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On the maximal operator associated with the free Schrödinger equation

by

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Abstract. For $d > 1$, let $(S_d f)(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^d} \widehat{f}(\xi)\,d\xi$, $x \in \mathbb{R}^n$, where $\widehat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$, and $(S_d^* f)(x) = \sup_{0<t<1} |(S_d f)(x,t)|$ its maximal operator. P. Sjölin ([11]) has shown that for radial f, the estimate

$$(*)\qquad \Big(\int_{|x|<R} |(S_d^* f)(x)|^p\,dx\Big)^{1/p} \le C_R \|f\|_{H_{1/4}}$$

holds for $p = 4n/(2n-1)$ and fails for $p > 4n/(2n-1)$. In this paper we show that for non-radial f, $(*)$ fails for $p > 2$. A similar result is proved for a more general maximal operator.

1. Introduction. Consider the integral operator

$$(1.1)\qquad (S_d f)(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^d} \widehat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n,\ t > 0,$$

$d > 1$, where $\widehat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx.$$

In the case $d = 2$, $u(x,t) = (2\pi)^{-n}(S_2 f)(x,t)$ is the formal solution of the free Schrödinger equation $\Delta u = i\,\frac{\partial u}{\partial t}$, $u(x,0) = f(x)$. In order to obtain optimal function spaces for which solutions exist a.e. on the boundary $t = 0$, one is led to the study of regularity of the corresponding (local) maximal operator $(S_2^* f)(x) = \sup_{0<t<1} |(S_2 f)(x,t)|$, specifically one requires estimates of the form

$$(1.2)\qquad \Big\{\int_{|x|<R} |(S_d^* f)(x)|^p\,dx\Big\}^{1/p} \le C_R \|f\|_{H_s},$$

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where H_s denote the Sobolev spaces defined by

$$H_s = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H_s} = \Big\{ \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \Big\}^{1/2} < \infty \Big\}.$$

Here $\mathcal{S}'(\mathbb{R}^n)$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ ([14]).

Considerable literature is devoted to the study of the estimate (1.2) (cf. [1]–[7], [10]–[12], [15] and the literature cited there). In fact, if $s < 1/4$, $p \ge 1$, then (1.2) fails ([8], [15]), while in the case $n = 1$, the more general estimate

$$\Big\{ \int_{\mathbb{R}} |(S_d^* f)(x)|^4 \, dx \Big\}^{1/4} \le C \|f\|_{H_{1/4}} \tag{1.3}$$

holds ([6]). Moreover, (1.3) is sharp in the sense that the L^4-norm on the left cannot be replaced by an L^p-norm for $p \ne 4$ ([13]). For $n \ge 2$, only partial results regarding the estimate (1.2) are available (cf. [4], [8], [10], [11], [13], [15]). However, if f is radial, $n \ge 2$, $s = 1/4$, then P. Sjölin ([11]) proved that (1.2) is satisfied with $p = 4n/(2n-1)$ and if $p > 4n/(2n-1)$, $s = 1/4$, (1.2) fails. This result together with (1.3) might suggest that for $p = 4n/(2n-1)$, $s = 1/4$, the local estimate (1.2) is satisfied for non-radial f on $\mathbb{R}^n$, $n \ge 3$. That this is not the case follows from the following result:

THEOREM 1.1. *If $n \ge 3$, $p > 2$, $\alpha > 0$, then*

$$\sup_{f \in \mathcal{S}(\mathbb{R}^n)} \Big[\Big\{ \int_{|x|<R} |(S_d^* f)(x)|^p |x|^\alpha \, dx \Big\}^{1/p} / \|f\|_{H_{1/4}} \Big] = \infty \tag{1.4}$$

for $d > 1$.

Since $\int_{|x|<R} |(S_d^* f)(x)|^p \, dx \ge \frac{1}{R^\alpha} \big(\int_{|x|<R} |(S_d^* f)(x)|^p |x|^\alpha \, dx \big)$ it is clear from (1.4) that (1.2) cannot hold for $p > 2$ and $s = 1/4$.

As alluded to in the abstract, we shall prove a corresponding result for a more general maximal operator. Suppose Q is a real polynomial such that $\deg Q \ge 1$ and the leading coefficient of Q is positive. Let $\Omega(\xi) = Q(|\xi|)$, $\xi \in \mathbb{R}^n$, $n \ge 3$ and

$$(S_\Omega^* f)(x) = \sup_{0<t<1} \Big| \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it\Omega(\xi)} \widehat{f}(\xi) \, d\xi \Big|.$$

THEOREM 1.2. *If $n \ge 3$, $p > 2$, $\alpha > 0$, then*

$$\sup_{f \in \mathcal{S}(\mathbb{R}^n)} \Big[\Big\{ \int_{|x|<R} |(S_\Omega^* f)(x)|^p |x|^\alpha \, dx \Big\}^{1/p} / \|f\|_{H_{1/4}} \Big] = \infty. \tag{1.5}$$

As in the case of Theorem 1.1, it follows that for $f \in \mathcal{S}(\mathbb{R}^n)$, $n \ge 3$, the estimate

$$\Big\{ \int_{|x|<R} |(S_\Omega^* f)(x)|^p \, dx \Big\}^{1/p} \le C_R \|f\|_{H_{1/4}} \tag{1.6}$$

fails for $p > 2$ and $R > 0$.

The proofs of Theorems 1.1 and 1.2 follow from a series of lemmas involving properties and estimates of spherical harmonics. These results, some of which are of independent interest, are given in the next section.

We conclude the introduction by listing some definitions and notations.

(i) A homogeneous harmonic polynomial in $\mathbb{R}^n$ is a polynomial which is homogeneous and satisfies the Laplace equation in $\mathbb{R}^n$. The restriction of a homogeneous harmonic polynomial of order k to the unit sphere S^{n-1} in $\mathbb{R}^n$ is called a spherical harmonic of order k on S^{n-1} (cf. [14, Ch. 4]).

