Hardy spaces of conjugate temperatures

by

MARTHA GUZMÁN-PARTIDA (México, D.F.)

Abstract. We define Hardy spaces of pairs of conjugate temperatures on \( \mathbb{R}^2_+ \) using the equations introduced by Kochenn and Sazger. As in the holomorphic case, the Hilbert transform relates both components. We demonstrate that the boundary distributions of our Hardy spaces of conjugate temperatures coincide with the boundary distributions of Hardy spaces of holomorphic functions.

1. Introduction. The Hardy spaces of holomorphic functions \( \mathcal{H}^p_{\text{hol}}(\mathbb{R}^2_+) \) were originally defined by Stein and Weiss as spaces of harmonic functions \( u \) such that the vector \( F(x, t) = u(x, t) + iv(x, t) \), where \( v \) is the conjugate of \( u \), is uniformly in \( L^p(\mathbb{R}) \), that is, \( \int_{-\infty}^{\infty} |F(x, t)|^p dx \lesssim C < \infty \), independently of \( t > 0 \). The Theorem of Burkholder, Gundy and Silverstein characterized \( \mathcal{H}^p_{\text{hol}}(\mathbb{R}^2_+) \) as a space of harmonic functions without appealing to any notion of conjugacy: a tempered distribution \( f \) is in \( \text{Re} \mathcal{H}^p_{\text{hol}}(\mathbb{R}^2_+) \) if and only if it is the boundary distribution corresponding to a real harmonic function \( u(x, t) \) in \( \mathbb{R}^2_+ \) such that its maximal function \( u^*(x) = \sup_{|y-x|\leq t} |u(y, t)| \) belongs to \( L^p \). Fefferman and Stein showed that the following properties are equivalent for a tempered distribution \( f \) in \( \mathbb{R} \):

(a) \( f = \lim_{t \to 0} u(\cdot, t) \) in \( \mathcal{S}' \) for some \( u \in \mathcal{H}^p_{\text{hol}}(\mathbb{R}^2_+) \).

(b) \( \sup_{t>0} |f \ast \varphi_t(x)| \) is in \( L^p \) for some \( \varphi \in \mathcal{S} \) such that \( \int_{-\infty}^{\infty} \varphi = 1 \), where \( \varphi_t(x) = t^{-1} \varphi(x/t) \).

(c) \( \sup_{|y-x|\leq t} |f \ast \varphi_t(y)| \) is in \( L^p \) for some \( \varphi \) as above.

(d) \( P_t(f)(x) = \sup_{|y-x|\leq t} |u(y, t)| \) is in \( L^p \), where \( u(y, t) = P_t \ast f(y) \) and \( P_t \) is the Poisson kernel.

This theorem shows that \( \mathcal{H}^p_{\text{hol}}(\mathbb{R}^2_+) \) may be defined without having recourse to the Poisson kernel, and in fact, independently of \( \varphi \in \mathcal{S} \). In this work, we characterize Hardy spaces in terms of temperature functions using the notion of conjugacy introduced by Kochenn and Sazger. In particular, we study spaces of temperature functions \( u \) such that \( F(x, t) = u(x, t) + iv(x, t) \).

1991 Mathematics Subject Classification: Primary 42B30, 42A50; Secondary 35K05.
is uniformly in $L^p$, where $v$ is the conjugate of $u$. We show that the boundary distributions in this case are the same as in the holomorphic case.

2. Hardy spaces of temperature functions. We write

$$H^p(\mathbb{R}^2_+) = \left\{ u \in C^2(\mathbb{R}^2_+) : \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \right\},$$

$$H^p(\mathbb{R}^2_+) = \{ u \in H(\mathbb{R}^2_+) : ||u||_{H^p} < \infty \},$$

where

$$||u||_{H^p} = \sup_{t>0} ||u(\cdot, t)||_p, \quad 0 < p \leq \infty.$$ If the context does not cause confusion we shall write $H$ and $H^p$, respectively. We shall call the elements of $H$ temperature functions. $K(x, t)$ will denote the Gauss-Weierstrass kernel.

For $1 \leq p \leq \infty$, we have the following representations for $u \in H^p$:

(a) $u \in H^p$ if and only if $u(x, t) = K(\cdot, t) * f(x)$, where $f \in L^p$, $1 < p \leq \infty$.

(b) $u \in H^1$ if and only if $u(x, t) = K(\cdot, t) * \mu(x)$, where $\mu \in M(\mathbb{R})$, the space of Borel measures on $\mathbb{R}$.

(See [4, Th. 5] for $1 < p \leq \infty$ and use [6, Ch. VIII, Th. 10.2] and the Banach-Alaoglu Theorem for $p = 1$.)

