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Weak type (1, 1) multipliers on LCA groups

by

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Abstract. In [ABB] Asmar, Berkson and Bourgain prove that for a sequence $\{\phi_j\}_{j=1}^{\infty}$ of weak type (1, 1) multipliers in $\mathbb{R}^n$ and a function $k \in L^1(\mathbb{R}^n)$ the weak type (1, 1) constant of the maximal operator associated with $\{k * \phi_j\}_j$ is controlled by that of the maximal operator associated with $\{\phi_j\}_j$. In [ABG] this theorem is extended to LCA groups with an extra hypothesis: the multipliers must be continuous. In this paper we prove a more general version of this last result without assuming the continuity of the multipliers. The proof arises after simplifying the one in [ABB] which becomes then extensible to LCA groups.

1. Introduction. Let G be a locally compact Hausdorff abelian group (LCA group) and Γ the dual group of G. For each function $\phi \in L^\infty(\Gamma)$ we denote by T_ϕ the associated bounded operator in $L^2(G)$ given by $\widehat{T_\phi f} = \phi\widehat{f}$. We say that $\phi \in L^\infty(\Gamma)$ is a *weak type* (1, 1) *multiplier* in Γ (and we write $\phi \in M_1^{(w)}(\Gamma)$) if T_ϕ is of weak type (1, 1) on $(L^1 \cap L^2)(G)$. In this case T_ϕ extends to a bounded operator from $L^1(G)$ into $L^{1,\infty}(G)$. If $\{\phi_j\}_j$ is a sequence in $M_1^{(w)}(\Gamma)$ we denote by $N_1^{(w)}(\{\phi_j\}_j)$ the weak type (1, 1) norm (possibly ∞) of the maximal operator

$$T^* f = \sup_j |T_{\phi_j} f|, \quad f \in L^1(G). \tag{1.1}$$

Also we shall denote by $(M(\Gamma), \|\cdot\|)$ the Banach space of complex regular (necessarily finite) measures on Γ. The main result in this paper (proved in §2) is stated as follows.

THEOREM 1.2. *Let* $\mu \in M(\Gamma)$ *and* $\{\phi_j\}_{j=1}^{\infty} \subset M_1^{(w)}(\Gamma)$. *Then*

$$\{\mu * \phi_j\}_{j=1}^{\infty} \subset M_1^{(w)}(\Gamma)$$

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and also,

$$N_1^{(w)}(\{\mu * \phi_j\}_{j=1}^{\infty}) \le C\|\mu\| N_1^{(w)}(\{\phi_j\}_{j=1}^{\infty}),$$

where C is a universal constant.

The constant C will be defined by (2.10). If we consider L^p multipliers $(1 \le p < \infty)$ instead of weak type $(1,1)$ ones this result is an easy consequence of Minkowski's inequality for integrals. In [ABB] this "convolution theorem" is proved for the case $G = \mathbb{R}^n$ and $\mu = k \in L^1(\mathbb{R}^n)$. The proof given there is not directly applicable to the more general setting of LCA groups and in fact the constant C in that proof depends on the dimension n. In a more recent paper ([ABG]) the theorem is extended to LCA groups with an additional hypothesis: the multipliers must be continuous functions. A slight modification in the original argument in [ABB] allows us to obtain the more general version as stated above.

The key ingredients in the proof are the next two known lemmas which we state without proof. By combining them one can produce some weak type inequalities. Simple proofs for that results can be found in [GR] and for the best constants in Lemma 1.3 we refer to [Sz] and [Sk].

LEMMA 1.3 (Khinchin's inequality). *For each $p \in (0,\infty)$ there exist positive constants A_p and B_p such that for each finite sequence $\{c_j\}_{j=1}^J \subset \mathbb{C}$,*

$$A_p\Big(\sum_{j=1}^J |c_j|^2\Big)^{1/2} \le \Big(\int_{D^J}\Big|\sum_{j=1}^J c_j\varepsilon_j\Big|^p d\varepsilon\Big)^{1/p} \le B_p\Big(\sum_{j=1}^J |c_j|^2\Big)^{1/2},$$

where $D = \{-1,1\}$, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_J) \in D^J$ and $d\varepsilon$ is the Haar measure on D^J with total mass 1.