(ii) The Bessel function J_m of order $m > -1/2$ is defined in integral form by

$$J_m(r) = \frac{1}{\Gamma(1/2)\Gamma(m+1/2)} (r/2)^m \int_{-1}^{1} e^{irs} (1-s^2)^{(2m-1)/2} \, ds,$$

where $r \ge 0$.

(iii) The well-known Stirling formula

$$\lim_{r\to\infty} [\Gamma(r+1)/[(r/e)^r(\sqrt{2\pi r})]] = 1$$

is required in the sequel, where the Γ-function is defined by

$$\Gamma(r) = \int_0^\infty e^{-x} x^{r-1} \, dx, \quad r > 0.$$

(iv) The Legendre polynomials $P_n(z)$ of order n, $n = 0, 1, 2, \ldots$, are given by

$$P_n(z) = \frac{1}{2^n n!} \Big\{ \frac{d^n}{dz^n} (z^2-1)^n \Big\}.$$

The notations for C^∞, C_0^∞ and the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ are standard. Also, $L^p(X, d\mu)$ are the Lebesgue spaces with norm $\|\cdot\|_{L^p(X,d\mu)} = \|\cdot\|_p$. $[a,b]^n$ are n copies of $[a,b]$. $\doteq$ means "defined by" and C are constants (sometimes with subscripts) which may be different at different places.

2. Proofs. For the proof of Theorem 1.1, a number of technical lemmas are required. The first lemma is an immediate consequence of (2.19) and Theorem 3.10 of [14, Ch. 4].

LEMMA 2.1. *If Y is a spherical harmonic of order k, then*

$$\int_{S^{n-1}} e^{-is(x'\cdot\xi')}\, Y(\xi')\, d\sigma(\xi') = (2\pi)^{n/2} i^{-k} s^{(2-n)/2} J_{(n+2k-2)/2}(s)\, Y(x'). \tag{2.1}$$

Here $s>0$, $x',\xi'\in S^{n-1}$ and $d\sigma$ denotes the surface measure on S^{n-1}.

LEMMA 2.2. *If $H_n^{(k)}$, $n\ge 3$, is the vector space of spherical harmonics Y of order k on S^{n-1} and $p>2$, then*

$$\sup_{Y\in H_n^{(k)}} \|Y\|_p/\|Y\|_2 \ge C\, k^{(1/2-1/p)/2}, \tag{2.2}$$

for some $C>0$ independent of k. The norms are the usual L^p-norms over S^{n-1}.

Proof. We prove (2.2) for $n\ge 4$ first. This is done by constructing a particular spherical harmonic $Y_n^{(k)}\in H_n^{(k)}$ satisfying

$$\|Y\|_p/\|Y\|_2 \ge C k^{(1/2-1/p)/2}. \tag{2.3}$$

The functions $Y_{m,k}(\theta,\varphi) = e^{im\varphi}(\sin\theta)^{|m|} P_k^{(|m|)}(\cos\theta)$, $m=0,\pm1,\pm2,\ldots,\pm k$, are spherical harmonics of order k on S^2 and form a basis for $H_3^{(k)}$, where $P_k(z)$ is the Legendre polynomial of order k and $P_k^{(l)}(z) = \frac{d^l}{dz^l}(P_k(z))$ (cf. [16, Theorem (6.7)]). Since $P_k^{(k)}(\cos\theta) = C_k$, a constant, it follows that $Y_{k,k}(\theta,\varphi) = C_k e^{ik\varphi}(\sin\theta)^k$. Let

$$Q_k(x_1,x_2,x_3) = (x_1^2+x_2^2+x_3^2)^{k/2} Y_{k,k}(\theta,\varphi) \tag{2.4}$$

be the homogeneous harmonic polynomial of order k associated with $Y_{k,k}$. Then we define a homogeneous harmonic polynomial of order k in $\mathbb{R}^n$ by $\widetilde{Q}_k(x_1,x_2,\ldots,x_n) = Q_k(x_1,x_2,x_3)$. The polar coordinates in $\mathbb{R}^n$ are given by

$$x_1 = r\cos\theta_1,$$
$$x_j = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{j-1}\cos\theta_j,\qquad 2\le j\le n-1,$$
$$x_n = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1},$$

where $0\le\theta_j\le\pi$, $1\le j\le n-2$, $0\le\theta_{n-1}\le 2\pi$, $r>0$. Restricting $\widetilde{Q}_k$ to S^{n-1}, we obtain a spherical harmonic $Y_n^{(k)} \doteq Y_n^{(k)}(\theta_1,\theta_2,\ldots,\theta_{n-1})$ of order k.

We claim that

$$|Y_n^{(k)}| = C_k(\sin\theta_1)^k[1-(\sin\theta_2)^2(\sin\theta_3)^2]^{k/2}. \tag{2.5}$$

