u)\overline{k(x,v)}\,dx\Big| = \Big|\int e(x)\varphi(x,u)\overline{\varphi(x,v)}(v-u)^{-2}\partial_x^2 e^{ix(u-v)}\,dx\Big| \leq C|u-v|^{-2}\lambda(u)\lambda(v).$$

These two inequalities combined together give

$$A(u,v) \leq \frac{C}{1+|u-v|^2}\lambda(u)\lambda(v)$$

and

$$a_{mn} \leq \frac{C}{1+|m-n|^2},$$

which completes the argument.

We separated Propositions 5.1 and 5.2 to make more transparent the role of the conditions imposed on the kernel k. Together, these two propositions give rise to the following

THEOREM 5.3. *For a kernel k satisfying conditions* (5.1)–(5.3) *we have the following description of its extended domain*:

$$\widetilde{D}_K = l^2(L^1(\mathbb{R}_+,\lambda)),$$

where λ is the weight appearing in (5.2).

Remarks. Condition (5.3) could be stated in a more general form: it is sufficient to assume that $\mathbb{R}_+ \ni x \to \varphi(x,\cdot) \in L^0$ be bounded.

The argument above gives a very inaccurate idea as to the range of the operator $\widetilde{K}$; it would also be of interest to determine the images of $\widetilde{K}$ restricted to spaces L^p and other subspaces of $\widetilde{D}_K$.

We next consider some concrete cases.

When $\varphi = 1$ we get the well known case of the Fourier transform [FS], $\widetilde{D}_F = l^2(L^1)$.

When $\varphi(x,y) = |x-y|^d$ with $\mathrm{Re}(d) > -1$ or $\varphi(x,y) = (1+|x-y|)^d$ then, it is easy to check that $\psi(x,y) = \mathrm{Im}(d)\ln|x-y|$ satisfies (5.1) and we conclude that $\widetilde{D}_K$ is as in Theorem 5.3 with $\lambda(y) = (1+y)^{\mathrm{Re}(d)}$.

In the next example we make use of the change of variables observations (vii) and (viii).

Let $k(x,y) = \exp(ix^a y^b)|x-y|^d$ where d is a complex number with $\mathrm{Re}(d) > -1$.

Without changing $\widetilde{D}_K$ we may, using (vii), replace the kernel k by $k(x,y) = \exp(ixy^b)|x-y|^d$ and conclude changing the variable $\widetilde{y} = y^b$ and using (viii) that $f \in \widetilde{D}_K$ if and only if

$$\sum_{l=0}^{\infty}\Big[\int_{l^{1/b}}^{(l+1)^{1/b}}\lambda(y)|f(y)|\,dy\Big]^2 < \infty,$$

where $\lambda(y) = 1$ for $y \in [0,1]$ and $\lambda(y) = y^{\mathrm{Re}(d)}$ for $y \in (1,\infty)$.

The above condition describes the compressed l^2-L^1 amalgam space with weight λ, the "compression" of the amalgam being determined by $y^{1/b}$.

A similar result holds for kernels of the form $\varphi(x,y)\exp[i\alpha(x)\beta(y)]$, where α and β are increasing functions and φ satisfies suitable growth and differentiability conditions. In this case the "compression" of the amalgam is determined by the function β^{-1}.

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Received July 24, 1995
Revised version July 24, 1996 (3505)

STUDIA MATHEMATICA 122 (3) (1997)

On some vector balancing problems

by

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Abstract. Let V be an origin-symmetric convex body in $\mathbb{R}^n$, $n \ge 2$, of Gaussian measure $\gamma_n(V) \ge 1/2$. It is proved that for every choice $u_1, \ldots, u_n$ of vectors in the Euclidean unit ball B_n, there exist signs $\varepsilon_j \in \{-1, 1\}$ with $\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n \in (c \log n)V$. The method used can be modified to give simple proofs of several related results of J. Spencer and E. D. Gluskin.

1. Introduction. Let $\mathcal{C}_n$ denote the class of all origin-symmetric convex bodies in $\mathbb{R}^n$, $n \ge 2$. Following W. Banaszczyk [2], for each pair $U, V \in \mathcal{C}_n$ we define $\beta(U, V)$ as the smallest $r > 0$ satisfying the following condition: given $u_1, \ldots, u_n \in U$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that $\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n \in rV$.

Several "vector balancing" results, proved by various authors for quite different purposes, can be described as estimates on $\beta(U, V)$ for specific choices of U, V, or both of them:

(a) W. Banaszczyk [2] established a general lower bound for $\beta(U, V)$ in terms of the volumes of U, V: for some absolute constant $c > 0$, and for any $U, V \in \mathcal{C}_n$, one has

$$\beta(U, V) \ge c\sqrt{n}(|U|/|V|)^{1/n}.$$

(b) I. Bárány and V. S. Grinberg [5] show that $\beta(U, U) \le 2n$ for every $U \in \mathcal{C}_n$.

(c) The vector form of a well-known result of J. Beck and T. Fiala [6] states that $\beta(B_1^n, Q_n) \le 2$, where B_1^n is the unit ball of ℓ_1^n and Q_n is the unit cube in $\mathbb{R}^n$.

(d) J. Spencer [11] and E. D. Gluskin [7] have proved independently that $\beta(Q_n, Q_n) \le c\sqrt{n}$, where $c > 0$ is an absolute constant.

(e) We write B_n for the Euclidean unit ball in $\mathbb{R}^n$. Suppose that $\mathcal{E} \in \mathcal{C}_n$ is an ellipsoid with principal semiaxes $a_1, \ldots, a_n$. W. Banaszczyk [3] proves that $\beta(B_n, \mathcal{E}) = (a_1^{-2} + \ldots + a_n^{-2})^{1/2}$.

1991 *Mathematics Subject Classification*: Primary 52C07.