Now, we analyze the growth of functions in $H^p$. If $u \in H^p$ for $1 < p < \infty$, Flett's estimate in [4, Th. 2(iv)] gives for $t > 0$ and $x \in \mathbb{R}$,

$$|u(x, t)| \leq C ||f||_p t^{-1/(2p)},$$

where $u(x, t) = K(\cdot, t) * f(x), f \in L^p$. Also, an application of Flett's result in [4, Th. 2(2)] and (v) gives for $t > 0$ and $x \in \mathbb{R}$,

$$|D_t u(x, t)| \leq C ||f||_p t^{-1-1/(2p)}.$$ An immediate consequence of estimates (1) and (2) [4, Th. 2(iv)] is that $u$ and $D_t u$ are bounded in each proper half-plane $\{(x, t) \in \mathbb{R}^2_+ : t \geq t_0 > 0\}$.

It remains to analyze the case $0 < p \leq 1$. We will require Lemma 1 below, whose proof follows the main ideas of the harmonic case [5, p. 172] adapted to our situation. Before we state it, we recall some known facts that will be used in this part.

Let $Q = (0, 1) \times (0, 1), T = T_1 \cup T_2 \cup T_3$ (the parabolic boundary of $Q$), where $T_1 = (0, 1) \times (0, 1), T_2 = \{1\} \times (0, 1), T_3 = (0, 1) \times \{0\}$, and let $\lambda$ be the one-dimensional Lebesgue measure on $T$.

For $t > 0$, let

$$\theta(x, t) = \sum_{n=1}^{\infty} K(x + 2n, t), \quad \varphi(x, t) = -2D_x \theta(x, t),$$

and for $t \leq 0$, let $K = \theta = \varphi = 0$.

We consider the kernels $K_i$ on $T_i, i = 1, 2, 3$, given by

$$K_1(x, t; 0, \tau) = \varphi(x, t - \tau), \quad 0 \leq \tau < 1,$$

$$K_2(x, t; 1, \tau) = \varphi(1 - x, t - \tau), \quad 0 \leq \tau < 1,$$

$$K_3(x, t; \xi, 0) = \varphi(\theta(x - \xi, t) - \theta(x, t), \quad 0 < \xi < 1,$$

and then consider the heat kernel $K(x, t; \xi, \tau)$ on $Q \times T$ defined as the union of $K_1, K_2, K_3$. It is well known that every temperature function on $Q$ which is continuous on $Q$ can be written on $(0, 1) \times (0, 1)$ as

$$u(x, t) = \int_{\mathcal{T}} K(x, t; \xi, \tau) u(\xi, \tau) d\lambda(\xi, \tau).$$

We shall also consider for every $0 < r < 1$ the mapping $T_r(x, t) = (x, r t)$, where

$$x_r = r x + (1 - r)/2, \quad t_r = r^2 t + (1 - r)^2.$$ Notice that $Q = \{x_r : (x_r, t_r) \in T_r; \xi, \tau \in \mathcal{T}; 0 < r < 1\}$.

**Lemma 1.** Let $u$ be a temperature function on a rectangle $R = (a, b) \times (c, d) \subset \mathbb{R}^2_+$, where $d - c = (b - a)^2$, and let $u$ be continuous on $R$. If $(x_0, t_0)$ is the middle point of the upper boundary of $R$, then for every $p$ with $0 < p \leq 1$,

$$\{|u(x_0, t_0)|^p \leq C_p \left( \int_R |u(x, t)|^p dx dt \right)^{1/p},$$

where $|R|$ is the area of $R$ and $C_p$ is a constant depending only on $p$.

**Proof.** As the mapping $\Psi : Q \to (a, b) \times (c, d)$,

$$\xi = ((b - a) \xi + a, (b - d) \xi + c),$$

makes $u \circ \Psi$ a temperature function since $d - c = (b - a)^2$, there is no loss of generality in supposing that $R = Q$ and $\int_Q |u(x, t)|^p dx dt = |Q| = 1$. We notice that $u \circ T_r$ is a temperature function on $Q$ for $0 < r < 1$, because $T_r(x, t) = \left(t - \frac{1}{2} + \frac{1}{2} r t \right)(1 - t - 1) + 1$. For $r > 0$ let

$$m_p(r) = \left( \int_{T_r} |u(T_r(\xi, \tau))|^p d\lambda(\xi, \tau) \right)^{1/p},$$

and

$$m_{\infty}(r) = \sup\{u(T_r(\xi, \tau)) : (\xi, \tau) \in \mathcal{T}\}.$$ We may also assume that $m_{\infty}(r) = 1$ for every $r \in (0, 1)$ since otherwise the maximum principle for temperature functions on a rectangle would immediately give us the required inequality.

First, we notice that

$$m_1(r) \leq m_{\infty}(r)^{1-p} m_p(r)^p.$$
We have the representation

\[ u(T,r,t) = \int_\mathbb{R} K(x,r,t,\xi) u(T_r(\xi,\tau)) d\lambda(\xi,\tau). \]

For \((x,t) \in Q^s = [(1-r)/2, (1+r)/2] \times [(1-r^2),1]\), we have

\[ u(x,t) = \int_\mathbb{R} K(T_r^{-1}(x,t)\xi,\tau) u(T_r(\xi,\tau)) d\lambda(\xi,\tau), \]

and since \(T_r \circ T_s = T_{rs}\), it follows that for \((x,t) \in Q^s\), \(0 < s < r < 1\),
\[ u(x,t) = \int_\mathbb{R} K(T_s/T_r^{-1}(x,t)\xi,\tau) u(T_r(\xi,\tau)) d\lambda(\xi,\tau). \]

It is enough to analyze the behavior of \(K(T_s/T_r^{-1}(x,t)\xi,\tau)\) for \((x,t) \in Q\) because \(T_r^{-1}(x,t) \in Q\).