Since $r^2 = x_1^2+\ldots+x_n^2 = 1$ and in $\mathbb{R}^3$, $x_1 = R\cos\theta$, $x_2 = R\sin\theta\cos\varphi$, $x_3 = R\sin\theta\sin\varphi$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$, $R\ge 0$, we have

$$\begin{aligned} R^2 &= x_1^2+x_2^2+x_3^2\\ &= (\cos\theta_1)^2 + (\sin\theta_1)^2(\cos\theta_2)^2 + [(\sin\theta_1)(\sin\theta_2)(\cos\theta_3)]^2\\ &= 1-[(\sin\theta_1)(\sin\theta_2)(\sin\theta_3)]^2 \end{aligned}$$

and $\cos\theta = x_1/R = (\cos\theta_1)[1-(\sin\theta_1)^2(\sin\theta_2)^2(\sin\theta_3)^2]^{-1/2}$. Moreover,

$$\begin{aligned} \sin\theta &= [1-(\cos\theta)^2]^{1/2} = [1-(\cos\theta_1/R)^2]^{1/2} = [R^2-(\cos\theta_1)^2]^{1/2}/R\\ &= (1/R)[1-(\sin\theta_1)^2(\sin\theta_2)^2(\sin\theta_3)^2-(\cos\theta_1)^2]^{1/2}\\ &= (1/R)(\sin\theta_1)[1-(\sin\theta_2)^2(\sin\theta_3)^2]^{1/2} \end{aligned}$$

and therefore from (2.4),

$$|Y_n^{(k)}| = R^k C_k(\sin\theta)^k = C_k(\sin\theta_1)^k[1-(\sin\theta_2)^2(\sin\theta_3)^2]^{k/2},$$

which proves (2.5). The spherical harmonic so constructed will be shown to satisfy (2.3).

In fact, on S^{n-1},

$$\begin{aligned} \|Y_n^{(k)}\|_p &= \Big\{\int_{S^{n-1}} |Y_n^{(k)}(\theta_1,\theta_2,\ldots,\theta_{n-1})|^p\, d\sigma\Big\}^{1/p}\\ &= B_n^{1/p} C_k\Big[\int_0^\pi (\sin\theta_1)^{pk+n-2}\, d\theta_1\Big]^{1/p}\\ &\quad\times\Big[\int_0^\pi\int_0^\pi [1-(\sin\theta_2)^2(\sin\theta_3)^2]^{pk/2}\, d\mu(\theta_2,\theta_3)\Big]^{1/p} \end{aligned}$$

where

$$B_n \doteq \int_0^{2\pi}\Big(\int_{[0,\pi]^{n-4}} (\sin\theta_{n-2})(\sin\theta_{n-3})^2\ldots(\sin\theta_4)^{n-5}\, d\theta_4\ldots d\theta_{n-2}\Big)\, d\theta_{n-1}$$

and

$$d\mu(\theta_2,\theta_3) = (\sin\theta_2)^{n-3}(\sin\theta_3)^{n-4}\, d\theta_2 d\theta_3.$$

By Hölder's inequality with index $p/2$, $p>2$,

$$\begin{aligned} &\Big\{\int_0^\pi\int_0^\pi [1-(\sin\theta_2)^2(\sin\theta_3)^2]^k\, d\mu(\theta_2,\theta_3)\Big\}^{1/2}\\ &\quad\le\Big\{\int_0^\pi\int_0^\pi [1-(\sin\theta_2)^2(\sin\theta_3)^2]^{pk/2}\, d\mu(\theta_2,\theta_3)\Big\}^{1/p}\Big\{\int_0^\pi\int_0^\pi d\mu(\theta_2,\theta_3)\Big\}^{1/2-1/p}, \end{aligned}$$

so that

$$\frac{\|Y_n^{(k)}\|_p}{\|Y_n^{(k)}\|_2} \ge C_{n,p} \frac{\left[\int_0^\pi (\sin\theta_1)^{pk+n-2}\,d\theta_1\right]^{1/p}}{\left[\int_0^\pi (\sin\theta_1)^{2k+n-2}\,d\theta_1\right]^{1/2}} \tag{2.6}$$

where $C_{n,p} = \{B_n \int_0^\pi \int_0^\pi d\mu(\theta_2,\theta_3)\}^{1/p-1/2}$. Since

$$\int_0^\pi (\sin\theta)^\alpha\,d\theta = \sqrt{\pi}\,\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)},$$

$\alpha > -1$ (cf. [9, p. 194]), it follows that the right side of (2.6) is equal to

$$C_{n,p}\pi^{(1/p-1/2)/2}\left[\frac{\Gamma((pk+n-1)/2)}{\Gamma((pk+n-1)/2+1/2)}\right]^{1/p} \times \left[\frac{\Gamma((2k+n-1)/2)}{\Gamma((2k+n-1)/2+1/2)}\right]^{-1/2}. \tag{2.7}$$

But since

$$\lim_{x\to\infty} \frac{\Gamma(x+1/2)}{\Gamma(x)x^{1/2}} = 1 \quad \text{(cf. [9, p. 203])}$$

there exists a constant $C > 0$ such that

$$\frac{\Gamma((pk+n-1)/2)}{\Gamma((pk+n-1)/2+1/2)} \ge Ck^{-1/2} \quad (k \ge 1) \tag{2.8}$$

and

$$\frac{\Gamma((2k+n-1)/2)}{\Gamma((2k+n-1)/2+1/2)} \ge Ck^{-1/2} \quad (k \ge 1). \tag{2.9}$$

Substituting (2.8), (2.9) into (2.7), we see that the right side of (2.6) is greater than or equal to

$$C_{n,p}[C\sqrt{\pi}]^{1/p-1/2}(k^{-1/2})^{1/p}(k^{-1/2})^{-1/2} = C_{n,p}(C\sqrt{\pi})^{1/p-1/2}k^{(1/2-1/p)/2},$$

which proves (2.3).

If $n = 3$, the spherical harmonics are simply $Y_{k,k}(\theta,\varphi) = C_k\, e^{ik\varphi}(\sin\theta)^k$ and (2.5) becomes $C_k(\sin\theta)^k$. The rest of the argument is then almost the same as in the case $n \ge 4$. We omit the details.

