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Hardy spaces of conjugate temperatures

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Abstract. We define Hardy spaces of pairs of conjugate temperatures on $\mathbb{R}^2_+$ using the equations introduced by Kochneff and Sagher. As in the holomorphic case, the Hilbert transform relates both components. We demonstrate that the boundary distributions of our Hardy spaces of conjugate temperatures coincide with the boundary distributions of Hardy spaces of holomorphic functions.

1. Introduction. The Hardy spaces of holomorphic functions $\mathbf{H}^p_{\mathrm{hol}}(\mathbb{R}^2_+)$ were originally defined by Stein and Weiss as spaces of harmonic functions u such that the vector $F(x,t)=u(x,t)+iv(x,t)$, where v is the conjugate of u, is uniformly in $L^p(\mathbb{R})$, that is, $\int_{-\infty}^{\infty}|F(x,t)|^p\,dx\le C<\infty$, independently of $t>0$. The Theorem of Burkholder, Gundy and Silverstein characterized $\mathbf{H}^p_{\mathrm{hol}}(\mathbb{R}^2_+)$ as a space of harmonic functions without appealing to any notion of conjugacy: a tempered distribution f is in $\operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}(\mathbb{R}^2_+)$ if and only if it is the boundary distribution corresponding to a real harmonic function $u(x,t)$ in $\mathbb{R}^2_+$ such that its maximal function $u^*(x)=\sup_{|y-x|<t}|u(y,t)|$ belongs to L^p. Fefferman and Stein showed that the following properties are equivalent for a tempered distribution f in $\mathbb{R}$:

(a) $f=\lim_{t\to0}u(\cdot,t)$ in $\mathcal{S}'$ for some $u\in\mathbf{H}^p_{\mathrm{hol}}(\mathbb{R}^2_+)$.

(b) $\sup_{t>0}|f*\varphi_t(x)|$ is in L^p for some $\varphi\in\mathcal{S}$ such that $\int_{-\infty}^{\infty}\varphi=1$, where $\varphi_t(x)=t^{-1}\varphi(x/t)$.

(c) $\sup_{|y-x|<t}|f*\varphi_t(y)|$ is in L^p for some φ as above.

(d) $P^*_\nabla(f)(x)=\sup_{|y-x|<t}|u(y,t)|$ is in L^p, where $u(y,t)=P_t*f(y)$ and P_t is the Poisson kernel.

This theorem shows that $\mathbf{H}^p_{\mathrm{hol}}(\mathbb{R}^2_+)$ may be defined without having recourse to the Poisson kernel, and in fact, independently of $\varphi\in\mathcal{S}$. In this work, we characterize Hardy spaces in terms of temperature functions using the notion of conjugacy introduced by Kochneff and Sagher. In particular, we study spaces of temperature functions u such that $F(x,t)=u(x,t)+iv(x,t)$

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is uniformly in L^p, where v is the conjugate of u. We show that the boundary distributions in this case are the same as in the holomorphic case.

2. Hardy spaces of temperature functions. We write

$$H(\mathbb{R}^2_+) = \left\{u \in C^2(\mathbb{R}^2_+) : \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\right\},$$
$$H^p(\mathbb{R}^2_+) = \{u \in H(\mathbb{R}^2_+) : \|u\|_{H^p} < \infty\},$$

where

$$\|u\|_{H^p} = \sup_{t>0} \|u(\cdot,t)\|_p, \quad 0 < p \le \infty.$$

If the context does not cause confusion we shall write H and H^p, respectively. We shall call the elements of H *temperature functions*. $K(x,t)$ will denote the Gauss–Weierstrass kernel.

For $1 \le p \le \infty$, we have the following representations for $u \in H^p$:

(a) $u \in H^p$ if and only if $u(x,t) = K(\cdot,t) * f(x)$, where $f \in L^p$, $1 < p \le \infty$.

(b) $u \in H^1$ if and only if $u(x,t) = K(\cdot,t) * \mu(x)$, where $\mu \in M(\mathbb{R})$, the space of Borel measures on $\mathbb{R}$.

(See [4, Th. 5] for $1 < p \le \infty$ and use [6, Ch. VIII, Th. 10.2] and the Banach–Alaoglu Theorem for $p = 1$.)

Now, we analyze the growth of functions in H^p. If $u \in H^p$ for $1 < p < \infty$, Flett's estimate in [4, Th. 2(iv)] gives for $t > 0$ and $x \in \mathbb{R}$,

$$|u(x,t)| \le C\|f\|_p t^{-1/(2p)}, \tag{1}$$

where $u(x,t) = K(\cdot,t) * f(x)$, $f \in L^p$. Also, an application of Flett's result in [4, Th. 2(ii) and (v)] gives for $t > 0$ and $x \in \mathbb{R}$,

$$|D_t u(x,t)| \le C\|f\|_p t^{-1-1/(2p)}. \tag{2}$$

An immediate consequence of estimates (1) and (2) [4, Th. 2(iv)] is that u and $D_t u$ are bounded in each proper half-plane $\{(x,t) \in \mathbb{R}^2_+ : t \ge t_0 > 0\}$.

