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Spaces of holomorphic mappings on Banach spaces with a Schauder basis

by

JORGE MUJICA (Campinas)

Abstract. We show that if U is a balanced open subset of a separable Banach space with the bounded approximation property, then the space $\mathcal{H}(U)$ of all holomorphic functions on U, with the Nachbin compact-ported topology, is always bornological.

Introduction. Let E be a complex Banach space, and let $\mathcal{H}(U)$ denote the vector space of all holomorphic functions on an open subset U of E. Let τ_ω denote the compact-ported topology on $\mathcal{H}(U)$ introduced by Nachbin [18], and let τ_δ denote the bornological topology on $\mathcal{H}(U)$ introduced by Coeuré [3] and Nachbin [19], [20]. τ_δ is always the bornological topology associated with τ_ω, and the question as to whether these topologies coincide was mentioned explicitly by Nachbin in [19], [20], but its significance was implicit also in the works of Coeuré [3] and Dineen [5], because of its connection with the study of holomorphic continuation.

The first partial answers to this question were given by Dineen, who proved in [7] that $\tau_\omega \neq \tau_\delta$ on $\mathcal{H}(E)$ when $E = l^\infty$, whereas he proved in [8] that $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$ whenever U is a balanced open subset of a Banach space with an unconditional Schauder basis. Shortly afterwards Coeuré [4] modified Dineen's proof to show that $\tau_\omega = \tau_\delta$ on $\mathcal{H}(E)$ whenever E is a homogeneous Banach space in the sense of Katznelson's book [14]. Homogeneous Banach spaces include the space $L^1[0, 2\pi]$, which, by a result of Pełczyński [23], does not have an unconditional Schauder basis.

In this paper we extend Dineen's result by proving that $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$ whenever U is a balanced open subset of a Banach space with an arbitrary Schauder basis. By combining this result with a result of Johnson *et al.* [12] and Pełczyński [24], it follows that $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$ whenever U is a balanced open subset of a separable Banach space with the bounded approximation property. And this extends also Coeuré's result.

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Our proof follows the pattern of Dineen's original proof. Dineen's proof is extremely elaborate, and is so tightly packed, that gives the impression of not leaving room for improvement. Dineen's proof relies heavily on some estimates of the relative sizes of certain open sets, denoted by $B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}$. Dineen obtained those estimates by using some properties of unconditional Schauder bases, and a cursory examination of his proof gives the impression that the hypothesis of an unconditional basis is essential for his proof. However, a more careful analysis shows that it is still possible to obtain suitable estimates on the sets $B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}$ in Banach spaces with an arbitrary Schauder basis. The estimates obtained in this more general setting are not as good as those obtained in the case of an unconditional basis, but they are still good enough to make the proof work.

Dineen's original proof in [8] is very difficult to follow, and it is not easier to follow in his book [9] either. I hope that the way the proof is presented here, it will be more readable.

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1. Notation and terminology. $\mathbb{N}$ denotes the set of all strictly positive integers, whereas $\mathbb{N}_0$ denotes the set $\mathbb{N}\cup\{0\}$. The letters E and F always represent complex Banach spaces. If $m\in\mathbb{N}_0$ then $\mathcal{L}^s(^mE;F)$ denotes the Banach space of all symmetric, continuous, F-valued m-linear mappings on E^m, whereas $\mathcal{P}(^mE;F)$ denotes the Banach space of all continuous, F-valued m-homogeneous polynomials on E. If U is an open subset of E, then $\mathcal{H}(U;F)$ denotes the vector space of all F-valued holomorphic mappings on U. When $F=\mathbb{C}$ we write $\mathcal{L}^s(^mE)$, $\mathcal{P}(^mE)$ and $\mathcal{H}(U)$ instead of $\mathcal{L}^s(^mE;\mathbb{C})$, $\mathcal{P}(^mE;\mathbb{C})$ and $\mathcal{H}(U;\mathbb{C})$. Given a mapping $f:U\to F$ and a set $A\subset U$, we shall set $\|f\|_A=\sup_{x\in A}\|f(x)\|$.

A seminorm p on $\mathcal{H}(U;F)$ is said to be *ported* by a compact set $K\subset U$ if for each open set V with $K\subset V\subset U$, there is $c>0$ such that $p(f)\le c\|f\|_V$ for every $f\in H(U;F)$. The topology τ_ω on $\mathcal{H}(U;F)$ is defined by all such seminorms.