Now, we get an estimate of \(K_3\). Since (see [2])
\[ \vartheta(x,y,t) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) \sin(n\pi y), \]
we obtain for \(0 < q < 1\),
\[ |K_3(x,t;\xi,\tau)| \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{(1-q^2)^{n/2}} \leq \frac{1}{3} (1-q)^{-2}. \]

To get an estimate of \(K_1\), we first notice that
\[ \varphi(x,t) = -2D_z K(x,t) + U(x,t), \]
where \(U(x,t) = \sum_{n \neq 0} -2D_z K(x+2n,t)\), which is smooth and bounded. Since
\[ -2D_z K(x,t) \leq \int (x/t) K(x,t) \quad \text{if} \quad t > 0, \]
\[ 0 \quad \text{if} \quad t \leq 0, \]
and \((x/t)^{e^{-x^2/(4t)}} \leq C/e^x\) for \(t > 0, \beta > 0, x \in \mathbb{R}\), we have for \(0 < q < 1\),
\[ |K_1(x,t;\xi,\tau)| \leq C \frac{e^{x/(1-q^2/2)}}{e^{x/(1-q^2/2)^2}} + C \leq C(1-q)^{-2}. \]

It remains to analyze \(K_2\). As before, for \(0 < q < 1\),
\[ |K_2(x,t;\xi,\tau)| \leq C(1-q)^{-2}. \]

Collecting all the estimates above, we obtain from (4), for \(q = s/r\) and \((x,t) \in Q^s\),
\[ |u(x,t)| \leq M(1-s/r)^{-3} \int \lambda(T_r(\xi,\tau)) d\lambda(\xi,\tau), \]

where \(M\) is a positive constant. Inequality (5) implies
\[ m_\infty(s) \leq M(1-s/r)^{-3} m_1(r). \]

Choosing \(s = r^a\) with \(a > 1\) we obtain from (3) and (6),
\[ m_\infty(r^a) \leq M(1-r^{a-1})^{-3} m_\infty(r)^{1-p(m)}(r)^p. \]

Taking logarithms, multiplying by \(1/r\) and then integrating with respect to \(r\), we get
\[ \int_1^{1/2} \log m_\infty(r^a) dr/r \leq C_a + (1-p) \int_1^{1/2} \log m_\infty(r) dr/r \]
\[ + \int_1^{1/2} \log m_\infty(r)^p dr/r, \]
where \(C_a\) is the real constant \(\int_1^{1/2} \log M dr/r + 2 \int_1^{1/2} \log(1-r^{a-1})^{-1} dr/r\).

Now, we notice that
\[ 1 = \int \int_{|u(x,t)|^p} dx dt = \sum_{i=1}^3 \int \int_{Q_i} |u(x,t)|^p dx dt, \]
where
\[ Q_1 = \{(x,t) : 0 \leq x \leq 1, (1-4(x-1/2)^2) \leq t \leq 1\}, \]
\[ Q_2 = \{(x,t) : 1/2 \leq x \leq 1, (1-4(x-1/2)^2) \leq t \leq 1\}, \]
\[ Q_3 = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq (1-4(x-1/2)^2)\}. \]

If we consider the mapping \(T : [0,1] \times [0,1] \rightarrow Q_1\),
\[ T(\xi,\tau) = ((1-\xi)/2, \xi^2+1-\xi^2) \]
we get
\[ \int \int_{Q_1} |u(x,t)|^p dx dt = \int_0^{1/2} \int_0^{1/2} |u(T(\xi,\tau))|^p d\xi d\tau \]
\[ = \int_0^{1/2} \int_0^{1/2} |u(T(\xi,\tau))|^p d\lambda(0,\tau) d\xi. \]

On the other hand, taking the mapping \(S : [0,1] \times [0,1] \rightarrow Q_2\),
\[ S(\xi,\tau) = ((1+\xi)/2, \xi^2+1-\xi^2) \]
we see that
\[ \int \int_{Q_2} |u(x,t)|^p dx dt = \int_0^{1/2} \int_0^{1/2} |u(T(\xi,\tau))|^p d\lambda(1,\tau) d\xi. \]

