STUDIA MATHEMATICA 122 (2) (1997)

On log-subharmonicity of singular values of matrices

by

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Abstract. Let F be an analytic function from an open subset Ω of the complex plane into the algebra of $n\times n$ matrices. Denoting by $s_1,\ldots,s_n$ the decreasing sequence of singular values of a matrix, we prove that the functions $\log s_1(F(\lambda))+\ldots+\log s_k(F(\lambda))$ and $\log^+ s_1(F(\lambda))+\ldots+\log^+ s_k(F(\lambda))$ are subharmonic on Ω for $1\le k\le n$.

0. Introduction. If f denotes a meromorphic function on the complex plane, the characteristic function $T(r,f)$ plays a fundamental role in Rolf Nevanlinna's theory to prove the First Fundamental Theorem.

In his attempts to adapt the classical Nevanlinna theory to the case of $n\times n$ matrices, O. Nevanlinna [4, 5] introduced the characteristic functions for matrices, that is, the functions which are denoted by ψ_k $(1\le k\le n)$ in Theorem 2. In order to apply the Maximum Principle, it is a natural question to ask if these functions are subharmonic. The aim of this paper is to answer this question of O. Nevanlinna in the affirmative (Theorem 2). The paper also contains a slightly different version (Theorem 1) which is the main ingredient in the proof of Theorem 2.

1. The results. Given an $n\times n$ matrix M we denote by $\lambda_1(M),\ldots$ $\ldots,\lambda_n(M)$ its n eigenvalues, each one counted with its multiplicity, and ordered in such a way that

$$|\lambda_1(M)|\ge|\lambda_2(M)|\ge\ldots\ge|\lambda_n(M)|.$$

We recall that the n *singular values* of M are

$$s_1(M)=\lambda_1((M^*M)^{1/2})=\|M\|\ge s_2(M)=\lambda_2((M^*M)^{1/2})\ge\ldots$$
$$\ge s_n(M)=\lambda_n((M^*M)^{1/2}).$$

They form a decreasing sequence of positive numbers. These singular values are invariant under multiplication of M by a unitary matrix. Moreover, by

1991 *Mathematics Subject Classification*: 15A18, 31A05.

H. Weyl's famous inequalities (see [3] for instance) we have $\prod_{i=1}^k |\lambda_i(M)| \le \prod_{i=1}^k s_i(M)$ for every k such that $1 \le k \le n$.

Let F be an analytic function from an open subset Ω of the complex plane into the algebra of $n \times n$ matrices. Because $|\lambda_1(M)|$ is the spectral radius of M, $z \to \log|\lambda_1(F(z))|$ is subharmonic on Ω (see [1], Theorem 3.4.7). Are the corresponding statements for $\lambda_2, \ldots, \lambda_n$ also true? Taking

$$F(z) = \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix} \in M_2(\mathbb{C}),$$

we have $|\lambda_1(F(z))| = \max(|1+z|, |1-z|)$ and $|\lambda_2(F(z))| = \min(|1+z|, |1-z|)$, so $\log|\lambda_2(F(z))|$ is not subharmonic on the plane because $|\lambda_2(F(z))|$ is not, as it violates the mean inequality on a circle of centre 0 and radius $\varepsilon > 0$ small enough (see [2], §2, for calculations). Nevertheless as a consequence of a slightly more general theorem of [2], we can conclude that $z \to \log|\lambda_1(F(z))| + \ldots + \log|\lambda_k(F(z))|$ is subharmonic on Ω for every k such that $1 \le k \le n$. In the Lemma of §2 we shall give a proof of this result in the case of matrices. But what can be said about the analytic properties of the functions $z \to \log s_i(F(z))$ for $i = 1, \ldots, n$? Because $s_1(M) = \|M\|$ it is clear that $z \to \log s_1(F(z))$ is subharmonic on Ω (see [1], Lemma 3.4.6). Unfortunately, the functions $z \to \log s_i(F(z))$ $(i = 2, \ldots, n)$ are not subharmonic in general. To see this take

$$F(z) = \begin{pmatrix} 1 & 1 \\ 0 & z \end{pmatrix} \in M_2(\mathbb{C}).$$

Simple calculations show that $s_1(F(z)) = \max(1, |z|)$ and $s_2(F(z)) = \min(1, |z|)$. This last function is not subharmonic because it violates the mean inequality on the unit circle; consequently, its logarithm is not subharmonic either.