The topology τ_δ on $\mathcal{H}(U;F)$ is defined by all seminorms p such that, for each countable, open cover $(V_n)_{n=1}^\infty$ of U, there are $N\in\mathbb{N}$ and $c>0$ such that $p(f)\le c\|f\|_{\bigcup_{n=1}^N V_n}$ for every $f\in\mathcal{H}(U;F)$.

We refer to the books of Dineen [9] or the author [17] for background information on infinite-dimensional complex analysis.

2. Preparatory lemmas. Throughout this section E denotes a Banach space with a monotone, normalized, Schauder basis (e_n). Let (z_n) denote

the sequence of coordinate functionals, and let (T_n) denote the sequence of canonical projections, that is, $T_nx=\sum_{j=1}^n z_j(x)e_j$. Then

$$\Big\|\sum_{j=p}^q z_j(x)e_j\Big\|\le 2\|x\|$$

whenever $x\in E$ and $p\le q$. Let B_E denote the open unit ball of E.

Let $(q_i)_{i=1}^\infty$ be a strictly increasing sequence in $\mathbb{N}$, and let $(\beta_i)_{i=1}^\infty$ be a decreasing sequence of strictly positive numbers such that $\sum_{i=1}^\infty\beta_i<\infty$. Consider the following sets:

$$K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}=\Big\{\sum_{i=1}^j\beta_i\zeta_i\sum_{n=q_{i-1}+1}^{q_i}z_n(x)e_n : x\in B_E,\ |\zeta_i|=1\Big\}\ (\text{where } q_0=0),$$

$$K^{(q_i)}_{(\beta_i)}=\Big\{\sum_{i=1}^\infty\beta_i\zeta_i\sum_{n=q_{i-1}+1}^{q_i}z_n(x)e_n : x\in B_E,\ |\zeta_i|=1\Big\},$$

$$B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}=\Big\{\sum_{i=1}^j\beta_i\zeta_i\sum_{n=q_{i-1}+1}^{q_i}z_n(x)e_n+\beta_{j+1}\zeta_{j+1}\sum_{n=q_j+1}^{\infty}z_n(x)e_n : x\in B_E,\ |\zeta_i|=1\Big\}.$$

2.1. LEMMA. (a) $B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}\subset K^{(q_i)}_{(\beta_i)}+2\beta_{j+1}B_E$.

(b) *The set* $K^{(q_i)}_{(\beta_i)}$ *is relatively compact.*

Proof. (a) Clearly

$$B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}\subset K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}+2\beta_{j+1}B_E.$$

On the other hand, since $T_{q_j}x\in B_E$ whenever $x\in B_E$, we see that $K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}\subset K^{(q_i)}_{(\beta_i)}$ and (a) follows.

(b) Since $K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}$ lies in a finite-dimensional subspace and

$$K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}\subset 2\Big(\sum_{i=1}^j\beta_i\Big)B_E,$$

we see that $K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}$ is relatively compact. On the other hand,

$$K^{(q_i)}_{(\beta_i)}\subset K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}+2\Big(\sum_{i=j+1}^\infty\beta_i\Big)B_E.$$

Thus, since each $K^{q_1\ldots q_j}_{\beta_1\ldots\beta_j}$ is precompact, so is $K^{(q_i)}_{(\beta_i)}$.

2.2. Lemma. *We have*
$$B_E \subset 2\frac{j+1}{\beta_{j+1}} B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}.$$

Proof. For each $\lambda = (\lambda_1, \ldots, \lambda_{j+1}) \in \mathbb{C}^{j+1}$ we define $A_\lambda \in \mathcal{L}(E;E)$ by
$$A_\lambda x = \sum_{i=1}^{j} \lambda_i \sum_{n=q_{i-1}+1}^{q_i} z_n(x)e_n + \lambda_{j+1} \sum_{n=q_j+1}^{\infty} z_n(x)e_n.$$
Then
$$\|A_\lambda\| \le 2\sum_{i=1}^{j+1} |\lambda_i|.$$
If $\beta = (\beta_1, \ldots, \beta_{j+1})$ and $\alpha = (\alpha_1, \ldots, \alpha_{j+1})$, where $\beta_i\alpha_i = 1$ for every i, then $A_\beta A_\alpha = I$ and
$$A_\beta(B_E) \subset B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}.$$
It follows that
$$B_E \subset \|A_\alpha\| A_\beta(B_E) \subset 2\Big(\sum_{i=1}^{j+1} \frac{1}{\beta_i}\Big) B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}} \subset 2\frac{j+1}{\beta_{j+1}} B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}.$$

