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Dept. Matemàtica Aplicada i Anàlisi
Univ. de Barcelona
08071 Barcelona, Spain
E-mail: raposo@cerber.mat.ub.es

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Semi-Browder operators and perturbations

by

VLADIMIR RAKOČEVIĆ (Niš)

Abstract. An operator in a Banach space is called upper (resp. lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (resp. descent). An operator in a Banach space is called semi-Browder if it is upper semi-Browder or lower semi-Browder. We prove the stability of the semi-Browder operators under commuting Riesz operator perturbations. As a corollary we get some results of Grabiner [6], Kaashoek and Lay [8], Lay [11], Rakočević [15] and Schechter [16].

Let X be an infinite-dimensional complex Banach space and denote the set of bounded (resp. compact) linear operators on X by $B(X)$ (resp. $K(X)$). For T in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Set $N^\infty(T) = \bigcup_n N(T^n)$, $R^\infty(T) = \bigcap_n R(T^n)$, $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an *index* $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. It is well known that the index is a continuous function on the set of semi-Fredholm operators. Let $\Phi_+(X)$ (resp. $\Phi_-(X)$) denote the set of *upper* (*resp. lower*) *semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). It is well known that $\Phi_+(X)$ and $\Phi_-(X)$ are open semigroups in $B(X)$ (see [1], [7]). Recall that $a(T)$ (resp. $d(T)$), the *ascent* (*resp. descent*) of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ (resp. $R(T^n) = R(T^{n+1})$). If no such n exists, then $a(T) = \infty$ (resp. $d(T) = \infty$). An operator T is called *upper semi-Browder* if $T \in \Phi_+(X)$ and $a(T) < \infty$; T is called *lower semi-Browder* if $T \in \Phi_-(X)$ and $d(T) < \infty$ [7, Definition 7.9.1]. Let $\mathcal{B}_+(X)$ (resp. $\mathcal{B}_-(X)$) denote the set of upper (resp. lower) semi-Browder operators. An operator in a Banach space is called *semi-Browder* if it is upper semi-Browder or lower semi-Browder. Semi-Browder

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operators were studied by many authors; see e.g. [6], [7], [9], [10], [14], [15], [19], [22]. The name was introduced in [7]. An operator T is *Browder* if it is both upper semi-Browder and lower semi-Browder [7, Definition 7.7.1]. Let $\mathcal{B}(X)$ denote the set of Browder operators, i.e., $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

For $T \in B(X)$ set

$$d_+(T) = \operatorname{dist}(T, B(X) \setminus \Phi_+(X)),$$
$$d_-(T) = \operatorname{dist}(T, B(X) \setminus \Phi_-(X)).$$

Hence, $d_+(T) > 0$ if and only if $T \in \Phi_+(X)$, and $d_-(T) > 0$ if and only if $T \in \Phi_-(X)$. The *semi-Fredholm radii* of the operator T ([20]) are

$$r_+(T) = \sup\{\varepsilon \ge 0 : T - \lambda I \in \Phi_+(X) \text{ for } |\lambda| < \varepsilon\},$$
$$r_-(T) = \sup\{\varepsilon \ge 0 : T - \lambda I \in \Phi_-(X) \text{ for } |\lambda| < \varepsilon\}.$$

Let us remark that $r_+(T) \ge d_+(T)$ and $r_-(T) \ge d_-(T)$.

The fact that $K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the *Calkin algebra* over X as the quotient algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|.$$

We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Let $r_e(T) = \lim \|\pi(T^n)\|^{1/n}$ be the *essential spectral radius* of T. An operator $T \in B(X)$ is a *Riesz* operator if and only if $r_e(T) = 0$ [1, Theorem 3.3.1], i.e., if and only if $\pi(T)$ is quasinilpotent in $C(X)$. Let $R(X)$ denote the set of Riesz operators in $B(X)$.

Let us recall that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets in $B(X)$ [10, Satz 4], but not stable under finite-rank perturbations [1, pp. 13–14]. In this paper, among other things, we generalize Grabiner's well known theorem [6, Theorem 2] and our recent result [15, Theorem 1] on perturbations of semi-Fredholm operators with finite ascent or descent (see Corollaries 3 and 4 below). Now our arguments are based on the observation that both [6, Theorem 2] and [15, Theorem 1] have been presented in the global form, i.e., they have been stated for all semi-Fredholm operators with finite ascent or descent, while the perturbation results have been in the local form, i.e., they have depended on the particular choice of semi-Fredholm operator.