The aim of this short note is to prove the following two results.

THEOREM 1. *Let F be an analytic function from an open subset Ω of the complex plane into $M_n(\mathbb{C})$. For $1 \le k \le n$ the functions $\varphi_k(z) = \sum_{i=1}^k \log s_i(F(z))$ and $e^{\varphi_k(z)}$ are subharmonic on Ω.*

We recall that $\log^+ r = \max(0, \log r)$.

THEOREM 2. *Let F be an analytic function from an open subset Ω of the complex plane into $M_n(\mathbb{C})$. For $1 \le k \le n$ the functions $\psi_k(z) = \sum_{i=1}^k \log^+ s_i(F(z))$ and $e^{\psi_k(z)}$ are subharmonic on Ω.*

2. The proofs

LEMMA. *Let F be an analytic function from an open subset Ω of the complex plane into $M_n(\mathbb{C})$. For $1 \le k \le n$ the functions $\sigma_k(z) = \sum_{i=1}^k \log|\lambda_i(F(z))|$ are subharmonic on Ω.*

Proof. First step. Let $z_0 \in \Omega$ be fixed, $\alpha_0 \in \operatorname{Sp} F(z_0)$ and $s > 0$ such that the disk $B(\alpha_0, s)$ isolates α_0 from the rest of the spectrum. There exists $r > 0$ such that $|z - z_0| < r$ implies $z \in \Omega$ and $\operatorname{Sp} F(z) \cap \partial B(\alpha_0, s) = \emptyset$. From the Scarcity Theorem ([1], Theorem 3.4.25) there exists an integer $m(\alpha_0)$ such that $\operatorname{Sp} F(z) \cap B(\alpha_0, s)$ has exactly $m(\alpha_0)$ points for $0 < |z - z_0| < r$. It is easy to see that this integer $m(\alpha_0)$ is in fact the multiplicity of $F(\lambda_0)$ at α_0. This comes from the fact that $m(\alpha_0) \le \operatorname{Mult}(F(\lambda_0), \alpha_0)$ and $n = \sum_{\alpha \in \operatorname{Sp} F(z_0)} m(\alpha) \le \sum_{\alpha \in \operatorname{Sp} F(z_0)} \operatorname{Mult}(F(\lambda_0), \alpha) = n$. If we denote by $\alpha_1(z), \ldots, \alpha_m(z)$ the elements of $\operatorname{Sp} F(z) \cap B(\alpha_0, s)$, for $0 < |z - z_0| < r$, then the function h defined by $h(z_0) = \alpha_0^m$, $h(z) = \alpha_1(z) \ldots \alpha_m(z)$ for $0 < |z - z_0| < r$, is continuous on $B(z_0, r)$ and holomorphic on $B(z_0, r) \backslash \{z_0\}$ so, by Radó's Extension Theorem, h is holomorphic on all $B(z_0, r)$. Consequently, we have

$$m \log|\alpha_0| \le \frac{1}{2\pi\varrho} \int_{|z-z_0|=\varrho} (\log|\alpha_1(z)| + \ldots + \log|\alpha_m(z)|)\,|dz| \tag{1}$$

for $0 < \varrho < r$. Moreover, by the continuity of the spectrum on $M_n(\mathbb{C})$ it is easy to conclude that the functions σ_k are continuous on Ω.