Finally, we consider the mapping \(W : [0,1] \times [0,1] \rightarrow Q_3\),
\[ W(\xi,\tau) = (\xi + (1-\tau)/2, (1-\tau)^2), \]
and we obtain
\[
\int_0^1 \int_{Q_{t_0}} |u(x, t)|^p \, dx \, dt = 2 \int_{0}^{1} \int_{0}^{r} |u(T_r(\xi, 0))|^p \, d\lambda(\xi, 0) \, dr.
\]
Collecting all the integrals, it follows from (8) that
\[
\int_0^1 r^2 m_p(r)^p \, dr = \int_0^1 r^2 |u(T_r(\xi, \tau))|^p \, d\lambda(\xi, \tau) \, dr \leq C.
\]
Then
\[
\int_0^1 \log m_p(r)^p \, dr / r \leq 8 \int_0^1 r^2 m_p(r)^p \, dr \leq C.
\]
Using inequality (9), making a change of variable in the integral on the left hand side of (7) and choosing \(a < 1/(1 - p)\) we can write
\[
\int_0^{1/2a} \log m_{\infty}(r)^p \, dr / r \leq C_p,
\]
where \(C_p\) is a constant depending only on \(p\). This inequality implies that there exists \(r_0 \in [1/2a, 1]\) such that \(m_{\infty}(r_0) \leq M_p\), a constant depending only on \(p\), and from the maximum principle the assertion follows.

**Remark.** Notice that Lemma 1 is also true for \(1 < p < \infty\), since it is valid for \(p = 1\).

**Theorem 1.** Let \(u \in H^p \) for \(0 < p \leq 1\). Then there exists a constant \(C\) depending only on \(p\) such that for every \((x, t) \in \mathbb{R}^2\),
\[
|u(x, t)| \leq C \|u\|_{H^p} t^{-1/(2p)}.
\]
In particular, \(u(x, t)\) is bounded in each proper half-plane \(\{(x, t) : t \geq t_0 > 0\}\). In fact, \(u(x, t) \rightarrow 0\) if \((x, t) \rightarrow \infty\) in some such half-plane.

**Proof.** Let \((x_0, t_0) \in \mathbb{R}^2\) and \(R_0 = (x_0 - \sqrt{t_0} (2, \sqrt{2}), x_0 + \sqrt{t_0} (2, \sqrt{2})) \times (t_0/2, t_0)\). By using Lemma 1, the proof of this result is exactly the same as the proof of the harmonic case given in [5, p. 174].

Next, we analyze the growth of \(D_t u\) for \(u \in H^p\), \(0 < p \leq 1\). Let \(t > 0\) and \(x \in \mathbb{R}\). Fix \(t_0 > 0\) such that \(t/2 < t_0 < t\). Estimate (10) implies
\[
u(x, t) = \int_{-\infty}^{\infty} K(x \cdot t - y, -t_0) u(y, t_0) \, dy
\]
and therefore using [7, Lemma 3, (18)] we obtain
\[
|D_t u(x, t)| \leq C \|u\|_{H^p} t_0^{-1/(2p)} \int_{-\infty}^{\infty} \min \left\{ \frac{1}{|x - y|^2}, \frac{1}{(t - t_0)^{3/2}} \right\} dy
\]

Letting \(t_0 \rightarrow t/2\) we finally get
\[
|D_t u(x, t)| \leq C \|u\|_{H^p} t^{-1/(2p)}.
\]

Now we analyze the boundary behavior of \(u \in H^p\), \(0 < p < \infty\). In the case \(1 < p < \infty\), \(u(\cdot, t) \rightarrow f\) in the \(L^p\) norm as \(t \rightarrow 0\) (see [4, Th. 2(xi)]), where \(u(x, t) = K(\cdot, t) * f(x)\), \(f \in L^p\). In the case \(p = 1\), \(u(\cdot, t) \rightarrow \mu\) as \(t \rightarrow 0\) in the weak* topology of \(M(\mathbb{R})\), where \(u(x, t) = K(\cdot, t) * \mu(x)\) with \(\mu \in M(\mathbb{R})\). Therefore, if \(u \in H^p\), \(1 < p < \infty\), the family \(\{u(\cdot, t)\}_{t > 0}\) converges in \(S'\) as \(t \rightarrow 0\) and the boundary distribution uniquely determines \(u\). Now suppose that \(u \in H^p\), \(0 < p < 1\). Theorem 1 implies that each \(u(\cdot, t)\) is a bounded function, hence a tempered distribution. In fact, \(\{u(\cdot, t)\}_{t > 0}\) converges in \(S'\), as stated in the following theorem.

**Theorem 2.** Let \(u \in H^p\), \(0 < p < \infty\). Then \(\lim_{t \rightarrow 0} u(\cdot, t) \equiv f\) exists in \(S'\) and \(f\) uniquely determines \(u\).

**Proof.** It remains to consider the case \(0 < p < 1\). Since estimate (10) holds, taking \(a = b = 0\) and \(c = 1/(2p)\) in [4, Th. 17], we see that \(u(x, t) = K(\cdot, t) * f(x)\) where \(f \in S'\). Writing \(F^p \equiv \hat{F}\), for each \(f \in S\) we have \(u(\cdot, t) \circ \hat{F} = F^p(\cdot, t) \circ \hat{F} = (e^{-\alpha \cdot t \cdot \cdot})^p F \circ \hat{F} - \cdot \circ \hat{F} \rightarrow \cdot \circ \hat{F} = \cdot + t \rightarrow 0\). It is clear that \(f\) uniquely determines \(u\).