The main result of this paper is the following theorem.

THEOREM 1. *Suppose that* $T, S \in B(X)$ *and* $TS = ST$. *Then*

(1.1) $\quad T \in \mathcal{B}_+(X)$ *and* $r_e(S) < r_+(T) \Rightarrow T + S \in \mathcal{B}_+(X)$,

(1.2) $\quad T \in \mathcal{B}_-(X)$ *and* $r_e(S) < r_-(T) \Rightarrow T + S \in \mathcal{B}_-(X)$.

Proof. To prove (1.1) suppose that $T \in \mathcal{B}_+(X)$ and $r_e(S) < r_+(T)$. Let us remark that $\lim d_+(T^n)^{1/n} = r_+(T)$ ([2], [4], [13], [17], [18], [20]).

From $r_e(S) < r_+(T)$, there is an n with $\|\pi(S^n)\| < d_+(T^n)$. Hence there is a compact operator $K \in B(X)$ such that $\|S^n - K\| < d_+(T^n)$. Then we obtain

(1.3) $$0 < d_+(T^n) - \|S^n - K\| \le d_+(T^n - S^n + K).$$

Hence $T^n - S^n + K \in \Phi_+(X)$, and therefore $T^n - S^n \in \Phi_+(X)$. Using the hypothesis that T and S commute, we have $T^n - S^n = (T^{n-1} - ST^{n-2} + \ldots)(T - S)$. Now, [1, Corollary 1.3.4] shows that $T - S \in \Phi_+(X)$. Let us remark that the argument above shows that $T + \lambda S \in \Phi_+(X)$ for all $\lambda \in [0,1]$. Hence, by [3, Theorem 3], there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that

(1.4) $$\overline{N^\infty(T_\lambda)} \cap R^\infty(T_\lambda) = \overline{N^\infty(T_\mu)} \cap R^\infty(T_\mu)$$

in the open disc $S(\lambda)$ with center λ and radius ε. Formula (1.4) says, in effect, that $\overline{N^\infty(T_\lambda)} \cap R^\infty(T_\lambda)$ is a locally constant function of λ on the interval $[0,1]$. Now, since every locally constant function on a connected set like $[0,1]$ is constant, and $a(T) < \infty$ implies that $N^\infty(T) \cap R^\infty(T) = \overline{N^\infty(T)} \cap R^\infty(T) = \{0\}$ [19, Proposition 1.6(i)], we conclude that $\overline{N^\infty(T+S)} \cap R^\infty(T+S) = \{0\}$. Thus $N^\infty(T+S) \cap R^\infty(T+S) = \{0\}$, and again by [19, Proposition 1.6(i)] it follows that $a(T+S) < \infty$. This completes the proof of (1.1).

To prove (1.2) suppose that $T \in \mathcal{B}_-(X)$ and $r_e(S) < r_-(T)$. Now $T^* \in \mathcal{B}_+(X^*)$ and $T^*S^* = S^*T^*$. Since $r_e(S^*) = r_e(S)$ and $r_+(S^*) = r_-(S)$, by (1.1) we have $T^* + S^* \in \mathcal{B}_+(X^*)$, i.e., $T + S \in \mathcal{B}_-(X)$. ■

Let us remark that the commutativity condition in Theorem 1 is essential, even for finite-dimensional perturbation S [1, pp. 13–14].

COROLLARY 2. *Suppose that* $T \in B(X)$, $S \in R(X)$ *and* $TS = ST$. *Then*

(2.1) $\quad T \in \mathcal{B}_+(X) \Rightarrow T + S \in \mathcal{B}_+(X)$,

(2.2) $\quad T \in \mathcal{B}_-(X) \Rightarrow T + S \in \mathcal{B}_-(X)$.

Proof. From Theorem 1. ■

Now as a corollary, we get the main result of S. Grabiner [6, Theorem 2]; (see also [7, Theorem 7.9.2]). Our formulation of that result is somehow different from that of S. Grabiner's, but resembles that of Theorem 1.

COROLLARY 3. *Suppose that* $T \in B(X)$, $S \in K(X)$ *and* $TS = ST$. *Then*

(3.1) $\quad T \in \mathcal{B}_+(X) \Rightarrow T + S \in \mathcal{B}_+(X)$,

(3.2) $\quad T \in \mathcal{B}_-(X) \Rightarrow T + S \in \mathcal{B}_-(X)$.