It remains to analyze the case $0 < p \le 1$. We will require Lemma 1 below, whose proof follows the main ideas of the harmonic case [5, p. 172] adapted to our situation. Before we state it, we recall some known facts that will be used in this part.

Let $Q = (0,1) \times (0,1)$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (the parabolic boundary of Q), where $\Gamma_1 = \{0\} \times [0,1)$, $\Gamma_2 = \{1\} \times [0,1)$, $\Gamma_3 = (0,1) \times \{0\}$, and let λ be the one-dimensional Lebesgue measure on Γ.

For $t > 0$, let

$$\theta(x,t) = \sum_{n=-\infty}^{\infty} K(x + 2n, t), \quad \varphi(x,t) = -2D_x\theta(x,t),$$

and for $t \le 0$, let $K = \theta = \varphi = 0$.

We consider the kernels K_i on Γ_i, $i = 1,2,3$, given by

$$\begin{aligned} K_1(x,t;0,\tau) &= \varphi(x, t-\tau), && 0 \le \tau < 1,\\ K_2(x,t;1,\tau) &= \varphi(1-x, t-\tau), && 0 \le \tau < 1,\\ K_3(x,t;\xi,0) &= \theta(x-\xi,t) - \theta(x+\xi,t), && 0 < \xi < 1, \end{aligned}$$

and then consider the heat kernel $K(x,t;\xi,\tau)$ on $Q \times \Gamma$ defined as the union of K_1, K_2, K_3. It is well known that every temperature function on Q which is continuous on $\overline{Q}$ can be written on $(0,1) \times (0,1)$ as

$$u(x,t) = \int_\Gamma K(x,t;\xi,\tau) u(\xi,\tau)\, d\lambda(\xi,\tau).$$

We shall also consider for every $0 < r < 1$ the mapping $T_r(x,t) = (x_r, t_r)$, where

$$x_r = rx + (1-r)/2, \quad t_r = r^2 t + (1 - r^2).$$

Notice that $Q = \{T_r(\xi,\tau) : (\xi,\tau) \in \Gamma,\ 0 < r < 1\}$.

LEMMA 1. *Let u be a temperature function on a rectangle $R = (a,b) \times (c,d) \subset \mathbb{R}^2_+$ where $d - c = (b-a)^2$, and let u be continuous on $\overline{R}$. If (x_0,t_0) is the middle point of the upper boundary of R, then for every p with $0 < p \le 1$,*

$$|u(x_0,t_0)|^p \le C_p \frac{1}{|R|} \iint_R |u(x,t)|^p\, dx\, dt,$$

where $|R|$ is the area of R and C_p is a constant depending only on p.

Proof. As the mapping $\Psi : Q \to (a,b) \times (c,d)$,

$$(\xi,\tau) \mapsto ((b-a)\xi + a, (d-c)\tau + c),$$

makes $u \circ \Psi$ a temperature function since $d - c = (b-a)^2$, there is no loss of generality in supposing that $R = Q$ and $\iint_Q |u(x,t)|^p\, dx\, dt = |Q| = 1$. We notice that $u \circ T_r$ is a temperature function on Q for $0 < r < 1$, because $T_r(x,t) = (r(x - 1/2) + 1/2, r^2(t-1) + 1)$. For $r > 0$ let

$$m_p(r) = \Big[\int_\Gamma |u(T_r(\xi,\tau))|^p\, d\lambda(\xi,\tau)\Big]^{1/p},$$

and

$$m_\infty(r) = \sup\{|u(T_r(\xi,\tau))| : (\xi,\tau) \in \Gamma\}.$$

We may also assume that $m_\infty(r) \ge 1$ for every $r \in (0,1)$ since otherwise the maximum principle for temperature functions on a rectangle would immediately give us the required inequality.

First, we notice that

$$m_1(r) \le m_\infty(r)^{1-p} m_p(r)^p. \tag{3}$$

We have the representation

$$u(T_r(x,t)) = \int_\Gamma K(x,t;\xi,\tau)u(T_r(\xi,\tau))\,d\lambda(\xi,\tau).$$

For $(x,t) \in Q^{(r)} = [(1-r)/2,(1+r)/2] \times [(1-r^2),1]$ we have

$$u(x,t) = \int_\Gamma K(T_r^{-1}(x,t);\xi,\tau)u(T_r(\xi,\tau))\,d\lambda(\xi,\tau),$$

and since $T_r \circ T_s = T_{rs}$, it follows that for $(x,t) \in Q^{(s)}$, $0 < s < r < 1$,

$$u(x,t) = \int_\Gamma K(T_{s/r}(T_s^{-1}(x,t));\xi,\tau)u(T_r(\xi,\tau))\,d\lambda(\xi,\tau). \tag{4}$$

It is enough to analyze the behavior of $K(T_{s/r}(x,t);\xi,\tau)$ for $(x,t) \in Q$ because $T_s^{-1}(x,t) \in Q$.