**3. Hardy spaces of conjugate pairs of temperature functions.** Kochneff and Saghier introduced in [7] the class \(A_H\) that we shall use to define our Hardy spaces of pairs. For real functions \(u, v \in C^1(\mathbb{R}^2)\), they write \(u + iv \in A_H\) if

(a) \(D_t^{1/2} u\) and \(D_t^{1/2} v\) exist on \(\mathbb{R}^2\), and

(b) the following equations hold:
\[
D_t u(x, t) = -i D_t^{1/2} u(x, t), \quad i D_t^{1/2} u(x, t) = D_t v(x, t),
\]
where \(D_t^{1/2}\) is Weyl’s fractional derivative operator of order \(1/2\) with respect to \(t\).

We recall that (see [9]) Weyl’s fractional integral of order \(\alpha > 0\) is defined by
\[
D^{-\alpha} f(t) = \frac{\gamma(\alpha)}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-2} \, ds
\]
and the fractional derivative of order \( \alpha > 0 \) is
\[
D^\alpha f(t) = \frac{e^{i\pi \alpha}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f^{(n)}(s)(s-t)^{\alpha-1} \, ds,
\]
where \( \alpha = n - \vec{\alpha}, \, n \in \mathbb{N} \) and \( 0 < \vec{\alpha} \leq \frac{1}{2} \). Of course, \( f \) must be sufficiently regular to ensure that the integral converges.

In [7] it is shown that
1. If \( u + iv \in \mathcal{AH} \) then \( u, v \in H \).
2. \( K + i\mathcal{H}K \in \mathcal{AH} \) where \( K \) is the Gauss–Weierstrass kernel and \( \mathcal{H}K \) is the Hilbert transform of \( K \) with respect to the spatial variable.
3. \( g * K + i\mathcal{H}(g * K) \in \mathcal{AH} \) where \( g \in L^p \) for \( 1 < p < \infty \) and the convolution is taken with respect to the spatial variable.

**Definition 1.** For \( 0 < p < \infty \) we define
\[
\mathcal{H}^p = \left\{ F = u + iv \in \mathcal{AH} : \sup_{t>0} \int_{-\infty}^{\infty} |F(x,t)|^p \, dx < \infty \right\}.
\]

When \( u \in \mathcal{H}^p \), estimates (2) and (11) in the previous section imply
\[
|D^{1/2}_x u(x,t)| \leq CB(1/2, 1/2 + 1/(2p))\|u\|_{\mathcal{H}^{p-1/2}} \quad \text{for} \quad 0 < p < \infty,
\]
where \( B \) is the beta function. Therefore, any \( u \in \mathcal{H}^p \), \( 0 < p < \infty \), has fractional derivative \( D^{1/2}_x u \). Moreover, if \( u, v \) are real functions in \( \mathcal{H}^p \) and satisfy condition (b) in the definition of the class \( \mathcal{AH} \), then \( u + iv \in \mathcal{H}^p \).

**Theorem 3.** For \( 1 < p < \infty \),
\[
\mathcal{H}^p = \{u + i\mathcal{H}u : u \in \mathcal{H}^p, u \text{ real}\},
\]
where \( \mathcal{H}u \) is taken with respect to the spatial variable.

**Proof.** If \( u + i\mathcal{H}u \) is in the set on the right hand side, then [7, Ths. 5, 6] implies that \( u + i\mathcal{H}u \in \mathcal{AH} \), and since \( \mathcal{H} \) is bounded from \( L^p \) to \( L^p \) for \( p > 1 \), it follows that \( u + i\mathcal{H}u \in \mathcal{H}^p \).

Conversely, let \( F = u + iv \in \mathcal{H}^p \) and \( u(x,t) = K(\cdot,t) * f(x) \) for some \( f \in L^p \). Since \( K + i\mathcal{H}K \in \mathcal{AH} \) we have
\[
D_x u(x,t) = \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} D_x K(x - y, s + t) f(y) \, dy \right) s^{-1/2} \, ds
\]
\[
= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} D_x K(x - y, s + t) s^{-1/2} \, ds \right) f(y) \, dy
\]
\[
= i \int_{-\infty}^{\infty} D_x^{1/2} K(x - y, t) f(y) \, dy
\]

\[
= \int_{-\infty}^{\infty} D_x \mathcal{H}K(x - y, t) f(y) \, dy = D_x \mathcal{H}(K(\cdot, t) * f)(x),
\]
hence \( v(x,t) = \mathcal{H}u(x,t) + C(t) \) and since \( v \in \mathcal{H}^p \) we infer that \( v(x,t) = \mathcal{H}u(x,t) \). As in [7, p. 46] the application of Fubini’s Theorem is justified by
\[
\left| D_x K(x - y, s + t) \right| s^{-1/2} \, ds \leq C \min \left\{ \frac{1}{|x - y|^2}, \frac{1}{(s + t)^{3/2}} \right\} \leq \frac{1}{(s + t)^{3/2}} \leq C \min \left\{ \frac{1}{|x - y|^2}, \frac{1}{t} \right\},
\]
and the differentiation with respect to the spatial variable under the integral sign is valid since \( f \in L^p \) and
\[
|D_x \mathcal{H}K(x,t)| \leq C \min \left\{ \frac{1}{|x|^2}, \frac{1}{t} \right\} \quad \text{by [7, Lemma 3]}.
\]