LEMMA 2.3. *If $m \ge 3/2$ and $r > 1$, then there exist constants C_m, $\overline{C}_m$ and a bounded function ψ_m such that*

$$J_m(r) = (C_m e^{-ir} + \overline{C}_m e^{ir})r^{-1/2} + \psi_m(r)r^{-3/2}, \tag{2.10}$$

where

$$|C_m| = |\overline{C}_m| = (2\pi)^{-1/2} \quad \textit{and} \quad |\psi_m(r)| \le C(2m)^{m+1}.$$

Proof. The proof is essentially that of Lemma 3.11 in [14, Ch. 4], except that the bound of ψ_m was not given explicitly there. We give the argument

here for completeness only. Since

$$J_m(r) = \frac{1}{\Gamma(1/2)\Gamma((2m+1)/2)}(r/2)^m I,$$

where $I = \int_{-1}^1 e^{irs}(1-s^2)^{(2m-1)/2}\,ds$, as in the proof of Lemma 3.11 of [14, Ch. 4] we obtain $I = I_1 - I_2$, where

$$I_1 = \int_0^\infty e^{ir(iy-1)}(y^2+2iy)^{m-1/2}\,dy,$$

and

$$I_2 = \int_0^\infty e^{ir(iy+1)}(y^2-2iy)^{m-1/2}\,dy.$$

Now

$$(y^2 \pm 2iy)^{m-1/2} = \begin{cases} y^{m-1/2}(\pm 2i)^{m-1/2} + A_m^\pm(y)\,y^{m+1/2}, & 0 \le y \le 1,\\ y^{m-1/2}(\pm 2i)^{m-1/2} + B_m^\pm(y)\,y^{2m-1}, & 1 < y < \infty,\end{cases}$$

where

$$\begin{cases} |A_m^\pm(y)| \le (m+1)3^m & (0 \le y \le 1),\\ |B_m^\pm(y)| \le (m+1)3^m & (1 \le y < \infty).\end{cases} \tag{2.11}$$

In fact,

$$\begin{aligned} |A_m^\pm(y)| &= \left|\frac{(y^2 \pm 2iy)^{m-1/2} - y^{m-1/2}(\pm 2i)^{m-1/2}}{y^{m+1/2}}\right| \\ &= \left|\frac{(y \pm 2i)^{m-1/2} - (\pm 2i)^{m-1/2}}{y}\right| \\ &\le \sup_{0<y<1} |(m-1/2)(y \pm 2i)^{m-3/2}| \le (m+1)3^m \end{aligned}$$

and

$$\begin{aligned} |B_m^\pm(y)| &= \left|\frac{(y^2 \pm 2iy)^{m-1/2} - y^{m-1/2}(\pm 2i)^{m-1/2}}{y^{2m-1}}\right| \\ &= \left|\frac{(y \pm 2i)^{m-1/2} - (\pm 2i)^{m-1/2}}{y^{m-1/2}}\right| \\ &\le (1 + 4/y^2)^{(m-1/2)/2} + 2^{m-1/2} \le (m+1)3^m. \end{aligned}$$

Using (2.11) in I_1 and I_2, the argument is then identical to [14, Lemma 3.11, Ch. 4]. The result follows.

The construction of the sequence of "bump functions" in the following lemma is similar to that of [15, Thm. 4].

LEMMA 2.4. *There exists a sequence* $\{\Phi_m\}_{m=1}^{\infty}$, $\Phi_m \in C_0^{\infty}(\mathbb{R})$, *such that*

$$\begin{cases} 0 \le \Phi_m \le 1, \quad \operatorname{supp} \Phi_m = [2^m, 2^m + 2^{m/2}], \\ \Phi_m \equiv 1 \quad \text{on } [a_m, b_m] \subset [2^m, 2^m + 2^{m/2}], \end{cases} \tag{2.12}$$

where

$$a_m = (2^m + 2^{m/2-1})[1 - \delta_m(1-\delta_m)],$$
$$b_m = (2^m + 2^{m/2-1})[1 + \delta_m(1-\delta_m)],$$
$$\delta_m = (1 + 2^{m/2+1})^{-1},$$

and

$$\int_{-\infty}^{\infty} |\Phi_m'(r)|\, dr = 2. \tag{2.13}$$

Proof. If

$$\varphi(r) = \begin{cases} e^{-1/r}, & r > 0, \\ 0, & r \le 0, \end{cases}$$

then $\varphi \in C^{\infty}(\mathbb{R})$ and $\varphi'(r) \ge 0$. Next, for $a < b$, define $\varphi_{a,b}$ and $\psi_{a,b}$ by

$$\varphi_{a,b}(r) = (\varphi(r-a))[\varphi(r-a) + \varphi(b-r)]^{-1},$$
$$\psi_{a,b}(r) = (\varphi(b-r))[\varphi(r-a) + \varphi(b-r)]^{-1}.$$

Then $\varphi_{a,b}$, $\psi_{a,b} \in C^{\infty}(\mathbb{R})$ and direct computations show that $\varphi'_{a,b}(r) \ge 0$, $\psi'_{a,b}(r) \le 0$ and

$$\varphi_{a,b}(r) = 1 - \psi_{a,b}(r) = \begin{cases} 0 & \text{if } r \le a, \\ 1 & \text{if } r \ge b. \end{cases}$$

Now for $m \in \mathbb{N}$, define

$$\Phi_m(r) = \begin{cases} \varphi_{2^m, a_m}(r) & \text{if } -\infty < r \le a_m, \\ 1 & \text{if } a_m \le r \le b_m, \\ \psi_{b_m, 2^m + 2^{m/2}}(r) & \text{if } b_m \le r < \infty. \end{cases}$$

Then (2.12) holds and since

$$\int_{-\infty}^{\infty} |\Phi_m'(r)|\, dr = \int_{2^m}^{2^m+2^{m/2}} |\Phi_m'(r)|\, dr = \int_{2^m}^{a_m} \Phi_m'(r)\, dr - \int_{b_m}^{2^m+2^{m/2}} \Phi_m'(r)\, dr = 1 + 1 = 2,$$

so does (2.13).

We are now in a position to prove the main results.