Second step. We fix $z_0 \in \Omega$ and suppose that k is the sum of the multiplicities of the first p spectral values of $F(z_0)$. Applying the formula (1) p times and the fact that $|\lambda_1(F(z))| \ldots |\lambda_k(F(z))|$ is greater than any product of k moduli of elements of $\operatorname{Sp} F(z)$, we conclude that

$$\sigma_k(F(z_0)) \le \frac{1}{2\pi\varrho} \int_{|z-z_0|=\varrho} \sigma_k(F(z))\,|dz| \tag{2}$$

for $0 < \varrho < r$. This means that σ_k is locally subharmonic at z_0.

Third step. We fix $z_0 \in \Omega$ and suppose that l is the greatest sum of the first multiplicities in the spectrum of $F(z_0)$ which is less than k. Suppose that $l < k < l + m$ where m is the multiplicity of $\lambda_k(F(z_0))$ (the cases $l = k$ or $k = l + m$ have been studied in the second step). By formula (1) we have

$$\begin{aligned} m \log|\lambda_k(F(z_0))| \\ \le \frac{1}{2\pi\varrho} \int_{|z-z_0|=\varrho} (\log|\lambda_k^1(F(z))| + \ldots + \log|\lambda_k^m(F(z))|)\,|dz| \end{aligned} \tag{3}$$

for $0 < \varrho < r$, where $\lambda_k^1(F(z)), \ldots, \lambda_k^m(F(z))$ denote the m points of $\operatorname{Sp} F(z) \cap B(\alpha_0, s)$ with $\alpha_0 = \lambda_k(F(z_0))$. Denote by $\alpha_{l+1}(z) \ge \ldots \ge \alpha_k(z)$ the $k - l$ greatest moduli among $\lambda_k^1(F(z)), \ldots, \lambda_k^m(F(z))$. The $m - k + l$ remaining moduli are less than or equal to $\alpha_k(z) \le (\alpha_{l+1}(z) \ldots \alpha_k(z))^{1/k-l}$. Consequently, by (3), we obtain

$$(4)\quad m\log|\lambda_k(F(z_0))| \le \frac{1}{2\pi\varrho}\int\limits_{|z-z_0|=\varrho}\left(1+\frac{m-k+l}{k-l}\right)(\log\alpha_{l+1}(z)+\ldots+\log\alpha_n(z))\,|dz|.$$

Hence

$$(5)\quad (k-l)\log|\lambda_k(F(z_0))| \le \frac{1}{2\pi\varrho}\int\limits_{|z-z_0|=\varrho}(\log\alpha_{l+1}(z)+\ldots+\log\alpha_n(z))\,|dz|.$$

Applying (1) we conclude that

$$(6)\qquad \sigma_l(F(z_0)) \le \frac{1}{2\pi\varrho}\int\limits_{|z-z_0|=\varrho}\Big(\sum_{j=1}^{l}\log|\nu_j(F(z))|\Big)\,|dz|,$$

where the $\nu_j(F(z))$ are the l elements of $\operatorname{Sp} F(z)$ in the union of the $B(\lambda_i(F(z_0)), s)$, $i=1,\ldots,l$. Taking the union of the l elements $\nu_j(F(z))$ with the $k-l$ elements of $\operatorname{Sp} F(z)\cap B(\lambda_k(F(z_0)), s)$ which have for moduli $\alpha_{l+1}(z),\ldots,\alpha_k(z)$, it is possible to reorder this set of k elements in such a way that their moduli are respectively less than or equal to $|\lambda_1(F(z))|,\ldots,|\lambda_k(F(z))|$. Adding (5) and (6) we conclude that $\sigma_k(F(z))$ satisfies locally the mean inequality (2) at z_0.