**Corollary 1.** For \( 0 < p \leq 1 \),
\[
\mathcal{H}^p \subset \{ u + i\mathcal{H}u : u \in \mathcal{D}^p, u \text{ real} \}.
\]

**Proof.** Let \( f = u + iv \in \mathcal{H}^p \) and fix \( t_0 > 0 \). Then
\[
\begin{align*}
u_{u}(x,t) & \equiv u(x, t + t_0) = K(\cdot, t + t_0) * u(x, t_0)(x), \\
u_{v}(x,t) & \equiv v(x, t + t_0) = K(\cdot, t + t_0) * v(x, t_0)(x).
\end{align*}
\]

Theorem 1 implies that \( u(\cdot, t + t_0) \in L^p \cap L^\infty \subset L^q \) for every \( q > 1 \), and since \( u_{t_0} + iv_{t_0} \in \mathcal{AH} \), it follows that \( u_{t_0} + iv_{t_0} \in \mathcal{H}^q \). Then using Theorem 3 we obtain \( v(\cdot, t + t_0) = \mathcal{H}u_{t_0}(\cdot, t + t_0) \) for every \( t > 0 \). Since \( t \) and \( t_0 \) are arbitrary, we conclude that for any \( s > 0 \), \( v(\cdot, s) = \mathcal{H}u(\cdot, s) \).

For \( F \in \mathcal{H}^p \), \( 0 < p < \infty \), we define
\[
\|F\|_\mathcal{H}^p = \sup_{t>0} \left( \int_{-\infty}^{\infty} |F(x,t)|^p \, dx \right)^{1/p};
\]

\( F \mapsto \|F\|_\mathcal{H}^p \) is a norm for \( 1 < p < \infty \) and \( F \mapsto \|F\|_\mathcal{H}^p \) is a \( p \)-norm for \( 0 < p < 1 \).

**Theorem 4.** \( \mathcal{H}^p \) is complete for every \( 0 < p < \infty \).

**Proof.** Let \( (F_n)_{n=1}^\infty \) be a Cauchy sequence in \( \mathcal{H}^p \), \( F_n = u_n + iv_n \). By estimates (1) and (10),
\[
\left| (u_n - u_m)(x, t) \right| \leq Ct^{1-2/p} \|F_n - F_m\|_{\mathcal{H}^p},
\]
where \( 0 < p < \infty \). Then \( (u_n)_{n=1}^\infty \) and \( (v_n)_{n=1}^\infty \) are temperature functions which converge uniformly on compact subsets of \( \mathbb{R}^2 \) to temperature functions \( u(x,t) \) and \( v(x,t) \), respectively. It suffices to show that \( u \) and \( v \) satisfy
condition (b) in the definition of \(A'\), since letting \(F = u + iv\) and \(\varepsilon > 0\) we get
\[
\left\| F_n(x, t) - F(x, t) \right\|_p dx \leq \liminf_{n \to \infty} \left\| F_n(x, t) - F_m(x, t) \right\|_p dx \leq \varepsilon
\]
for \(n\) arbitrarily large. The rest of the proof is a routine computation. \(\qed\)

The last part of this section is devoted to characterizing our \(H^p\) spaces. We denote by \(H^p_{\text{hol}}\) the classical Hardy spaces of holomorphic functions:
\[
H^p_{\text{hol}} = \left\{ F = u + iv : F \text{ is holomorphic on } \mathbb{R}^2 \right\}
\]
and
\[
\sup_{t \geq 0} \int_{-\infty}^{\infty} |F(x, t)|^p dx < \infty
\]
with the \(p\)-norm
\[
\|F\|_{H^p_{\text{hol}}} = \sup_{t \geq 0} \int_{-\infty}^{\infty} |F(x, t)|^p dx
\]
and the norm
\[
\|F\|_{H^p_{\text{hol}}} = \sup_{t \geq 0} \int_{-\infty}^{\infty} |F(x, t)|^p dx
\]
if \(0 < p < 1\)

It is well known that for \(1 < p < \infty\),
\[
(H^p_{\text{hol}}, \|\cdot\|_{H^p_{\text{hol}}}) \cong \left( \{ f + i\mathcal{H}f : f \in \text{Re } L^p \}, \|f\|_p + \|\mathcal{H}f\|_p \right).
\]

Theorem 3, [4, Th. 2(x)] and continuity of \(\mathcal{H}\) from \(L^p\) to \(L^p\) imply
\[
(H^p, \|\cdot\|_{H^p}) \cong \left( H^p_{\text{hol}}, \|\cdot\|_{H^p_{\text{hol}}} \right), \quad 1 < p < \infty.
\]

For \(0 < p \leq 1\), we define the space \(\text{Re } H^p\) whose elements are the boundary distributions \(\text{Re } F(x)\) corresponding to the functions \(F \in H^p\) with the \(p\)-norm \(\text{Re } F(x) \rightarrow \|F\|_{H^p}\). As in the holomorphic case (see [5, p. 236]), it can be shown that \(\text{Re } H^p \hookrightarrow \mathcal{S}'\).