LEMMA 1.4 (Kolmogorov condition). *For each $p \in (0,1)$ and $f \in L^2(G)$,*

$$\begin{aligned}\|f\|_{L^{1,\infty}(G)} &\le \sup\{|E|^{(p-1)/p}\|f\|_{L^p(E)} : E \subset G \text{ compact},\ |E| > 0\}\\ &\le (1-p)^{-1/p}\|f\|_{L^{1,\infty}(G)}.\end{aligned}$$

The remainder of this paper is organized as follows. In §2 we prove Theorem 1.2 using the above two lemmas. In § 3 we give an l^p vector-valued extension of (1.2) and also an application to singular integral theory.

2. Proof of Theorem 1.2.

As a substitute of what in $\mathbb{R}^n$ is the Schwartz class of functions we are going to use in G the class

$$SL^1(G) = \{f \in L^1(G) : \widehat{f} \in L^1(\Gamma)\}.$$

It is easy to see that $SL^1(G)$ is a dense subspace of $L^p(G)$ for all $p \in [1,\infty)$ and of $(C_0, \|\cdot\|_\infty)$, where C_0 is the class of functions vanishing at infinity. Also, the Fourier transform is one-to-one from $SL^1(G)$ into $SL^1(\Gamma)$.

Proof of Theorem 1.2. For $f \in (L^1 \cap L^2)(G)$ let

$$T^{**}f(x) = \sup_j |T_{\mu*\phi_j}f(x)|, \quad x \in G. \tag{2.1}$$

We have to find a number $C \in (0,\infty)$ such that

$$\|T^{**}f\|_{L^{1,\infty}(G)} \le C\|\mu\|N_1^{(w)}(\{\phi_j\}_j)\|f\|_{L^1(G)} \tag{2.2}$$

for all $f \in (L^1 \cap L^2)(G)$. By Fatou's lemma we may suppose that the sequence $\{\phi_j\}_j$ is finite: $\{\phi_j\}_{j=1}^J$. But in this case it is easy to see (using again Fatou and other standard arguments) that we only need to prove (2.2) for all $f \in SL^1(G)$. Then

$$T_{\mu*\phi_j}f(x) = \int_\Gamma (\mu * \phi_j)(y)\widehat{f}(y)\gamma_y(x)\,dy, \quad x \in G, \tag{2.3}$$

where $\gamma_y(x)$ denotes the action of the character $y \in \Gamma$ on $x \in G$. So $T_{\mu*\phi_j}f$ is in this case a well defined function and belongs to the class $C_0(G)$. Also it follows from (2.3) that if $\mu_n \to \mu$ in $M(\Gamma)$ then $T_{\mu_n*\phi_j}f(x) \to T_{\mu*\phi_j}f(x)$ for all $x \in G$ and all $j = 1,\ldots,J$. Thus by Fatou's lemma and the regularity of the measures in $M(G)$ we may assume that $\operatorname{supp}\mu = K$ compact. Now pick $p \in (0,1)$. In order to prove (2.2) we only need to show, by Lemma 1.4, that

$$\|T^{**}f\|_{L^p(E)} \le C|E|^{(1-p)/p}\|\mu\|N_1^{(w)}(\{\phi_j\}_{j=1}^\infty)\|f\|_{L^1(G)} \tag{2.4}$$

for all compact sets $E \subset G$.

Note that

$$T_{\mu*\phi_j}f(x) = \int_\Gamma T_{\phi_j}(\gamma_{-z}f)(x)\gamma_z(x)\,d\mu(z). \tag{2.5}$$

We want to express the integral in (2.5) as a Riemann sum. To accomplish this we make use of the following technical result to be proved later.

LEMMA 2.6. *For $j = 1,\ldots,J$ and $x \in G$ let*

$$F_{j,x}(z) = |T_{\phi_j}(\gamma_{-z}f)(x)|^2, \quad z \in \Gamma.$$

Then for each $n = 1,2,\ldots$ there exists a finite family $\{V_i^n\}_{i=1}^{I_n}$ of pairwise disjoint measurable sets in Γ such that

(i) $K \subset \bigcup_{i=1}^{I_n} V_i^n$,

(ii) *if $i = 1,\ldots,I_n$ and $z_1, z_2 \in V_i^n$ then $|F_{j,x}(z_1) - F_{j,x}(z_2)| < 1/n$ for $j = 1,\ldots,J$ and $x \in G$.*