For each $q \in \mathbb{N}$ let E_1^q be the subspace of E generated by $\{e_n : n \le q\}$, and let E^∞_{q+1} be the closed subspace of E generated by $\{e_n : n > q\}$. Thus we have a canonical decomposition $E = E_1^q \oplus E^\infty_{q+1}$.

Let $P \in \mathcal{P}(^mE;F)$, $P = \widehat{A}$, with $A \in \mathcal{L}^s(^mE;F)$. Given $q \in \mathbb{N}$ and $0 \le l \le m$, let $P_l^q \in \mathcal{P}(^mE;F)$ be defined by
$$P_l^q(z) = \binom{m}{l} A x^{m-l} y^l$$
for every $z = x + y \in E_1^q \oplus E^\infty_{q+1}$. Thus
$$P = \sum_{l=0}^{m} P_l^q$$
and it follows from the Cauchy integral formula that
$$\|P_l^q(x+y)\| \le \sup_{|\zeta|=1} \|P(x+\zeta y)\|.$$

2.3. Lemma. *Let $P \in \mathcal{P}(^mE;F)$ and $0 \le l \le m$. Then*
$$\|P_l^{q_{j+1}}\|_{B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}} \le \Big(\frac{\beta_{j+1}}{\beta_{j+2}}\Big)^l \|P\|_{B^{q_1\ldots q_{j+1}}_{\beta_1\ldots\beta_{j+2}}}.$$

Proof. Given $z \in B^{q_1\ldots q_j}_{\beta_1\ldots\beta_{j+1}}$, write $z = x + y$, with $x \in E_1^{q_{j+1}}$ and $y \in E^\infty_{q_{j+1}+1}$. Then
$$\|P_l^{q_{j+1}}(z)\| = \binom{m}{l}\|Ax^{m-l}y^l\| = \Big(\frac{\beta_{j+1}}{\beta_{j+2}}\Big)^l \binom{m}{l} \Big\| Ax^{m-l}\Big(\frac{\beta_{j+2}}{\beta_{j+1}}y\Big)^l\Big\|$$
$$= \Big(\frac{\beta_{j+1}}{\beta_{j+2}}\Big)^l \Big\| P_l^{q_{j+1}}\Big(x + \frac{\beta_{j+2}}{\beta_{j+1}}y\Big)\Big\| \le \Big(\frac{\beta_{j+1}}{\beta_{j+2}}\Big)^l \|P_l^{q_{j+1}}\|_{B^{q_1\ldots q_{j+1}}_{\beta_1\ldots\beta_{j+2}}}$$
$$\le \Big(\frac{\beta_{j+1}}{\beta_{j+2}}\Big)^l \|P\|_{B^{q_1\ldots q_{j+1}}_{\beta_1\ldots\beta_{j+2}}}.$$

2.4. Lemma. *Let K be a compact subset of E. Then for each $\varepsilon > 0$ there are $c > 0$ and $q_\varepsilon \in \mathbb{N}$ such that*
$$\|P_l^q\|_K \le c^m \varepsilon^l \|P\|_{B_E}$$
whenever $P \in \mathcal{P}(^mE;F)$, $m \in \mathbb{N}$, $0 \le l \le m$ and $q \ge q_\varepsilon$.