Now, we get an estimate of K_3. Since (see [2])

$$\theta(x-y,t) - \theta(x+y,t) = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin(n\pi x)\sin(n\pi y),$$

we obtain for $0 < \varrho < 1$,

$$|K_3(x_\varrho,t_\varrho;\xi,0)| \le 2\sum_{n=1}^{\infty}\frac{1}{[\varrho^2 t + (1-\varrho^2)]n^2\pi^2} \le \frac{1}{3}(1-\varrho)^{-2}.$$

To get an estimate of K_1, we first notice that

$$\varphi(x,t) = -2D_x K(x,t) + U(x,t),$$

where $U(x,t) = \sum_{n\neq 0} -2D_x K(x+2n,t)$, which is smooth and bounded. Since

$$-2D_x K(x,t) \le \begin{cases} (x/t)K(x,t) & \text{if } t > 0,\\ 0 & \text{if } t \le 0,\end{cases}$$

and $(1/t^\beta)e^{-x^2/(4t)} \le C/|x|^{2\beta}$ for $t > 0$, $\beta > 0$, $x \in \mathbb{R}$, we have for $0 < \varrho < 1$,

$$|K_1(x_\varrho,t_\varrho;0,\tau)| \le C\frac{\varrho x + (1-\varrho)/2}{(\varrho x + (1-\varrho)/2)^3} + C \le C(1-\varrho)^{-2}.$$

It remains to analyze K_2. As before, for $0 < \varrho < 1$,

$$|K_2(x_\varrho,t_\varrho;1,\tau)| \le C(1-\varrho)^{-2}.$$

Collecting all the estimates above, we obtain from (4), for $\varrho = s/r$ and $(x,t) \in Q^{(s)}$,

$$|u(x,t)| \le M(1-s/r)^{-2}\int_\Gamma |u(T_r(\xi,\tau))|\,d\lambda(\xi,\tau), \tag{5}$$

where M is a positive constant. Inequality (5) implies

$$m_\infty(s) \le M(1-s/r)^{-2}m_1(r). \tag{6}$$

Choosing $s = r^a$ with $a > 1$ we obtain from (3) and (6),

$$m_\infty(r^a) \le M(1-r^{a-1})^{-2}m_\infty(r)^{1-p}m_p(r)^p.$$

Taking logarithms, multiplying by $1/r$ and then integrating with respect to r, we get

$$\int_{1/2}^{1}\log m_\infty(r^a)\,dr/r \le C_a + (1-p)\int_{1/2}^{1}\log m_\infty(r)\,dr/r + \int_{1/2}^{1}\log m_p(r)^p\,dr/r, \tag{7}$$

where C_a is the real constant $\int_{1/2}^1 \log M\,dr/r + 2\int_{1/2}^1 \log(1-r^{a-1})^{-1}\,dr/r$.

Now, we notice that

$$1 = \iint_Q |u(x,t)|^p\,dx\,dt = \sum_{i=1}^{3}\iint_{Q_i}|u(x,t)|^p\,dx\,dt, \tag{8}$$

where

$$Q_1 = \{(x,t) : 0 \le x \le 1/2,\ (1-4(x-1/2)^2) \le t \le 1\},$$
$$Q_2 = \{(x,t) : 1/2 \le x \le 1,\ (1-4(x-1/2)^2) \le t \le 1\},$$
$$Q_3 = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le (1-4(x-1/2)^2)\}.$$

If we consider the mapping $T : [0,1]\times[0,1] \to Q_1$,

$$T(\xi,\tau) = ((1-\xi)/2, \xi^2\tau + (1-\xi^2)),$$

we get

$$\iint_{Q_1}|u(x,t)|^p\,dx\,dt = \frac{1}{2}\int_0^1\int_0^1 \xi^2|u(T(\xi,\tau))|^p\,d\tau\,d\xi = \frac{1}{2}\int_0^1\int_{\Gamma_1}\xi^2|u(T_\xi(0,\tau))|^p\,d\lambda(0,\tau)\,d\xi.$$

On the other hand, taking the mapping $S : [0,1]\times[0,1] \to Q_2$,

$$S(\xi,\tau) = ((1+\xi)/2, \xi^2\tau + (1-\xi^2)),$$

we see that

$$\iint_{Q_2}|u(x,t)|^p\,dx\,dt = \frac{1}{2}\int_0^1\int_{\Gamma_2}\xi^2|u(T_\xi(1,\tau))|^p\,d\lambda(1,\tau)\,d\xi.$$

Finally, we consider the mapping $W : [0,1]\times[0,1] \to Q_3$,

$$W(\xi,\tau) = (\tau\xi + (1-\tau)/2, (1-\tau^2)),$$

and we obtain

$$\iint_{Q_3} |u(x,t)|^p\,dx\,dt = 2\int_0^1 \int_{\Gamma_3} \tau^2 |u(T_\tau(\xi,0))|^p\,d\lambda(\xi,0)\,d\tau.$$

Collecting all the integrals, it follows from (8) that

$$\int_0^1 r^2 m_p(r)^p\,dr = \int_0^1 \int_{\Gamma} r^2 |u(T_r(\xi,\tau))|^p\,d\lambda(\xi,\tau)\,dr \le C.$$

Then

$$\int_{1/2}^1 \log m_p(r)^p\,dr/r \le 8\int_0^1 r^2 m_p(r)^p\,dr \le C. \tag{9}$$

Using inequality (9), making a change of variable in the integral on the left hand side of (7) and choosing $a < 1/(1-p)$ we can write

$$\int_{1/2^a}^1 \log m_\infty(r)\,dr/r \le C_p,$$

where C_p is a constant depending only on p. This inequality implies that there exists $r_0 \in [1/2^a, 1]$ such that $m_\infty(r_0) \le M_p$, a constant depending only on p, and from the maximum principle the assertion follows. ■

Remark. Notice that Lemma 1 is also true for $1 < p < \infty$, since it is valid for $p = 1$.