In the holomorphic case, we denote by \(\text{Re } H^p_{\text{hol}}\) the analogous space.

Remark. We recall that we are viewing the elements of \(H^p_{\text{hol}}\) as harmonic functions \(u\) for which the vector function \(F(x, t)\) is uniformly in \(L^p\).

The classical result by Fefferman and Stein proved in [3, Th. 11] states that for \(0 < p < \infty\) and \(f \in \mathcal{S}'\), the following are equivalent:
\[
\begin{align*}
(a) \quad u^*(x) &= \sup_{\|\phi\|_p \leq 1} |\phi \ast u(y)| \in L^p \text{ for all } \phi \in S \text{ satisfying } \int_{-\infty}^{\infty} \phi = 1. \\
(b) \quad \text{The distribution } f \text{ arises as } f = \lim_{t \to 0} u(t, t) \text{ in } \mathcal{S}', \text{ where } u \in H^p_{\text{hol}}.
\end{align*}
\]
Moreover, \(\|u\|_{H^p_{\text{hol}}} \sim \|u^*\|_p\), where \(\sim\) means the standard equivalence of norms (\(p\)-norms).

In [10, Prop. 3, p. 123] the following characterization of \(\text{Re } H^p_{\text{hol}}\) is given:
If \(f \in \mathcal{S}'\) is restricted at infinity (that is, \(f \ast \varphi \in L^p\) for every \(\varphi \in S\) and for all \(r < \infty\) sufficiently large) and \(0 < p < \infty\), then \(f \in \text{Re } H^p_{\text{hol}}\) if and only if
\[
\sup_{t > 0} \|f \ast \varphi\|_p + \|\mathcal{H}f \ast \varphi\|_p \leq C < \infty,
\]
where \(\varphi \in S, \int_{-\infty}^{\infty} \varphi = 1, \varphi(t) = t^{-1} \varphi(x/t)\) and \(C\) is a constant. Of course, \(\mathcal{H}\) means the Hilbert transform of a distribution \(f\) that is restricted at infinity (see [10, p. 123]).

Before stating the main result of this paper, we need to prove the following lemma.

**Lemma 2.** Let \(f \in \mathcal{S}'\) and \(w(x, t) \equiv K(t, t) \ast f(x, t) \in \mathbb{R}^2\). If \(w^+(x) \equiv \sup_{y \geq 0} |w(x, t)| \in L^p\) then \(w^+(x) \equiv \sup_{y \geq 0} |w(y, t)| \in L^p\) and \(\|w^+\|_p \sim \|w^\|_p\) for \(0 < p < \infty\).

**Proof.** Let \(x \in \mathbb{R}\) and \(y(t) \in \mathbb{R}\). \(\{x, y\} : |x - y| < 1/2\). If \(R\) denotes the rectangle \(R(t, t) = \{y \in \mathbb{R}^2 : |y - t| < 1/2\}\), then according to Lemma 1 we have
\[
|w(y, t)| \leq \frac{C}{t^1/2} \int_{-\infty}^{\infty} |w(x, t)|^p dx dt'
\]
which implies \(\|w^+\|_p \sim \|w^\|_p\) for \(0 < p < \infty\).

The other inequality is immediate. \(\qed\)

**Theorem 5.** \(\text{Re } H^p = \text{Re } H^p_{\text{hol}}\). Moreover, \(H^p \cong H^p_{\text{hol}}\) for \(0 < p \leq 1\).