Proof. For every $z \in E$ and $q \in \mathbb{N}$ we may write $z = x_q + y_q$, with $x_q \in E_1^q$ and $y_q \in E^\infty_{q+1}$. Since the operators T_q converge to the identity uniformly on compact sets, given $\varepsilon > 0$ we can find $q_\varepsilon \in \mathbb{N}$ such that $\|y_q\| < \varepsilon$ for every $z \in K$ and $q \ge q_\varepsilon$. If $P \in \mathcal{P}(^mE;F)$, then
$$\|P_l^q(z)\| = \binom{m}{l}\|Ax_q^{m-l}y_q^l\|$$
$$= (\|z\|+1)^m\varepsilon^l \binom{m}{l}\Big\| A\Big(\frac{x_q}{\|z\|+1}\Big)^{m-l}\Big(\frac{y_q}{\varepsilon(\|z\|+1)}\Big)^l\Big\|$$
$$= (\|z\|+1)^m\varepsilon^l \Big\| P_l^q\Big(\frac{x_q}{\|z\|+1} + \frac{y_q}{\varepsilon(\|z\|+1)}\Big)\Big\|$$
$$\le (\|z\|+1)^m\varepsilon^l \|P_l^q\|_{B_E} \le (\|z\|+1)^m\varepsilon^l\|P\|_{3B_E}.$$
Thus it suffices to take $c = 3\sup_{z\in K}(\|z\|+1)$. Observe that c is independent of ε.

For the convenience of the reader we include a proof of the following known lemma (see [6, Lemma 3]), valid on any Banach space E.

2.5. Lemma. *Let $P_m \in \mathcal{P}(^mE;F)$ $(m \in \mathbb{N}_0)$ be such that*

(2.1) $$\lim_{m\to\infty} \|P_m\|_K^{1/m} = 0$$

for each compact set $K \subset E$. Then $\sum_{m=0}^\infty P_m \in \mathcal{H}(E;F)$ and

(2.2) $$\lim_{m\to\infty} p(P_m)^{1/m} = 0$$

for each continuous seminorm p on $(\mathcal{H}(E;F), \tau_\delta)$.

Proof. It follows from (2.1) and the classical Cauchy–Hadamard formula that $\sum_{m=0}^{\infty} |\lambda|^m \|P_m\|_K < \infty$ for every $\lambda \in \mathbb{C}$. Thus the series $\sum_{m=0}^{\infty} \lambda^m P_m$ converges uniformly on compact sets to a mapping $f_\lambda \in \mathcal{H}(E;F)$ for every $\lambda \in \mathbb{C}$. Hence the sequence $(\lambda^m P_m)$ is bounded in $(\mathcal{H}(E;F), \tau_\delta)$ for every $\lambda \in \mathbb{C}$. Consequently, $\sup_m |\lambda|^m p(P_m) < \infty$ for every $\lambda \in \mathbb{C}$. Therefore $\sum_{m=0}^{\infty} |\lambda|^m p(P_m) < \infty$ for every $\lambda \in \mathbb{C}$ and (2.2) follows from the classical Cauchy–Hadamard formula again.

3. The main result

3.1. THEOREM. *Let E be a Banach space with a Schauder basis. Then $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U;F)$ for every balanced, open set $U \subset E$ and every Banach space F.*

Proof. Without loss of generality we may assume that E has a monotone, normalized, Schauder basis, and will accordingly use the notation and terminology from the preceding section. We will first give the proof in detail in the case $U = E$, which is technically simpler, and will afterwards sketch the proof in the case $U \neq E$.

A. *Case $U = E$.* Let p be a continuous seminorm on $(\mathcal{H}(E;F), \tau_\delta)$. Since $E = \bigcup_{n=1}^{\infty} nB_E$, there are $N \in \mathbb{N}$ and $c_1 > 0$ such that

$$p(f) \le c_1 \|f\|_{NB_E}$$

for every $f \in \mathcal{H}(E;F)$. Hence

$$p(P) \le c_1 N^m \|P\|_{B_E} \le c_1 N^m \|P\|_{B^1_{11}} \tag{3.1}$$

for every $P \in \mathcal{P}(^mE;F)$ and $m \in \mathbb{N}_0$. Let (β_i) be a decreasing sequence of strictly positive numbers with $\beta_1 = \beta_2 = 1$ and $\sum_{i=1}^{\infty} \beta_i < \infty$. Let (γ_i) be a sequence of numbers with $\gamma_1 = 1$, $\gamma_i > 1$ for every $i \ge 2$ and $\prod_{i=1}^{\infty} \gamma_i = \gamma < \infty$. We will show the existence of a strictly increasing sequence (q_i) in $\mathbb{N}$, and a sequence (c_i) of strictly positive numbers such that

$$p(P) \le c_i (N\gamma_1 \dots \gamma_i)^m \|P\|_{B^{q_1 \dots q_i}_{\beta_1 \dots \beta_{i+1}}} \tag{3.2}$$