THEOREM 1. *Let $u \in H^p$ for $0 < p \le 1$. Then there exists a constant C depending only on p such that for every $(x,t) \in \mathbb{R}^2_+$,*

$$|u(x,t)| \le C\|u\|_{H^p} t^{-1/(2p)}. \tag{10}$$

In particular, $u(x,t)$ is bounded in each proper half-plane $\{(x,t) : t \ge t_0 > 0\}$. In fact, $u(x,t) \to 0$ if $(x,t) \to \infty$ in such a half-plane.

Proof. Let $(x_0, t_0) \in \mathbb{R}^2_+$ and $R_0 = (x_0 - \sqrt{t_0}/(2\sqrt{2}), x_0 + \sqrt{t_0}/(2\sqrt{2})) \times (t_0/2, t_0)$. By using Lemma 1, the proof of this result is exactly the same as the proof of the harmonic case given in [5, p. 174]. ■

Next, we analyze the growth of $D_t u$ for $u \in H^p$, $0 < p \le 1$. Let $t > 0$ and $x \in \mathbb{R}$. Fix $t_0 > 0$ such that $t/2 < t_0 < t$. Estimate (10) implies

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y, t-t_0) u(y,t_0)\,dy$$

and therefore using [7, Lemma 3, (18)] we obtain

$$|D_t u(x,t)| \le C\|u\|_{H^p} t_0^{-1/(2p)} \int_{-\infty}^{\infty} \min\left\{\frac{1}{|x-y|^3}, \frac{1}{(t-t_0)^{3/2}}\right\} dy$$

$$\begin{aligned} &= C\|u\|_{H^p} t_0^{-1/(2p)} \int_{|y-x|^3 > (t-t_0)^{3/2}} \frac{dy}{|y-x|^3} \\ &\quad + C\|u\|_{H^p} t_0^{-1/(2p)} \int_{|y-x|^3 < (t-t_0)^{3/2}} \frac{dy}{(t-t_0)^{3/2}} \\ &\le C\|u\|_{H^p} (t-t_0)^{-1-1/(2p)}. \end{aligned}$$

Letting $t_0 \to t/2$ we finally get

$$|D_t u(x,t)| \le C\|u\|_{H^p} t^{-1-1/(2p)}. \tag{11}$$

Now we analyze the boundary behavior of $u \in H^p$, $0 < p < \infty$. In the case $1 < p < \infty$, $u(\cdot,t) \to f$ in the L^p norm as $t \to 0$ (see [4, Th. 2(xi)]), where $u(x,t) = K(\cdot,t) * f(x)$, $f \in L^p$. In the case $p = 1$, $u(\cdot,t) \to \mu$ as $t \to 0$ in the weak* topology of $M(\mathbb{R})$, where $u(x,t) = K(\cdot,t) * \mu(x)$ with $\mu \in M(\mathbb{R})$. Therefore, if $u \in H^p$, $1 \le p < \infty$, the family $(u(\cdot,t))_{t>0}$ converges in $\mathcal{S}'$ as $t \to 0$ and the boundary distribution uniquely determines u. Now suppose that $u \in H^p$, $0 < p < 1$. Theorem 1 implies that each $u(\cdot,t)$ is a bounded function, hence a tempered distribution. In fact, $(u(\cdot,t))_{t>0}$ converges in $\mathcal{S}'$, as stated in the following theorem.

THEOREM 2. *Let $u \in H^p$, $0 < p < \infty$. Then $\lim_{t\to 0} u(\cdot,t) \equiv f$ exists in $\mathcal{S}'$ and f uniquely determines u.*

Proof. It remains to consider the case $0 < p < 1$. Since estimate (10) holds, taking $a = b = 0$ and $c = 1/(2p)$ in [4, Th. 17], we see that $u(x,t) = K(\cdot,t) * f(x)$ where $f \in \mathcal{S}'$. Writing $F\Psi \equiv \widehat{\Psi}$, for each $\varphi \in \mathcal{S}$ we have $\langle u(\cdot,t), \varphi\rangle = \langle Fu(\cdot,t), F^{-1}\varphi\rangle = \langle e^{-4\pi^2 t(\cdot)^2} Ff, F^{-1}\varphi\rangle \to \langle Ff, F^{-1}\varphi\rangle = \langle f, \varphi\rangle$ as $t \to 0$. It is clear that f uniquely determines u. ■

3. Hardy spaces of conjugate pairs of temperature functions. Kochneff and Sagher introduced in [7] the class $\mathcal{AH}$ that we shall use to define our Hardy spaces of pairs. For real functions $u, v \in C^1(\mathbb{R}^2_+)$, they write $u + iv \in \mathcal{AH}$ if

(a) $D_t^{1/2} u$ and $D_t^{1/2} v$ exist on $\mathbb{R}^2_+$, and
(b) the following equations hold:

$$D_x u(x,t) = -iD_t^{1/2} v(x,t), \quad iD_t^{1/2} u(x,t) = D_x v(x,t),$$

where $D_t^{1/2}$ is Weyl's fractional derivative operator of order $1/2$ with respect to t.