The function σ_k being continuous and locally subharmonic at each point of Ω, it is subharmonic on Ω. ∎

Proof of Theorem 1. By H. Weyl's inequalities and the invariance of the s numbers under multiplication by a unitary matrix, it is clear that

$$\max_U \prod_{i=1}^{k}|\lambda_i(UM)| \le \prod_{i=1}^{k}s_i(M),$$

where the maximum is taken over all unitary matrices. By the polar decomposition of M there exists U_0 unitary such that $M=(M^*M)^{1/2}U_0$ and obviously $\lambda_i(U_0^{-1}M)=s_i(M)$. So we have

$$(7)\qquad \max_U \prod_{i=1}^{k}|\lambda_i(UM)| = \prod_{i=1}^{k}s_i(M).$$

By the Lemma, the functions $\varphi_k^U(z)=\sum_{i=1}^{k}\log|\lambda_i(UF(z))|$ are subharmonic for every U unitary. Consequently, we have

$$(8)\qquad \varphi_k^U(z_0) \le \frac{1}{2\pi}\int_0^{2\pi}\varphi_k^U(z_0+re^{i\theta})\,d\theta$$

for $\overline{B}(z_0,r)\subset\Omega$. Taking $\max\varphi_k^U$ for all U unitary the same inequality (8) is true. Hence, from (7), $\varphi_k(z)$ satisfies (8) and, as it is continuous, it is subharmonic. The last part of the theorem comes from the fact that the exponential of a subharmonic function is subharmonic. ∎

Proof of Theorem 2. It is obvious that $\varphi_k(z)\le\psi_k(z)$. We prove the theorem by induction on k. For $k=1$, $\psi_1=\varphi_1^+$, so it is subharmonic. Suppose the result is true until k and we prove it for $k+1$. Let $\Omega'=\{z\in\Omega : s_{k+1}(F(z))<1\}$, which is open. If $z_0\in\Omega'$ then $\psi_{k+1}(z)=\psi_k(z)$ in a neighbourhood of z_0, so ψ_{k+1} is locally subharmonic on Ω', and consequently ψ_{k+1} is subharmonic on Ω'. If $z_0\in\overline{\Omega'}\setminus\Omega'$ then by continuity, and taking $\overline{B}(z_0,r)\subset\Omega$, we have, by the previous theorem,

$$\psi_{k+1}(z_0)=\varphi_{k+1}(z_0)\le\frac{1}{2\pi}\int_0^{2\pi}\varphi_{k+1}(z_0+re^{i\theta})\,d\theta \le\frac{1}{2\pi}\int_0^{2\pi}\psi_{k+1}(z_0+re^{i\theta})\,d\theta.$$

Consequently, ψ_{k+1} is locally subharmonic at z_0. Now if $z_0\in\Omega\setminus\overline{\Omega'}$ there are two cases. Either $s_1(F(z_0))\le1$, in which case $\psi_{k+1}(F(z_0))=0$, so the mean inequality is true on a small circle centred at z_0 because $\psi(z)\ge0$, or $s_1(F(z_0))>1$, in which case there exists a larger r $(1\le r\le k+1)$ such that $s_r(F(z_0))>1$. If $r=k+1$, then in a neighbourhood of z_0 we have $\psi_{k+1}(z)=\varphi_{k+1}(z)$, so, by the previous theorem, ψ_{k+1} is locally subharmonic at z_0. If $r\le k$ then $\psi_{k+1}(z_0)=\psi_k(z_0)+\log^+ s_{k+1}(F(z_0))=\psi_k(z_0)$. Consequently, by subharmonicity of ψ_k we have

$$\psi_{k+1}(z_0)=\psi_k(z_0)\le\frac{1}{2\pi}\int_0^{2\pi}\psi_k(z_0+re^{i\theta})\,d\theta \le\frac{1}{2\pi}\int_0^{2\pi}\psi_{k+1}(z_0+re^{i\theta})\,d\theta.$$

The function ψ_{k+1}, being locally subharmonic, is subharmonic on all Ω. ∎

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Received June 28, 1996 (3700)