for every $P \in \mathcal{P}(^mE;F)$, $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$. The sequences (q_i) and (c_i) will be found by induction. (3.1) shows that $q_1 = 1$ and c_1 satisfy (3.2) for $i = 1$. Assuming that we have found $q_1, \dots, q_j$ and $c_1, \dots, c_j$ that satisfy (3.2) for $i = j$, we will show the existence of q_{j+1} and c_{j+1} that satisfy (3.2) for $i = j+1$. Otherwise for each $k \in \mathbb{N}$ there is $P_k \in \mathcal{P}(^{m_k}E;F)$ such that

$$p(P_k) > k(N\gamma_1 \dots \gamma_{j+1})^{m_k} \|P_k\|_{B^{q_1 \dots q_j, q_j+k}_{\beta_1 \dots \beta_{j+2}}}. \tag{3.3}$$

We distinguish two cases.

(a) First assume that the sequence (m_k) is bounded. By passing to a subsequence we may assume that $m_k = m$ for every k. Then set

$$Q_k = \frac{P_k}{(N\gamma_1 \dots \gamma_{j+1})^m \|P_k\|_{B^{q_1 \dots q_j, q_j+k}_{\beta_1 \dots \beta_{j+2}}}}.$$

It follows from Lemma 2.2 that

$$\|Q_k\|_{B_E} \le (N\gamma_1 \dots \gamma_{j+1})^{-m} 2^m \left(\frac{j+2}{\beta_{j+2}}\right)^m$$

and hence (Q_k) is bounded in $\mathcal{P}(^mE;F)$, and thus bounded in $(\mathcal{H}(E;F), \tau_\delta)$. On the other hand, it follows from (3.3) that $p(Q_k) > k$, a contradiction.

(b) Next assume that the sequence (m_k) is unbounded. Then by passing to a subsequence we may assume that the sequence (m_k) is strictly increasing. Now each P_k admits a decomposition

$$P_k = \sum_{l=0}^{m_k} P^{q_j+k}_{kl},$$

where the polynomials $P^{q_j+k}_{kl} = (P_k)^{q_j+k}_l$ were defined before Lemma 2.3. Then by (3.3) for each k there exists l_k, with $0 \le l_k \le m_k$, such that

$$p(P^{q_j+k}_{kl_k}) > \frac{k}{m_k+1}(N\gamma_1 \dots \gamma_{j+1})^{m_k} \|P_k\|_{B^{q_1 \dots q_j, q_j+k}_{\beta_1 \dots \beta_{j+2}}}. \tag{3.4}$$

We now distinguish two subcases.

(i) First assume that $\lim_{k\to\infty} l_k/m_k = 0$. Then set

$$R_k = \frac{P^{q_j+k}_{kl_k}}{(N\gamma_1 \dots \gamma_j)^{m_k} \|P^{q_j+k}_{kl_k}\|_{B^{q_1 \dots q_j}_{\beta_1 \dots \beta_{j+1}}}}.$$

It follows from the induction hypothesis that $p(R_k) \le c_j$ and therefore

$$\limsup_{k\to\infty} p(R_k)^{1/m_k} \le 1.$$

On the other hand, it follows from (3.4) that

$$p(R_k) > \frac{k}{m_k+1} \gamma_{j+1}^{m_k} \frac{\|P_k\|_{B^{q_1 \dots q_j, q_j+k}_{\beta_1 \dots \beta_{j+2}}}}{\|P^{q_j+k}_{kl_k}\|_{B^{q_1 \dots q_j}_{\beta_1 \dots \beta_{j+1}}}},$$

and using Lemma 2.3 we get

$$p(R_k) > \frac{k}{m_k+1} \gamma_{j+1}^{m_k} \left(\frac{\beta_{j+2}}{\beta_{j+1}}\right)^{l_k}.$$

Since $\lim_{k\to\infty} l_k/m_k = 0$, it follows that
$$\limsup_{k\to\infty} p(R_k)^{1/m_k} \geq \gamma_{j+1} > 1,$$
a contradiction.