We recall that (see [9]) Weyl's *fractional integral* of order $\alpha > 0$ is defined by

$$D^{-\alpha} f(t) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_t^{\infty} f(s)(s-t)^{\alpha-1}\,ds$$

and the *fractional derivative* of order $\alpha > 0$ is

$$D^\alpha f(t) = \frac{e^{i\pi\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \int_t^\infty f^{(n)}(s)(s-t)^{\tilde{\alpha}-1}\,ds,$$

where $\alpha = n - \tilde{\alpha}$, $n \in \mathbb{N}$ and $0 < \tilde{\alpha} \le 1$. Of course, f must be sufficiently regular to ensure that the integral converges.

In [7] it is shown that

(1) If $u + iv \in \mathcal{AH}$ then $u, v \in H$.

(2) $K + i\mathcal{H}K \in \mathcal{AH}$ where K is the Gauss–Weierstrass kernel and $\mathcal{H}K$ is the Hilbert transform of K with respect to the spatial variable.

(3) $g * K + i\mathcal{H}(g * K) \in \mathcal{AH}$ where $g \in L^p$ for $1 \le p < \infty$ and the convolution is taken with respect to the spatial variable.

DEFINITION 1. For $0 < p < \infty$ we define

$$\mathbf{H}^p = \Big\{F = u + iv \in \mathcal{AH} : \sup_{t>0} \int_{-\infty}^\infty |F(x,t)|^p\,dx < \infty\Big\}.$$

When $u \in H^p$, estimates (2) and (11) in the previous section imply

$$|D_t^{1/2}u(x,t)| \le CB(1/2, 1/2 + 1/(2p))\|u\|_{H^p} t^{-1/2-1/(2p)} \quad \text{for } 0 < p < \infty,$$

where B is the beta function. Therefore, any $u \in H^p$, $0 < p < \infty$, has fractional derivative $D_t^{1/2}u$. Moreover, if u, v are real functions in H^p and satisfy condition (b) in the definition of the class $\mathcal{AH}$, then $u + iv \in \mathbf{H}^p$.

THEOREM 3. *For* $1 < p < \infty$,

$$\mathbf{H}^p = \{u + i\mathcal{H}u : u \in H^p, u \text{ real}\},$$

where $\mathcal{H}u$ *is taken with respect to the spatial variable.*

Proof. If $u + i\mathcal{H}u$ is in the set on the right hand side, then [7, Ths. 5, 6] implies that $u + i\mathcal{H}u \in \mathcal{AH}$, and since $\mathcal{H}$ is bounded from L^p to L^p for $p > 1$, it follows that $u + i\mathcal{H}u \in \mathbf{H}^p$.

Conversely, let $F = u + iv \in \mathbf{H}^p$ and $u(x,t) = K(\cdot,t) * f(x)$ for some $f \in L^p$. Since $K + i\mathcal{H}K \in \mathcal{AH}$ we have

$$D_x v(x,t) = \frac{-1}{\sqrt{\pi}} \int_0^\infty \Big(\int_{-\infty}^\infty D_t K(x-y, s+t) f(y)\,dy\Big) s^{-1/2}\,ds$$
$$= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^\infty \Big(\int_0^\infty D_t K(x-y, s+t) s^{-1/2}\,ds\Big) f(y)\,dy$$
$$= i \int_{-\infty}^\infty D_t^{1/2} K(x-y,t) f(y)\,dy$$

$$= \int_{-\infty}^\infty D_x \mathcal{H}K(x-y,t) f(y)\,dy = D_x\mathcal{H}(K(\cdot,t) * f)(x),$$

hence $v(x,t) = \mathcal{H}u(x,t) + C(t)$ and since $v \in H^p$ we infer that $v(x,t) = \mathcal{H}u(x,t)$. As in [7, p. 46] the application of Fubini's Theorem is justified by

$$\int_0^\infty |D_t K(x-y, s+t)| s^{-1/2}\,ds$$
$$\le C \int_0^\infty \min\Big\{\frac{1}{|x-y|^3}, \frac{1}{(s+t)^{3/2}}\Big\} s^{-1/2}\,ds \le C \min\Big\{\frac{1}{(x-y)^2}, \frac{1}{t}\Big\},$$

and the differentiation with respect to the spatial variable under the integral sign is valid since $f \in L^p$ and

$$|D_x\mathcal{H}K(x,t)| \le C \min\Big\{\frac{1}{x^2}, \frac{1}{t}\Big\} \quad \text{by [7, Lemma 3].} \ \blacksquare$$

COROLLARY 1. *For* $0 < p \le 1$,

$$\mathbf{H}^p \subset \{u + i\mathcal{H}u : u \in H^p, u \text{ real}\}.$$

Proof. Let $f = u + iv \in \mathbf{H}^p$ and fix $t_0 > 0$. Then

$$u_{t_0}(x,t) \equiv u(x, t+t_0) = K(\cdot,t) * u(\cdot,t_0)(x),$$
$$v_{t_0}(x,t) \equiv v(x, t+t_0) = K(\cdot,t) * v(\cdot,t_0)(x).$$