Proof. Since $K(X) \subset R(X)$, this follows from Corollary 2. ■

Recall that the *perturbation classes* associated with $\Phi_+(X)$ and $\Phi_-(X)$ are denoted, respectively, by $P(\Phi_+(X))$ and $P(\Phi_-(X))$, i.e.,

$$P(\Phi_+(X)) = \{T \in B(X) : T + S \in \Phi_+(X) \text{ for all } S \in \Phi_+(X)\}$$

and

$$P(\Phi_-(X)) = \{T \in B(X) : T + S \in \Phi_-(X) \text{ for all } S \in \Phi_-(X)\}.$$

Now as a corollary, we get the main result of V. Rakočević [15, Theorem 1].

COROLLARY 4. *Suppose that $T, K \in B(X)$ and $TK = KT$. Then*

$$T \in \mathcal{B}_+(X) \text{ and } K \in P(\Phi_+(X)) \Rightarrow T + K \in \mathcal{B}_+(X), \tag{4.1}$$

$$T \in \mathcal{B}_-(X) \text{ and } K \in P(\Phi_-(X)) \Rightarrow T + K \in \mathcal{B}_-(X). \tag{4.2}$$

Proof. Since $P(\Phi_+(X)) \cup P(\Phi_-(X)) \subset R(X)$ ([1, Theorems 5.5.9 and 5.6.9]), the assertion follows from Corollary 2. ■

Remark. Let us mention that in a series of papers ([2], [4], [5], [12], [17], [18], [20], [21]) the quantities d_+, d_-, r_+, r_-, and several other operational quantities have been studied characterizing upper and lower semi-Fredholm operators. Recently, M. González and A. Martinón [5] have proved that these quantities can be divided into three classes, in such a way that two of them are equivalent if they belong to the same class, and are comparable and not equivalent if they belong to different classes. From the proof of Theorem 1, it is clear that instead of d_+, d_-, r_+, r_-, we can use other appropriate operational quantities.

The set ~~of~~ upper (lower) semi-Browder operators and Browder operators define, respectively, the corresponding spectra, i.e., for $T \in B(X)$ set

$$\sigma_{\mathrm{ab}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_+(X)\},$$
$$\sigma_{\mathrm{db}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_-(X)\},$$
$$\sigma_{\mathrm{eb}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}(X)\}.$$

It is clear that $\sigma_{\mathrm{eb}}(T) = \sigma_{\mathrm{ab}}(T) \cup \sigma_{\mathrm{db}}(T)$. $\sigma_{\mathrm{eb}}(T)$ is the well known *Browder's essential spectrum* of T ([7], [11], [16]). $\sigma_{\mathrm{ab}}(T)$ and $\sigma_{\mathrm{db}}(T)$ are non-empty compact subsets of the complex plane $\mathbb{C}$ called *Browder's essential approximate point spectrum* of T and *Browder's essential defect spectrum* of T, respectively, ([14], [15], [22]). Let $\sigma(T), \sigma_{\mathrm{a}}(T)$ and $\sigma_{\mathrm{d}}(T)$ denote, respectively, the *spectrum*, *approximate point spectrum* and *approximate defect spectrum* of an element T of $B(X)$ (recall that $\sigma_{\mathrm{a}}(T) = \{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda I)x\| = 0\}$ and $\sigma_{\mathrm{d}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not onto}\}$). It is well known that

$$\sigma_{\mathrm{eb}}(T) = \bigcap_{\substack{TK=KT \\ K \in K(X)}} \sigma(T + K).$$

Recall that ([14], [15], [22])

$$\sigma_{\mathrm{ab}}(T) = \bigcap_{\substack{TK=KT \\ K \in K(X)}} \sigma_{\mathrm{a}}(T + K) \quad \text{and} \quad \sigma_{\mathrm{db}}(T) = \bigcap_{\substack{TK=KT \\ K \in K(X)}} \sigma_{\mathrm{d}}(T + K).$$

Let $\mathcal{S}$ be a subset of $B(X)$. A subset $\triangle$ of $\sigma(T)$ is said to *remain invariant under perturbations of T by operators in $\mathcal{S}$* if $\triangle \subset \bigcap_{S \in \mathcal{S}} \sigma(T + S)$ ([8]).