Proof of Theorem 1.1. Let $\widehat{f}(\xi) = \Phi_m(|\xi|)Y^{(k)}(\xi')$, where $Y^{(k)}(\xi')$ is the spherical harmonic of order k constructed from $\widetilde{Q}_k$ in Lemma 2.2 and $\{\Phi_m\}_{m=1}^{\infty}$ is the sequence of $C_0^{\infty}(\mathbb{R})$ functions constructed in Lemma 2.4. Let $R > 0$ be fixed and $r_0 = \min\{R, (\pi/2)(d-1)^{-1}\}$, $d > 1$. Let $r_0/2 < |x| < r_0$, and choose the integer m so large that

$$t(x) \doteq (|x|/d)(2^m + 2^{m/2-1})^{1-d} \in (0,1).$$

Since for $\widehat{f}(\xi) = \Phi_m(|\xi|)Y^{(k)}(\xi')$,

$$(S_d f)(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^d}\, \Phi_m(|\xi|)Y^{(k)}(\xi')\, d\xi = \int_0^{\infty} r^{n-1}\Phi_m(r)e^{itr^d} \int_{S^{n-1}} e^{ir|x|(x'\cdot\xi')}Y^{(k)}(\xi')\, d\sigma(\xi')\, dr,$$

where $x = |x|x'$, $\xi = r\xi'$, (2.1) of Lemma 2.1 shows that

$$(S_d f)(x,t) = (2\pi)^{n/2} i^{-k} |x|^{(2-n)/2} Y^{(k)}(-x') I(|x|, t)$$

where

$$I(|x|,t) \doteq \int_0^{\infty} r^{n/2} e^{itr^d} \Phi_m(r) J_{(n+2k-2)/2}(r|x|)\, dr.$$

So in particular with $t = t(x)$,

$$(S_d^* f)(x) \ge (2\pi)^{n/2} r_0^{(2-n)/2} |Y^{(k)}(x')| \cdot |I(|x|, t(x))|.$$

By Lemma 2.3, there exist constants C_k, $\overline{C}_k$ and a bounded function F_k such that for $t > 1$,

$$\begin{cases} |C_k| = |\overline{C}_k| = (2\pi)^{-1/2} \quad \text{and} \quad |F_k(t)| \le C(n+2k)^{n+2k}, \\ J_{(n+2k-2)/2}(t) = t^{-1/2}[C_k\, e^{-it} + \overline{C}_k\, e^{it}] + F_k(t)t^{-3/2}. \end{cases} \tag{2.14}$$

Hence

$$\begin{aligned} I(|x|, t(x)) &= \int_0^{\infty} r^{n/2} e^{it(x)r^d} \Phi_m(r) J_{(n+2k-2)/2}(r|x|)\, dr \\ &= C_k |x|^{-1/2} \int_0^{\infty} r^{(n-1)/2} e^{i(t(x)r^d - r|x|)} \Phi_m(r)\, dr \\ &\quad + \overline{C}_k |x|^{-1/2} \int_0^{\infty} r^{(n-1)/2} e^{i(t(x)r^d + r|x|)} \Phi_m(r)\, dr \\ &\quad + |x|^{-3/2} \int_0^{\infty} r^{(n-3)/2} e^{it(x)r^d} \Phi_m(r)\, F_k(r|x|)\, dr \\ &\doteq C_k|x|^{-1/2} I_1 + \overline{C}_k |x|^{-1/2} I_2 + |x|^{-3/2} I_3, \end{aligned}$$

respectively. Since $\frac{1}{2}r_0 < |x| < r_0$, the triangle inequality shows that

$$|I(|x|, t(x))| \ge (2\pi r_0)^{-1/2}|I_1| - (\pi r_0)^{-1/2}|I_2| - (2/r_0)^{3/2}|I_3|.$$

We now prove that

$$\text{(2.15)}\qquad \begin{cases} |I_1| \geq C_1 2^{mn/2}, \quad |I_2| \leq C_2 2^{m(n-1)/2}, \\ |I_3| \leq C_3 (n+2k)^{(n+2k)} 2^{m(n-2)/2}, \end{cases}$$

where $C_j > 0$, $j = 1, 2, 3$, are independent of m, k and x.

In I_1, let $r = (2^m + 2^{m/2-1})s$. Then

$$|I_1| \geq 2^{m(n+1)/2} \Big| \int_0^\infty s^{(n-1)/2} \Phi_m((2^m + 2^{m/2-1})s) e^{i\tau_m(s)}\, ds \Big|$$

where $\tau_m(s) = |x|(2^m + 2^{m/2-1})[s^d/d - s]$, and if $g_m(s) = \Phi_m((2^m + 2^{m/2-1})s)$, then by (2.12),

$$\text{(2.16)}\qquad \begin{cases} 0 \leq g_m(s) \leq 1, \text{ and with } \delta_m = (1+2^{m/2+1})^{-1}, \\ \operatorname{supp} g_m = [1-\delta_m, 1+\delta_m], \\ g_m(s) \equiv 1 \quad \text{if } 1-\delta_m(1-\delta_m) \leq s \leq 1+\delta_m(1-\delta_m). \end{cases}$$

Since $\tau_m'(1) = 0$, Taylor's formula shows that

$$\begin{aligned} &|\tau_m(s) - \tau_m(1)| \\ &\quad = \big|\tfrac{1}{2}(s-1)^2 \tau_m''(1+\theta(s-1))\big| \\ &\quad = \big|\tfrac{1}{2}|x|(2^m + 2^{m/2-1})(s-1)^2(d-1)[1+\theta(s-1)]^{d-2}\big| \quad (0 < \theta < 1). \end{aligned}$$