(ii) Next assume that $\limsup_{k\to\infty} l_k/m_k = \delta > 0$. Then by passing to a subsequence we may assume that $\lim_{k\to\infty} l_k/m_k = \delta > 0$. Let K be a compact subset of E. Then by Lemma 2.4 for each $\varepsilon > 0$ there are $c > 0$ and $k_\varepsilon \in \mathbb{N}$ such that
$$\|P_{kl_k}^{q_j+k}\|_K \leq c^{m_k}\varepsilon^{l_k}\|P_k\|_{B_E}$$
for every $k \geq k_\varepsilon$. By using Lemma 2.2 we get
$$\|P_{kl_k}^{q_j+k}\|_K \leq c^{m_k}\varepsilon^{l_k}2^{m_k}\left(\frac{j+2}{\beta_{j+2}}\right)^{m_k}\|P_k\|_{B_{\beta_1\ldots\beta_{j+2}}^{q_1\ldots q_j,q_j+k}}.$$
Now set
$$S_k = \frac{P_{kl_k}^{q_j+k}}{(N\gamma_1\ldots\gamma_{j+1})^{m_k}\|P_k\|_{B_{\beta_1\ldots\beta_{j+2}}^{q_1\ldots q_j,q_j+k}}}.$$
Then
$$\|S_k\|_K \leq (N\gamma_1\ldots\gamma_{j+1})^{-m_k}\left(2c\frac{j+2}{\beta_{j+2}}\right)^{m_k}\varepsilon^{l_k},$$
and since $\lim_{k\to\infty} l_k/m_k = \delta$, it follows that
$$\limsup_{k\to\infty}\|S_k\|_K^{1/m_k} \leq (N\gamma_1\ldots\gamma_{j+1})^{-1}2c\frac{j+2}{\beta_{j+2}}\varepsilon^\delta.$$
As $\varepsilon > 0$ was arbitrary, we have
$$\limsup_{k\to\infty}\|S_k\|_K^{1/m_k} = 0$$
for every compact set $K \subset E$. By Lemma 2.5, $\sum_{k=1}^\infty S_k \in \mathcal{H}(E;F)$ and
$$\lim_{k\to\infty} p(S_k)^{1/m_k} = 0.$$
On the other hand, it follows from (3.4) that $p(S_k) > k/(m_k+1)$, and therefore
$$\limsup_{k\to\infty} p(S_k)^{1/m_k} \geq 1,$$
a contradiction. Thus there are sequences (q_i) and (c_i) satisfying (3.2) for every i.

We now prove that p is ported by the compact set $L = \mathrm{clos}(\gamma^2 N K_{(\beta_i)}^{(q_i)})$. Indeed, given $\varepsilon > 0$, choose $j \in \mathbb{N}$ such that $2\gamma^2 N\beta_{j+1} < \varepsilon$. Then by Lemma 2.1,
$$\gamma^2 N B_{\beta_1\ldots\beta_{j+1}}^{q_1\ldots q_j} \subset \gamma^2 N(K_{(\beta_i)}^{(q_i)} + 2\beta_{j+1}B_E) \subset L + \varepsilon B_E.$$

Let $\sum_{m=0}^\infty P_m$ be the Taylor series at the origin of a mapping $f \in \mathcal{H}(E;F)$. Then
$$p\Big(\sum_{m=0}^M P_m\Big) \leq \sum_{m=0}^M p(P_m) \leq \sum_{m=0}^\infty c_j(N\gamma)^m\|P_m\|_{B_{\beta_1\ldots\beta_{j+1}}^{q_1\ldots q_j}}$$
$$\leq c_j\sum_{m=0}^\infty \gamma^{-m}\|P_m\|_{\gamma^2 NB_{\beta_1\ldots\beta_{j+1}}^{q_1\ldots q_j}} \leq \frac{c_j\gamma}{\gamma-1}\|f\|_{L+\varepsilon B_E}.$$
Thus
$$p(f) = \lim_{M\to\infty} p\Big(\sum_{m=0}^M P_m\Big) \leq \frac{c_j\gamma}{\gamma-1}\|f\|_{L+\varepsilon B_E},$$
and the proof in the case $U = E$ is complete.

B. *Case* $U \neq E$. Since U is a Lindelöf space, and since $\bigcup_{q=1}^\infty E_1^q$ is dense in E, we can easily find a sequence (a_n) in $U \cap (\bigcup_{q=1}^\infty E_1^q)$, and sequences (ϱ_n) and (r_n), with $\varrho_n > 1$ and $r_n > 0$, such that
$$U = \bigcup_{n=1}^\infty (a_n + r_nB_E) \quad\text{and}\quad \varrho_n^2(A_n + 7r_nB_E) \subset U$$
for every n, where $A_n = \{\lambda a_n : |\lambda| \leq 1\}$.