Theorem 1 implies that $u(\cdot,t_0), v(\cdot,t_0) \in L^p \cap L^\infty \subset L^q$ for every $q > 1$, and since $u_{t_0} + iv_{t_0} \in \mathcal{AH}$, it follows that $u_{t_0} + iv_{t_0} \in \mathbf{H}^q$. Then using Theorem 3 we obtain $v(\cdot, t+t_0) = \mathcal{H}u(\cdot, t+t_0)$ for every $t > 0$. Since t and t_0 are arbitrary, we conclude that for any $s > 0$, $v(\cdot,s) = \mathcal{H}u(\cdot,s)$. $\blacksquare$

For $F \in \mathbf{H}^p$, $0 < p < \infty$, we define

$$\|F\|_{\mathbf{H}^p} = \sup_{t>0} \Big[\int_{-\infty}^\infty |F(x,t)|^p\,dx\Big]^{1/p};$$

$F \mapsto \|F\|_{\mathbf{H}^p}$ is a norm for $1 \le p < \infty$ and $F \mapsto \|F\|_{\mathbf{H}^p}^p$ is a p-norm for $0 < p < 1$.

THEOREM 4. $\mathbf{H}^p$ *is complete for every* $0 < p < \infty$.

Proof. Let $(F_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbf{H}^p$, $F_n = u_n + iv_n$. By estimates (1) and (10),

$$|(u_n - u_m)(x,t)| \le Ct^{-1/(2p)}\|F_n - F_m\|_{\mathbf{H}^p},$$

where $0 < p < \infty$. Then $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are temperature functions which converge uniformly on compact subsets of $\mathbb{R}_+^2$ to temperature functions $u(x,t)$ and $v(x,t)$, respectively. It suffices to show that u and v satisfy condition (b) in the definition of $\mathcal{AH}$, since letting $F = u + iv$ and $\varepsilon > 0$ we get

$$\int_{-\infty}^{\infty} |F_n(x,t) - F(x,t)|^p\,dx \le \liminf \int_{-\infty}^{\infty} |F_n(x,t) - F_m(x,t)|^p\,dx \le \varepsilon$$

for n arbitrarily large. The rest of the proof is a routine computation. ∎

The last part of this section is devoted to characterizing our $\mathbf{H}^p$ spaces. We denote by $\mathbf{H}^p_{\mathrm{hol}}$ the classical Hardy spaces of holomorphic functions:

$$\mathbf{H}^p_{\mathrm{hol}} = \Big\{F = u + iv : F \text{ is holomorphic on } \mathbb{R}^2_+ \text{ and } \sup_{t>0} \int_{-\infty}^{\infty} |F(x,t)|^p\,dx < \infty\Big\}$$

with the p-norm

$$\|F\|^p_{\mathbf{H}^p_{\mathrm{hol}}} = \sup_{t>0} \int_{-\infty}^{\infty} |F(x,t)|^p\,dx \quad \text{if } 0 < p < 1$$

and the norm

$$\|F\|_{\mathbf{H}^p_{\mathrm{hol}}} = \sup_{t>0} \Big[\int_{-\infty}^{\infty} |F(x,t)|^p\,dx\Big]^{1/p} \quad \text{if } 1 \le p < \infty.$$

It is well known that for $1 < p < \infty$,

$$(\mathbf{H}^p_{\mathrm{hol}}, \|\cdot\|_{\mathbf{H}^p_{\mathrm{hol}}}) \cong (\{f + i\mathcal{H}f : f \in \operatorname{Re} L^p\}, \|f\|_p + \|\mathcal{H}f\|_p).$$

Theorem 3, [4, Th. 2(xi)] and continuity of $\mathcal{H}$ from L^p to L^p imply

$$(\mathbf{H}^p, \|\cdot\|_{\mathbf{H}^p}) \cong (\mathbf{H}^p_{\mathrm{hol}}, \|\cdot\|_{\mathbf{H}^p_{\mathrm{hol}}}), \quad 1 < p < \infty.$$

For $0 < p \le 1$, we define the space $\operatorname{Re}\mathbf{H}^p$ whose elements are the boundary distributions $\operatorname{Re} F(x)$ corresponding to the functions $F \in \mathbf{H}^p$ with the p-norm $\operatorname{Re} F(x) \mapsto \|F\|^p_{\mathbf{H}^p}$. As in the holomorphic case (see [5, p. 236]), it can be shown that $\operatorname{Re}\mathbf{H}^p \hookrightarrow \mathcal{S}'$.

In the holomorphic case, we denote by $\operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$ the analogous space.

Remark. We recall that we are viewing the elements of $\mathbf{H}^p_{\mathrm{hol}}$ as harmonic functions u for which the vector $F(x,t)$ is uniformly in L^p.

The classical result by Fefferman and Stein proved in [3, Th. 11] states that for $0 < p < \infty$ and $f \in \mathcal{S}'$, the following are equivalent:

(a) $u^*(x) = \sup_{|y-x|<t} |\varphi_t * f(y)| \in L^p$ for all $\varphi \in \mathcal{S}$ satisfying $\int_{-\infty}^{\infty} \varphi = 1$.