**Proof.** First, we will show that \(\text{Re } H^p = \text{Re } H^p_{\text{hol}}\).

Let \(f \in \text{Re } H^p\). Then there exists \(u + iv \in H^p_{\text{hol}}\) such that \(f = \lim_{t \to 0} u(t, t) \in \mathcal{S}'\). Fix \(t_0 > 0\). Set \(\mathcal{G}(x) \equiv \frac{1}{\sqrt{x^2 + t^2}} e^{-x^2/2}\). Then \(K(x, t) = \mathcal{G}(x, t)\), thus Corollary 1 implies \(u_{t_0}(t, t) = \mathcal{G}(x, t_0, t)\), \(u_{t_0}(t, t) = \mathcal{G}(x, t_0, t)\), and moreover
\[
\sup_{t_0 > 0} \left\| \mathcal{G}(x, t_0) \right\|_p + \left\| \mathcal{G}(x, t_0) \right\|_p < \infty.
\]
Since \(u(t, t_0) \in L^q\) for any \(q > 1\), it is a distribution restricted at infinity. Then [10, Prop. 3, p. 123] implies that \(u(t, t_0) \in \text{Re } H^p_{\text{hol}}\). Also, from
Lemma 2 and [3, Th. 11] we have
\[
\|G_{\mathcal{V}}(u, (t, 0))\|_{\mathcal{P}} \sim \|G^{+}(u, (t, 0))\|_{Re \mathcal{H}_{hol}} \sim \|u(\cdot, t_0)\|_{Re \mathcal{H}_{hol}}
\]
where
\[
G_{\mathcal{V}}^{+}(u(\cdot, t_0))(x) \equiv \sup_{|y| \leq \tau} |G_{\sqrt{2\tau}} * (u(\cdot, t_0))(y)|,
\]
\[
G^{+}(u(\cdot, t_0))(x) \equiv \sup_{t > 0} |G_{\sqrt{2t}} * (u(\cdot, t_0))(x)|.
\]
On the other hand, if we define
\[
F_{\mathcal{V}}(x, t) = P_{\mathcal{V}} * u(\cdot, t_0)(x) + i\mathcal{H}(P_{\mathcal{V}} * u(\cdot, t_0))(x)
\]
where \(P_{\mathcal{V}}\) is the Poisson kernel, we obtain a holomorphic function in \(\mathcal{H}_{hol}\) which satisfies \(F_{\mathcal{V}}(x, t) \to u(x, t_0) + i\mathcal{H}(P_{\mathcal{V}} * u(\cdot, t_0))\) as \(t \to 0\) a.e. on \(\mathbb{R}\); consequently (see [5, Cor. 1.2(c), p. 233]),
\[
\|F_{\mathcal{V}}(x, t_0)\|_{\mathcal{H}_{hol}} \sim \|u(\cdot, t_0) + iv(\cdot, t_0)\|_{\mathcal{P}}.
\]
Thus,
\[
\|u(\cdot, t_0)\|_{Re \mathcal{H}_{hol}} \sim \|F_{\mathcal{V}}(x, t_0)\|_{Re \mathcal{H}_{hol}} \leq C \|u(\cdot, t_0) + iv(\cdot, t_0)\|_{\mathcal{P}} \leq C \|f\|_{Re \mathcal{H}_{P}},
\]
Combining (12) and (13) we get
\[
\|G_{\mathcal{V}}^{+}(u(\cdot, t_0))\|_{Re \mathcal{H}_{hol}} \leq C \|u(\cdot, t_0)\|_{Re \mathcal{H}_{hol}} \leq C \|f\|_{Re \mathcal{H}_{P}}
\]
and since \(G_{\mathcal{V}}^{+}(f) = \lim_{t_0 \to 0} G_{\mathcal{V}}^{+}(u(\cdot, t_0))\) a.e. on \(\mathbb{R}\), an application of Fatou’s Lemma gives us
\[
\|G_{\mathcal{V}}^{+}(f)\|_{P} \leq C \|f\|_{Re \mathcal{H}_{P}},
\]
and hence
\[
\|f\|_{Re \mathcal{H}_{hol}} \leq C \|f\|_{Re \mathcal{H}_{P}}.
\]
This shows that \(f \in Re \mathcal{H}_{hol}\).

Conversely, let \(f \in Re \mathcal{H}_{hol}\). There exists \(u + iv \in \mathcal{H}_{hol}\) such that \(f = \lim_{t_0 \to 0} u(\cdot, t_0)\) in \(S'\); in fact, \(f = \lim_{t_0 \to 0} u(\cdot, t_0) + i\mathcal{H}(u_{1}(\cdot, t_0))\) in \(Re \mathcal{H}_{hol}\). From [10, Prop. 3, p. 123] we have
\[
\sup_{t > 0} \|f * G_{\sqrt{2t}}\|_{P} \leq \|\mathcal{H}(f * G_{\sqrt{2t}})\|_{P} \leq C < \infty.
\]
Consequently, the function \(u_{1}(x, t) = K(\cdot, t) * f(x)\) belongs to \(\mathcal{H}_{P}\). Moreover, \(u_{1}(x, t + t_0) = K(\cdot, t) * u_{1}(x, t_0)\) for all \(t, t_0 > 0\) and as \(u_{1}(\cdot, t_0) \in L^{P} \cap L^{\infty} \subset L^{q}\) for every \(q > 1\), it follows that \(\mathcal{H}u_{1}(\cdot, t_0) \in L^{q}\). This implies that \(u_{1}(x, t + t_0) + iv_{2}(x, t + t_0) \in \mathcal{A}\) where \(v_{2}(x, t + t_0) \equiv K(\cdot, t) * u_{1}(\cdot, t_0)(x)\). Moreover, \(u_{1} + iv_{2} \in \mathcal{H}\) because \(u_{2}(x, t + t_0) = K(\cdot, t) * \mathcal{H}f(x)\). Now, \(u_{1}(\cdot, t)\) converges to an element in \(Re \mathcal{H}_{P}\) and also \(u_{1}(\cdot, t) \to f \in S'\) as \(t \to 0\), and since \(Re \mathcal{H}_{P} \hookrightarrow S'\) we see that \(f \in Re \mathcal{H}_{P}\).