Let p be a continuous seminorm on $(\mathcal{H}(U;F),\tau_\delta)$. Then there are $N \in \mathbb{N}$ and $c_1 > 0$ such that
$$p(f) \leq c_1\|f\|_{\bigcup_{n=1}^N(a_n+r_nB_E)} \leq c_1\sup_{n\leq N}\|f\|_{A_n+r_nB_E}$$
for every $f \in \mathcal{H}(U;F)$. Hence

$$p(P) \leq c_1\sup_{n\leq N} r_n^m\|P\|_{r_n^{-1}A_n+B_E} \leq c_1\sup_{n\leq N} r_n^m\|P\|_{r_n^{-1}A_n+B_{11}^{q_1}} \tag{3.5}$$

for every $P \in \mathcal{P}(^mE;F)$ and $m \in \mathbb{N}_0$, where $q_1 \in \mathbb{N}$ is chosen so that $a_n \in E_1^{q_1}$ for every $n \leq N$. Let (β_i) be a decreasing sequence of strictly positive numbers with $\beta_1 = \beta_2 = 1$ and $\sum_{i=1}^\infty \beta_i < 3$. Let (γ_i) be a sequence of numbers with $\gamma_1 = 1$, $\gamma_i > 1$ for every $i \geq 2$ and $\prod_{i=1}^\infty \gamma_i = \gamma < \min_{n\leq N}\varrho_n$. We claim that there exist a strictly increasing sequence (q_i) in $\mathbb{N}$ and a sequence (c_i) of strictly positive numbers such that

$$p(P) \leq c_i\sup_{n\leq N}(r_n\gamma_1\ldots\gamma_i)^m\|P\|_{r_n^{-1}A_n+B_{\beta_1\ldots\beta_{i+1}}^{q_1\ldots q_i}} \tag{3.6}$$

for every $P \in \mathcal{P}(^mE;F)$, $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Inequality (3.5) shows that q_1 and c_1 satisfy (3.6) for $i = 1$, and the proof of the induction step can be achieved by following the footsteps of the corresponding proof in the case $U = E$. Instead of the estimate given by Lemma 2.3, one should use the estimate

$$\|P_l^{q_j+1}\|_{A+B^{q_1\dots q_j}_{\beta_1\dots\beta_{j+1}}} \le \left(\frac{\beta_{j+1}}{\beta_{j+2}}\right)^l \|P\|_{A+B^{q_1\dots q_{j+1}}_{\beta_1\dots\beta_{j+2}}},$$

which is valid for each balanced set $A \subset E_1^{q_j}$. This explains our choice of q_1.

Once (3.6) is established, we consider the compact set

$$L = \operatorname{clos} \bigcup_{n=1}^{N} \gamma^2(A_n + r_n K^{(q_i)}_{(\beta_i)}).$$

Since $K^{(q_i)}_{(\beta_i)} \subset 2(\sum_{i=1}^{\infty} \beta_i) B_E \subset 6B_E$, we see that

$$\gamma^2(A_n + r_n K^{(q_i)}_{(\beta_i)}) \subset \varrho_n^2(A_n + 6r_n B_E),$$

and therefore L is a compact subset of U. We will prove that p is ported by L. Indeed, given $\varepsilon > 0$ choose $j \in \mathbb{N}$ such that $2\gamma^2\beta_{j+1}\sup_{n\le N} r_n < \varepsilon$. Then by Lemma 2.1,

$$\gamma^2(A_n + r_n B^{q_1\dots q_j}_{\beta_1\dots\beta_{j+1}}) \subset \gamma^2(A_n + r_n K^{(q_i)}_{(\beta_i)} + 2\beta_{j+1} r_n B_E) \subset L + \varepsilon B_E.$$