(b) The distribution f arises as $f = \lim_{t\to 0} u(\cdot,t)$ in $\mathcal{S}'$, where $u \in \mathbf{H}^p_{\mathrm{hol}}$.

Moreover, $\|u\|^p_{\mathbf{H}^p_{\mathrm{hol}}} \sim \|u^*\|^p_p$, where $\sim$ means the standard equivalence of norms (p-norms).

In [10, Prop. 3, p. 123] the following characterization of $\operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$ is given: If $f \in \mathcal{S}'$ is *restricted at infinity* (that is, $f * \varphi \in L^r$ for every $\varphi \in \mathcal{S}$ and for all $r < \infty$ sufficiently large) and $0 < p < \infty$, then $f \in \operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$ if and only if

$$\sup_{t>0}\{\|f * \varphi_t\|_{L^p} + \|\mathcal{H}f * \varphi_t\|_{L^p}\} \le C < \infty,$$

where $\varphi \in \mathcal{S}$, $\int_{-\infty}^{\infty} \varphi = 1$, $\varphi_t(x) = t^{-1}\varphi(x/t)$ and C is a constant. Of course, $\mathcal{H}f$ means the Hilbert transform of a distribution f that is restricted at infinity (see [10, p. 123]).

Before stating the main result of this paper, we need to prove the following lemma.

Lemma 2. *Let $f \in \mathcal{S}'$ and $w(x,t) \equiv K(\cdot,t) * f(x)$ for $(x,t) \in \mathbb{R}^2_+$. If $w^+(x) \equiv \sup_{t>0} |w(x,t)| \in L^p$ then $w^*(x) \equiv \sup_{|y-x|<t^{1/2}} |w(y,t)| \in L^p$ and $\|w^*\|^p_p \sim \|w^+\|^p_p$ for $0 < p < \infty$.*

Proof. Let $x \in \mathbb{R}$ and $(y,t) \in \{(\xi,\tau) : |\xi - x| < \tau^{1/2}\}$. If R denotes the rectangle $(y - \sqrt{t}/(2\sqrt{2}), y + \sqrt{t}/(2\sqrt{2})) \times (t/2, t)$, then according to Lemma 1 we have

$$|w(y,t)|^{p/2} \le \frac{C}{t^{3/2}} \iint_R |w(z,t')|^{p/2}\,dz\,dt'$$
$$\le \frac{C}{t^{3/2}} \iint_R w^+(z)^{p/2}\,dz\,dt' \le \frac{C}{t^{1/2}} \int_{x-\sqrt{t}}^{x+\sqrt{t}} w^+(z)^{p/2}\,dz,$$

which implies $(w^*)^{p/2}(x) \le C\mathcal{M}((w^+)^{p/2})(x)$, where $\mathcal{M}$ is the Hardy–Littlewood maximal function, thus

$$\int_{\mathbb{R}} (w^*)^p(x)\,dx \le C \int_{\mathbb{R}} (\mathcal{M}((w^+)^{p/2}))^2(x)\,dx \le C \int_{\mathbb{R}} (w^+)^p(x)\,dx.$$

The other inequality is immediate. ∎

Theorem 5. $\operatorname{Re}\mathbf{H}^p = \operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$. *Moreover*, $\mathbf{H}^p \cong \mathbf{H}^p_{\mathrm{hol}}$ *for* $0 < p \le 1$.

Proof. First, we will show that $\operatorname{Re}\mathbf{H}^p = \operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$.

Let $f \in \operatorname{Re}\mathbf{H}^p$. Then there exists $u + iv \in \mathbf{H}^p$ such that $f = \lim_{t\to 0} u(\cdot,t)$ in $\mathcal{S}'$. Fix $t_0 > 0$. Set $\mathcal{G}(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then $K(x,t) = \mathcal{G}_{\sqrt{2t}}(x)$, thus Corollary 1 implies $u_{t_0}(\cdot,t) = \mathcal{G}_{\sqrt{2t}} * u(\cdot,t_0)$, $v_{t_0}(\cdot,t) = \mathcal{G}_{\sqrt{2t}} * \mathcal{H}u(\cdot,t_0)$, and moreover

$$\sup_{t>0}\{\|\mathcal{G}_{\sqrt{2t}} * u(\cdot,t_0)\|^p_p + \|\mathcal{G}_{\sqrt{2t}} * \mathcal{H}u(\cdot,t_0)\|^p_p\} < \infty.$$

Since $u(\cdot,t_0) \in L^q$ for any $q > 1$, it is a distribution restricted at infinity. Then [10, Prop. 3, p. 123] implies that $u(\cdot,t_0) \in \operatorname{Re}\mathbf{H}^p_{\mathrm{hol}}$. Also, from

Lemma 2 and [3, Th. 11] we have

$$\|\mathcal{G}_{\nabla}^{*}(u(\cdot,t_0))\|_p^p \sim \|\mathcal{G}^{+}(u(\cdot,t_0))\|_p^p \sim \|u(\cdot,t_0)\|^p_{\mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}} \tag{12}$$

where

$$\mathcal{G}_{\nabla}^{*}(u(\cdot,t_0))(x) \equiv \sup_{|y-x|<\sqrt{2t}} |\mathcal{G}_{\sqrt{2t}} * (u(\cdot,t_0))(y)|,$$

$$\mathcal{G}^{+}(u(\cdot,t_0))(x) \equiv \sup_{t>0} |\mathcal{G}_{\sqrt{2t}} * (u(\cdot,t_0))(x)|.$$