If m is large, then $1-\delta_m \leq s \leq 1+\delta_m$ and since $r_0/2 < |x| < r_0$ where $r_0 \leq (\pi/2)(d-1)^{-1}$, we have

$$\begin{aligned} |\tau_m(s) - \tau_m(1)| &\leq |x|(d-1)(2^m + 2^{m/2-1})\delta_m^2 \\ &\leq (\pi/2)(2^m + 2^{m/2-1})\delta_m^2 \leq \pi/4. \end{aligned}$$

Hence

$$\begin{aligned} |I_1| &\geq 2^{m(n+1)/2} \Big| \int_{1-\delta_m}^{1+\delta_m} s^{(n-1)/2} g_m(s) e^{i(\tau_m(s)-\tau_m(1))}\, ds \Big| \\ &= 2^{m(n+1)/2} \Big| \int_{1-\delta_m(1-\delta_m)}^{1+\delta_m(1-\delta_m)} + \int_{1-\delta_m}^{1-\delta_m(1-\delta_m)} + \int_{1+\delta_m(1-\delta_m)}^{1+\delta_m} \Big| \\ &\geq 2^{m(n+1)/2} \Big[\Big| \int_{1-\delta_m(1-\delta_m)}^{1+\delta_m(1-\delta_m)} \Big| - 3\delta_m^2 \Big] \\ &\geq 2^{m(n+1)/2} \Big[\Big| \int_{1-\delta_m(1-\delta_m)}^{1+\delta_m(1-\delta_m)} s^{(n-1)/2} \cos(\tau_m(s) - \tau_m(1))\, ds \Big| - 3\delta_m^2 \Big] \end{aligned}$$

$$\begin{aligned} &\geq 2^{m(n+1)/2} \big[(1/\sqrt{2})[1-\delta_m(1-\delta_m)]^{(n-1)/2}\, 2\delta_m(1-\delta_m) - 3\delta_m^2 \big] \\ &\geq C 2^{m(n+1)/2} \cdot 2^{-m/2} \geq C 2^{mn/2}, \end{aligned}$$

which proves the first estimate of (2.15).

In I_2, let $h_m(r) = \Phi_m(r) r^{(n-1)/2} [dt(x) r^{d-1} + |x|]^{-1}$. Then

$$\begin{aligned} |I_2| &= \Big| \int_0^\infty r^{(n-1)/2} e^{i(t(x)r^d + r|x|)}\, \Phi_m(r)\, dr \Big| \\ &= \Big| \int_0^\infty \frac{d}{dr}(e^{i(t(x)r^d + r|x|)}) h_m(r)\, dr \Big| \\ &= \Big| \int_0^\infty e^{i(t(x)r^d + r|x|)} h_m'(r)\, dr \Big| \leq \int_0^\infty |h_m'(r)|\, dr \end{aligned}$$

where we integrated by parts. But

$$\begin{aligned} h_m'(r) = {} & \big[\Phi_m'(r) r^{(n-1)/2} + \tfrac{1}{2}(n-1) r^{(n-3)/2} \Phi_m(r)\big][t(x) d r^{d-1} + |x|]^{-1} \\ & + \Phi_m(r) r^{(n-1)/2} d(1-d) t(x) r^{d-2} [t(x) d r^{d-1} + |x|]^{-2}, \end{aligned}$$

so

$$\begin{aligned} |h_m'(r)| \leq {} & (1/|x|)|\Phi_m'(r)| r^{(n-1)/2} + \tfrac{1}{2}(n-1)(1/|x|)\Phi_m(r) r^{(n-3)/2} \\ & + (1/|x|^2)\Phi_m(r) t(x) d(d-1) r^{(n-1)/2+d-2} \\ \leq {} & C\big[|\Phi_m'(r)| r^{(n-1)/2} + \Phi_m(r) r^{(n-3)/2} + 2^{m(1-d)} \Phi_m(r) r^{d-2+(n-1)/2}\big] \end{aligned}$$

since $r_0/2 < |x| < r_0$.

Recalling the support of Φ_m, we obtain

$$\begin{aligned} |I_2| \leq {} & C\Big\{ \Big(\int_{2^m}^{2^m+2^{m/2}} |\Phi_m'(r)|\, dr \Big) (2^m + 2^{m/2})^{(n-1)/2} \\ & + \int_{2^m}^{2^m+2^{m/2}} [r^{(m-3)/2} + 2^{m(1-d)} r^{(n-1)/2+d-2}]\, dr \Big\} \\ \leq {} & C\{2 \cdot 2^{(m+1)(n-1)/2} + 2 \cdot 2^{m(n-2)/2}\} \leq C 2^{m(n-1)/2}. \end{aligned}$$

Note that we used (2.13).

Finally, by (2.14),

$$\begin{aligned} |I_3| &\leq C \int_0^\infty r^{(n-3)/2} \Phi_m(r) |F_k(r|x|)|\, dr \leq C(n+2k)^{n+2k} \int_{2^m}^{2^m+2^{m/2}} r^{(n-3)/2}\, dr \\ &\leq C(n+2k)^{n+2k} 2^{m(n-2)/2}, \end{aligned}$$

which proves the estimates of (2.15).