Let $\sum_{m=0}^{\infty} P_m$ be the Taylor series at the origin of a mapping $f \in \mathcal{H}(U;F)$. Then

$$p\Big(\sum_{m=0}^{M} P_m\Big) \le \sum_{m=0}^{M} p(P_m) \le \sum_{m=0}^{\infty} c_j \sup_{n\le N} (r_n\gamma)^m \|P_m\|_{r_n^{-1}A_n + B^{q_1\dots q_j}_{\beta_1\dots\beta_{j+1}}}$$
$$\le c_j \sum_{m=0}^{\infty} \gamma^{-m} \sup_{n\le N} \|P_m\|_{\gamma^2(A_n + r_n B^{q_1\dots q_j}_{\beta_1\dots\beta_{j+1}})} \le \frac{c_j\gamma}{\gamma-1}\|f\|_{L+\varepsilon B_E}.$$

Thus

$$p(f) = \lim_{M\to\infty} p\Big(\sum_{m=0}^{M} P_m\Big) \le \frac{c_j\gamma}{\gamma-1}\|f\|_{L+\varepsilon B_E},$$

and the proof of the theorem is complete.

3.2. PROPOSITION. *Suppose E is topologically isomorphic to a complemented subspace of a Banach space G. If $\tau_\omega = \tau_\delta$ on $\mathcal{H}(V;F)$ for every balanced, open set $V \subset G$, then $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U;F)$ for every balanced, open set $U \subset E$.*

Proof. Let $J \in \mathcal{L}(E;G)$ and $P \in \mathcal{L}(G;E)$ be such that $P \circ J =$ identity. Given a balanced, open set $U \subset E$, consider the mappings

$$P^* : f \in \mathcal{H}(U;F) \to f \circ P \in \mathcal{H}(P^{-1}(U);F),$$
$$J^* : g \in \mathcal{H}(P^{-1}(U);F) \to g \circ J \in \mathcal{H}(U;F).$$

Then $J^* \circ P^* =$ identity, and the desired conclusion follows from the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{H}(U;F),\tau_\omega) & \xrightarrow{\text{id}} & (\mathcal{H}(U;F),\tau_\delta) \\ P^*\downarrow & & \downarrow J^* \\ (\mathcal{H}(P^{-1}(U);F),\tau_\omega) & \xrightarrow{\text{id}} & (\mathcal{H}(P^{-1}(U);F),\tau_\delta) \end{array}$$

By a result obtained independently by Johnson *et al.* [12] and Pełczyński [24], every separable Banach space with the bounded approximation property is topologically isomorphic to a complemented subspace of a Banach space with a Schauder basis. Thus Theorem 3.1 and Proposition 3.2 yield the following corollary.

3.3. COROLLARY. *Let E be a separable Banach space with the bounded approximation property. Then $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U;F)$ for every balanced open set $U \subset E$ and every Banach space F.*

We remark that the proof of Theorem 3.1 works equally well, with the obvious modifications, in the case of Banach spaces with a finite-dimensional Schauder decomposition. But since that would not take us beyond Corollary 3.3 anyway, we preferred to restrict Theorem 3.1 to the more familiar case of Banach spaces with a Schauder basis.

Let us remark that Chae [2] has conjectured that $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$ whenever U is an open subset of a separable Banach space. On the other hand, Aron *et al.* [1] have proposed a new approach to this problem by studying the behaviour of the topologies τ_ω and τ_δ with respect to holomorphic functions defined on quotient spaces. Anyway, neither Chae [2] nor Aron *et al.* [1] have exhibited any additional example of a Banach space where the problem would have a positive or a negative solution.

It is interesting to note that separable Banach spaces with the bounded approximation property form the largest class among Banach spaces for which several important problems in infinite-dimensional complex analysis are known to have positive solutions. Indeed, by a result of Gruman and Kiselman [10], extended by Hervier [11] and Noverraz [21], the Levi problem has a positive solution within this class, whereas, by a result of Josefson [13], it has a negative solution in the space $c_0(I)$, with I uncountable. Furthermore, Noverraz [22], Schottenloher [25] and the author [15], [16] have obtained various versions of the Oka–Weil approximation theorem within this class.

After this paper was written I learned that Theorem 3.1 and Corollary 3.3 were obtained independently and at the same time by Seán Dineen. His proof is entirely different from mine. His proof is based on a refinement of the methods in his paper *Holomorphic functions and Banach-nuclear decompositions of Fréchet spaces*, Studia Math. 113 (1995), 43–54. Dineen will publish his proof in his forthcoming book *Complex Analysis on Infinite Dimensional Spaces*.

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Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postal 6065
13081-970 Campinas, SP
Brazil
E-mail: mujica@ime.unicamp.br

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