On the other hand, if we define

$$F_{t_0}(x,t) = P_t * u(\cdot,t_0)(x) + i\mathcal{H}(P_t * u(\cdot,t_0))(x)$$

where P_t is the Poisson kernel, we obtain a holomorphic function in $\mathbf{H}^p_{\mathrm{hol}}$ which satisfies $F_{t_0}(x,t) \to u(x,t_0)+i\mathcal{H}u(x,t_0) \equiv u(x,t_0)+iv(x,t_0)$ as $t\to 0$ a.e. on $\mathbb{R}$; consequently (see [5, Cor. 1.2(c), p. 233]),

$$\|F_{t_0}\|^p_{\mathbf{H}^p_{\mathrm{hol}}} \sim \|u(\cdot,t_0)+iv(\cdot,t_0)\|_p^p.$$

Thus,

$$\|u(\cdot,t_0)\|^p_{\mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}} = \|F_{t_0}\|^p_{\mathbf{H}^p_{\mathrm{hol}}} \le C\|u(\cdot,t_0)+iv(\cdot,t_0)\|_p^p \le C\|f\|^p_{\mathrm{Re}\,\mathbf{H}^p}. \tag{13}$$

Combining (12) and (13) we get

$$\|\mathcal{G}_{\nabla}^{*}(u(\cdot,t_0))\|_p^p \le C\|u(\cdot,t_0)\|^p_{\mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}} \le C\|f\|^p_{\mathrm{Re}\,\mathbf{H}^p}$$

and since $\mathcal{G}_{\nabla}^{*}(f) = \lim_{t_0\to 0}\mathcal{G}_{\nabla}^{*}(u(\cdot,t_0))$ a.e. on $\mathbb{R}$, an application of Fatou's Lemma gives us

$$\|\mathcal{G}_{\nabla}^{*}(f)\|_p^p \le C\|f\|^p_{\mathrm{Re}\,\mathbf{H}^p},$$

and hence

$$\|f\|^p_{\mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}} \le C\|f\|^p_{\mathrm{Re}\,\mathbf{H}^p}. \tag{14}$$

This shows that $f\in \mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}$.

Conversely, let $f\in \mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}$. There exists $u+iv\in \mathbf{H}^p_{\mathrm{hol}}$ such that $f=\lim_{t\to 0}u(\cdot,t)$ in $\mathcal{S}'$, in fact, $f=\lim_{t\to 0}u(\cdot,t)$ in $\mathrm{Re}\,\mathbf{H}^p_{\mathrm{hol}}$. From [10, Prop. 3, p. 123] we have

$$\sup_{t>0}\{\|f*\mathcal{G}_{\sqrt{2t}}\|_p^p + \|\mathcal{H}f*\mathcal{G}_{\sqrt{2t}}\|_p^p\} \le C < \infty.$$

Consequently, the function $w_1(x,t)=K(\cdot,t)*f(x)$ belongs to H^p. Moreover, $w_1(x,t+t_0)=K(\cdot,t)*w_1(x,t_0)$ for all $t,t_0>0$ and as $w_1(\cdot,t_0)\in L^p\cap L^\infty \subset L^q$ for every $q>1$, it follows that $\mathcal{H}w_1(\cdot,t_0)\in L^q$. This implies that $w_1(x,t+t_0)+iw_2(x,t+t_0)\in \mathcal{AH}$ where $w_2(x,t+t_0)\equiv K(\cdot,t)*\mathcal{H}w_1(\cdot,t_0)(x)$. Moreover, $w_1+iw_2\in\mathbf{H}^p$ because $w_2(x,t+t_0)=K(\cdot,t+t_0)*\mathcal{H}f(x)$. Now, $w_1(\cdot,t)$ converges to an element in $\mathrm{Re}\,\mathbf{H}^p$ and also $w_1(\cdot,t)\to f$ in $\mathcal{S}'$ as $t\to 0$, and since $\mathrm{Re}\,\mathbf{H}^p \hookrightarrow \mathcal{S}'$ we see that $f\in\mathrm{Re}\,\mathbf{H}^p$.

To finish the proof, it is sufficient to notice from (14) that the bijective linear mapping $T:\mathbf{H}^p\to\mathbf{H}^p_{\mathrm{hol}}$, $F=u+iv\mapsto f=\lim_{t\to 0}u(\cdot,t)$ in $\mathcal{S}'$, is continuous, and the result follows by the Open Mapping Theorem. ∎

COROLLARY 2. *Let $f\in\mathcal{S}'$ and $0<p\le 1$. Then $f\in\mathrm{Re}\,\mathbf{H}^p$ if and only if the maximal function $u^*(x)=\sup_{|y-x|<\sqrt{2t}}|\mathcal{G}_{\sqrt{2t}}*f(y)|$ belongs to L^p.*

Proof. Just apply Theorem 5 and Lemma 2. ∎

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