Now let $m = (n+2k)^{n+2k}$ in (2.15). Then

$$(S_d^*f)(x) \ge C|Y^{(k)}(x')|\{(2\pi r_0)^{-1/2}|I_1| - (\pi r_0)^{-1/2}|I_2| - (2/r_0)^{3/2}|I_3|\}$$
$$\ge C|Y^{(k)}(x')|2^{(n/2)(n+2k)^{n+2k}}.$$

Hence for $p > 2$ and $\alpha > 0$,

$$(2.17)\quad \Big\{\int\limits_{|x|<R} |(S_d^*f)(x)|^p|x|^\alpha\,dx\Big\}^{1/p} \ge C\Big\{\int\limits_{r_0/2<|x|<r_0} |(S_d^*f)(x)|^p\,dx\Big\}^{1/p}$$
$$\ge C2^{(n/2)(n+2k)^{n+2k}}\|Y^{(k)}\|_p.$$

On the other hand,

$$\|f\|_{H_s} = \Big\{\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi\Big\}^{1/2}$$
$$= \Big\{\int\limits_0^\infty r^{n-1}(1+r^2)^s|\Phi_m(r)|^2\,dr\Big\}^{1/2}\|Y^{(k)}\|_2$$
$$\le C\Big\{\int\limits_{2^m}^{2^m+2^{m/2}} r^{n+2s-1}\,dr\Big\}^{1/2}\|Y^{(k)}\|_2 \le C2^{m(n/2+s-1/4)}\|Y^{(k)}\|_2$$
$$\le C2^{(n/2+s-1/4)(n+2k)^{n+2k}}\|Y^{(k)}\|_2,$$

and combining this with (2.17), we obtain

$$\Big\{\int\limits_{|x|<R}|(S_d^*f)(x)|^p|x|^\alpha\,dx\Big\}^{1/p}\Big/\|f\|_{H_s}$$
$$\ge C2^{(1/4-s)(n+2k)^{n+2k}}\{\|Y^{(k)}\|_p/\|Y^{(k)}\|_2\}$$
$$\ge Ck^{(1/2-1/p)/2} \to \infty \quad (k\to\infty)$$

if $s = 1/4$ and $p > 2$, where we applied Lemma 2.2. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let Q be the polynomial of Theorem 1.2 and $\Omega(\xi) = Q(|\xi|)$. If

$$(S_\Omega f)(x,t) = \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it\Omega(\xi)}\widehat{f}(\xi)\,d\xi$$

and $\widehat{f}(\xi) = \Phi_m(|\xi|)Y^{(k)}(\xi')$, where Φ_m is the bump function of Lemma 2.4 and $Y^{(k)}$ is the spherical harmonic $Y_n^{(k)}$ constructed from $\widetilde{Q}_k$ in Lemma 2.2, we obtain

$$(S_\Omega f)(x,t)$$
$$= \int\limits_0^\infty r^{n-1}\Phi_m(r)e^{itQ(r)}\int\limits_{S^{n-1}} e^{ir|x|(x'\cdot\xi')}Y^{(k)}(\xi')\,d\sigma(\xi')\,dr$$
$$= (2\pi)^{n/2}i^{-k}Y^{(k)}(-x')|x|^{(2-n)/2}\int\limits_0^\infty r^{n/2}e^{itQ(r)}\Phi_m(r)J_{(n+2k-2)/2}(r|x|)\,dr.$$

If $r_0/2 < |x| < r_0 < R$, where r_0 is so chosen that for $r_m = 2^m + 2^{m/2-1}$ and $\delta_m = (1+2^{m/2+1})^{-1}$ (cf. Lemma 2.4),

$$(2.18)\quad r_0r_m\delta_m^2 \sup_{r\in[2^m,2^m+2^{m/2}]}\{|r_mQ''(r)/Q'(r_m)|\} < \pi/2.$$

Let m be so large that $t(x) \doteq |x|/Q'(r_m) \in (0,1)$. Then

$$(S_\Omega^*f)(x) = \sup_{0<t<1}|(S_\Omega f)(x,t)|$$
$$\ge C|Y^{(k)}(x')|\Big|\int\limits_0^\infty r^{n/2}e^{it(x)Q(r)}\Phi_m(r)J_{(n+2k-2)/2}(r|x|)\,dr\Big|.$$

Using the asymptotic formula for the Bessel function (2.14) it follows as in the proof of Theorem 1.1 that

$$(S_\Omega^*f)(x) \ge C|Y^{(k)}(x')|\Big\{(2\pi r_0)^{-1/2}\Big|\int\limits_0^\infty r^{(n-1)/2}e^{i(t(x)Q(r)-r|x|)}\Phi_m(r)\,dr\Big|$$
$$-(\pi r_0)^{-1/2}\Big|\int\limits_0^\infty r^{(n-1)/2}e^{i(t(x)Q(r)+r|x|)}\Phi_m(r)\,dr\Big|$$
$$-(2/r_0)^{3/2}\int\limits_0^\infty r^{(n-3)/2}\,\Phi_m(r)|F_k(r|x|)|\,dr\Big\}$$
$$\doteq C|Y^{(k)}(\xi')|\{(2\pi r_0)^{-1/2}B_1 - (\pi r_0)^{-1/2}B_2 - (2/r_0)^{3/2}B_3\}$$

respectively. As in the proof of Theorem 1.1, one shows that

$$(2.19)\quad B_3 \le C(n+2k)^{n+2k}2^{m(n-2)/2}.$$

Next, we prove that

$$(2.20)\quad B_1 \ge C2^{mn/2}.$$

On making the change of variable $r = r_ms$ in the integral, we obtain

$$(2.21)\quad B_1 \ge 2^{m(n+1)/2}\Big|\int\limits_0^\infty s^{(n-1)/2}\Phi_m(r_ms)e^{i|x|[Q(r_ms)/Q'(r_m)-r_ms]}\,ds\Big|.$$