Now we can prove

THEOREM 5. *Suppose that $T \in B(X)$. Then $\sigma_{\mathrm{ab}}(T)$ (resp. $\sigma_{\mathrm{db}}(T)$) is the largest subset of the approximate point (resp. defect) spectrum of T which remains invariant under perturbations of T by Riesz operators R which commute with T, i.e.,*

$$\sigma_{\mathrm{ab}}(T) = \bigcap_{\substack{TS=ST \\ S \in R(X)}} \sigma_{\mathrm{a}}(T + S) \quad \text{and} \quad \sigma_{\mathrm{db}}(T) = \bigcap_{\substack{TS=ST \\ S \in R(X)}} \sigma_{\mathrm{d}}(T + S). \tag{5.1}$$

Proof. It is enough to prove (5.1) only for $\sigma_{\mathrm{ab}}(T)$, and in this case it is sufficient to prove "$\subset$". If $\lambda \notin \bigcap\{\sigma_{\mathrm{a}}(T + S) : TS = ST,\ S \in R(X)\}$, there is a Riesz operator S_0 such that $TS_0 = S_0T$ and $\lambda \notin \sigma_{\mathrm{a}}(T + S_0)$. Hence $T + S_0 - \lambda I \in \mathcal{B}_+(X)$, and by Corollary 2 we have $T - \lambda I \in \mathcal{B}_+(X)$. Thus $\lambda \notin \sigma_{\mathrm{ab}}(T)$. ■

Now as a corollary we get the well known theorem of D. Lay [11, Theorem 4] or M. A. Kaashoek and D. C. Lay [8, Theorem 4.1] for bounded operators.

COROLLARY 6. *Suppose that $T \in B(X)$. Then $\sigma_{\mathrm{eb}}(T)$ is the largest subset of the spectrum of T which remains invariant under perturbations of T by Riesz operators R which commute with T.*

Proof. From Theorem 5 and the fact that $\sigma(T) = \sigma_{\mathrm{a}}(T) \cup \sigma_{\mathrm{d}}(T)$. ■

Further, as an application of Theorem 1, or Corollary 2, we have

THEOREM 7. (i) *An operator $S \in B(X)$ satisfies*

$$\sigma_{\mathrm{ab}}(T + S) = \sigma_{\mathrm{ab}}(T) \tag{7.1}$$

for all $T \in B(X)$ which commute with S if and only if $S \in R(X)$.

(ii) *An operator $S \in B(X)$ satisfies*

$$\sigma_{\mathrm{db}}(T + S) = \sigma_{\mathrm{db}}(T) \tag{7.2}$$

for all $T \in B(X)$ which commute with S if and only if $S \in R(X)$.

Proof. It is enough to prove (i). If (7.1) holds for all T which commute with S, then $\sigma_{\mathrm{ab}}(O + S) = \sigma_{\mathrm{ab}}(O) = \{0\}$. Now, by [14, Corollary 2.5] we have $\sigma_{\mathrm{eb}}(S) = \{0\}$, hence $S \in R(X)$. Conversely, suppose that $S \in R(X)$, T commutes with S and $\lambda \notin \sigma_{\mathrm{ab}}(T)$. Thus $T - \lambda I \in \mathcal{B}_+(X)$, and by Corollary 2 we have $T + S - \lambda I \in \mathcal{B}_+(X)$, i.e., $\lambda \notin \sigma_{\mathrm{ab}}(T + S)$. This implies that $\sigma_{\mathrm{ab}}(T + S) \subset \sigma_{\mathrm{ab}}(T)$. Now, clearly $\sigma_{\mathrm{ab}}[(T + S) - S] \subset \sigma_{\mathrm{ab}}(T + S)$, and we have $\sigma_{\mathrm{ab}}(T + S) = \sigma_{\mathrm{ab}}(T)$. ∎

Finally, as a corollary we get a theorem of M. Schechter [16, Theorem 2.6].

COROLLARY 8. *An operator $S \in B(X)$ satisfies*

$$\sigma_{\mathrm{eb}}(T + S) = \sigma_{\mathrm{eb}}(T) \tag{8.1}$$

for all $T \in B(X)$ which commute with S if and only if $S \in R(X)$.

Proof. From Theorem 7 and the fact that $\sigma(T) = \sigma_{\mathrm{a}}(T) \cup \sigma_{\mathrm{d}}(T)$. ∎

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Department of Mathematics
Faculty of Philosophy
University of Niš
Ćirila and Metodija 2
18000 Niš, Serbia, Yugoslavia
E-mail: vrakoc@ban.junis.ni.ac.yu

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