Let $g_m(s) = \Phi_m(r_ms)$ and $\sigma_m(s) = |x|[Q(r_ms)/Q'(r_m) - r_ms]$. Then g_m satisfies (2.16) and by Taylor's expansion about $r_0 = 1$ and $1-\delta_m \le s \le 1+\delta_m$,

$$|\sigma_m(s)-\sigma_m(1)| = \left(\frac{|x|r_m^2}{2}\right)\left|\frac{Q''(r_m s_0)}{Q'(r_m)}\right|(s-1)^2, \quad s_0\in(1-\delta_m,1+\delta_m),$$
$$\le (|x|/2)\, r_m^2\, \delta_m^2 \sup_{s\in[1-\delta_m,1+\delta_m]} \{|Q''(r_m s)/Q'(r_m)|\} \le \pi/4$$

where the last estimate follows from (2.18). Hence by (2.21) and (2.16),

$$B_1 \ge 2^{m(n+1)/2}\Big\{\Big|\int_{1-\delta_m}^{1+\delta_m} s^{(n-1)/2} g_m(s)\, e^{i[\sigma_m(s)-\sigma_m(1)]}\, ds\Big|\Big\}$$
$$\ge 2^{m(n+1)/2}\Big\{\Big|\int_{1-\delta_m(1-\delta_m)}^{1+\delta_m(1-\delta_m)} s^{(n-1)/2} g_m(s)\, e^{i[\sigma_m(s)-\sigma_m(1)]}\, ds\Big| - 3\delta_m^2\Big\}$$
$$\ge 2^{m(n+1)/2}\Big\{\Big|\int_{1-\delta_m(1-\delta_m)}^{1+\delta_m(1-\delta_m)} s^{(n-1)/2} \cos[\sigma_m(s)-\sigma_m(1)]\, ds\Big| - 3\delta_m^2\Big\}$$
$$\ge C 2^{mn/2}$$

since $|\sigma_m(s)-\sigma_m(1)|\le\pi/4$. This proves (2.20).

Finally, we show that

$$B_2 \le C2^{m(n-1)/2}. \tag{2.22}$$

Write $f_m(r) = |x|^{-1}[1+Q'(r)/Q'(r_m)]^{-1}\, \Phi_m(r) r^{(n-1)/2}$. Then integration by parts yields

$$B_2 = \Big|\int_0^\infty r^{(n-1)/2} e^{i(t(x)Q(r)+r|x|)}\Phi_m(r)\, dr\Big| = \Big|\int_0^\infty e^{i(t(x)Q(r)+r|x|)} f_m'(r)\, dr\Big|.$$

But a straightforward calculation shows that

$$|f_m'(r)| \le C\{|\Phi_m'(r)|r^{(n-1)/2} + \Phi_m(r) r^{(n-3)/2} + \Phi_m(r) r^{(n-1)/2} r_m^{-1}\}$$

where we used the fact that $|Q''(r)/Q'(r_m)| \le C r_m^{-1}$ if $r\in[2^m, 2^m+2^{m/2}]$. Hence

$$|B_2| \le \int_0^\infty |f_m'(r)|\, dr$$
$$\le C\Big\{\Big(\int_{2^m}^{2^m+2^{m/2}} |\Phi_m'(r)|\, dr\Big)(2^m+2^{m/2})^{(n-1)/2}$$
$$+ \int_{2^m}^{2^m+2^{m/2}} \Phi_m(r)[r^{(n-3)/2} + r^{(n-1)/2} r_m^{-1}]\, dr\Big\}$$
$$\le C2^{m(n-1)/2},$$

which proves (2.22).

Choosing $m=(n+2k)^{n+2k}$ and using the estimates (2.19), (2.20) and (2.22), we get

$$(S_\Omega^* f)(x) \ge C|Y^{(k)}(x')|\, 2^{(n/2)(n+2k)^{n+2k}}$$

since r_0 was defined by (2.18) and $r_0/2<|x|<r_0$. Hence for $p>2$, $\alpha>0$, we have as in (2.17),

$$\Big\{\int_{|x|<R} |(S_\Omega^* f)(x)|^p |x|^\alpha\, dx\Big\}^{1/p} \ge C\Big\{\int_{r_0/2<|x|<r_0} |S_\Omega^* f(x)|^p\, dx\Big\}^{1/p}$$
$$\ge C\|Y^{(k)}\|_p 2^{(n/2)(n+2k)^{n+2k}}.$$

Now we proceed exactly as in the proof of Theorem 1.1 and the result follows.

Remark. In [13], P. Sjölin informed me that he has proved Theorem 1.1 in the case $d=2$, and $\alpha=0$. This special case was also proved in [17].

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The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$

by

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Abstract. The aim of this paper is to prove the statement announced in the title which can be reformulated in the following way. Let H be a separable infinite-dimensional Hilbert space and let $\Phi : B(H) \to B(H)$ be a continuous linear mapping with the property that for every $A \in B(H)$ there exists a sequence (Φ_n) of automorphisms of $B(H)$ (depending on A) such that $\Phi(A) = \lim_n \Phi_n(A)$. Then Φ is an automorphism. Moreover, a similar statement holds for the set of all surjective isometries of $B(H)$.

Introduction. If X is a Banach space, then we denote by $L(X)$ and $B(X)$ the algebras of all linear and bounded linear operators on X, respectively. $F(X)$ and $C(X)$ stand for the ideals of $B(X)$ consisting of all finite-rank and compact operators, respectively. A subset $\mathcal{E} \subset B(X)$ is called *topologically* [*algebraically*] *reflexive* if $T \in B(X)$ belongs to $\mathcal{E}$ whenever $Tx \in \overline{\mathcal{E}x}$ [$Tx \in \mathcal{E}x$] for all $x \in X$. This concept has proved very useful in the analysis of operator algebras.

The study of algebraic reflexivity of the subspace of derivations on operator algebras has been begun by Kadison [Kad2] and Larson and Sourour [LS] from a different point of view. Since then the problem of algebraic reflexivity of the sets of derivations and automorphisms has been investigated in full detail and the preliminary results have been improved significantly [Bre, BS1, BS2].

The notion of topological reflexivity is due to Loginov and Shul'man [LoS], although they defined it only for the case of subspaces. Nevertheless, surprisingly enough, from the two fundamental concepts of derivations and automorphisms, the problem of topological reflexivity has so far been

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