Now consider the sets V_i^n given by Lemma 2.6 and for each i and n pick $z_i^n \in V_i^n$. It follows that if Λ_i^n is the characteristic function of V_i^n, then

$$\sum_{i=1}^{I_n} |T_{\phi_j}(\gamma_{-z_i^n}f)(x)|^2\Lambda_i^n(z) \xrightarrow[n]{} |T_{\phi_j}(\gamma_{-z}f)(x)|^2$$

for $z \in K$, $x \in G$ and $j = 1, \ldots, J$. Thus by the Cauchy–Schwarz inequality, Fatou's lemma and Khinchin's inequality (1.3) we obtain from (2.5),

$$(2.7)\quad |T_{\mu*\phi_j} f(x)| \le \|\mu\|^{1/2}\Big\{ \int_\Gamma |T_{\phi_j}(\gamma_{-z} f)(x)|^2\, d|\mu|(z)\Big\}^{1/2}$$
$$\le \|\mu\|^{1/2} \liminf_{n\to\infty} \Big(\sum_{i=1}^{I_n} |\mu|(V_i^n) |T_{\phi_j}(\gamma_{-z_i^n} f)(x)|^2\Big)^{1/2}$$
$$\le \frac{\|\mu\|^{1/2}}{A_p} \liminf_n \Big\{ \int_{D^{I_n}} \Big| \sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2} T_{\phi_j}(\gamma_{-z_i^n} f)(x)\varepsilon_i\Big|^p\, d\varepsilon\Big\}^{1/p}$$
$$= \frac{\|\mu\|_1^{1/2}}{A_p} \liminf_n \Big\{ \int_{D^{I_n}} \Big| T_{\phi_j}\Big(\sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2}\varepsilon_i \gamma_{-z_i^n} f\Big)(x)\Big|^p\, d\varepsilon\Big\}^{1/p}.$$

Since $|T_{\phi_j}|$ can be majorized by T^* (see (1.1)), it follows, by taking the supremum over j on the left side of (2.7) and recalling the definition (2.1), that

$$|T^{**} f(x)| \le \frac{\|\mu\|_1^{1/2}}{A_p} \liminf_n \Big\{ \int_{D^{I_n}} \Big(T^*\Big(\sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2}\varepsilon_i \gamma_{-z_i^n} f\Big)(x)\Big)^p\, d\varepsilon\Big\}^{1/p}$$

Then by Fubini's theorem and Jensen's inequality one has for all compact sets $E \subset G$,

$$(2.8)\quad \|T^{**} f\|_{L^p(E)}$$
$$\le \frac{\|\mu\|^{1/2}}{A_p} \liminf_n \Big\| \Big\{ \int_{D^{I_n}} \Big(T^*\Big(\sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2}\varepsilon_i \gamma_{-z_i^n} f\Big)(\cdot)\Big)^p\, d\varepsilon\Big\}^{1/p}\Big\|_{L^p(E)}$$
$$\le \frac{\|\mu\|^{1/2}}{A_p} \liminf_n \int_{D^{I_n}} \Big\| T^*\Big(\sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2}\varepsilon_i \gamma_{-z_i^n} f\Big)\Big\|_{L^p(E)}\, d\varepsilon.$$

Using again Lemma 1.4, Fubini and Khinchin (note that the best constant B_1 is 1) we have

$$(2.9)\quad \|T^{**} f\|_{L^p(E)} \le C_p A_p^{-1} \|\mu\|^{1/2} |E|^{(1-p)/p} N_1^{(w)}(\{\phi_j\}_j)$$
$$\times \liminf_n \int_G |f(x)| \int_{D^{I_n}} \Big| \sum_{i=1}^{I_n} |\mu|(V_i^n)^{1/2}\varepsilon_i \gamma_{-z_i^n}(x)\Big|\, d\varepsilon\, dx$$
$$\le C_p A_p^{-1} \|\mu\|^{1/2} |E|^{(1-p)/p} N_1^{(w)}(\{\phi_j\}_j) \|f\|_1 \liminf_n \Big(\sum_{i=1}^{I_n} |\mu|(V_i^n)\Big)^{1/2}$$
$$= C_p A_p^{-1} \|\mu\| \cdot |E|^{(1-p)/p} N_1^{(w)}(\{\phi_j\}_j) \|f\|_1,$$

where $C_p = (1-p)^{-1/p}$. Thus we obtain (2.4) with constant

$$(2.10)\quad C = \inf_{0<p<1} (1-p)^{-1/p} A_p^{-1}. \ \blacksquare$$

Proof of Lemma 2.6. Let $B = \max_{1\le j\le J} \|\phi_j\|_\infty$. Note that

$$(2.11)\quad T_{\phi_j}(\gamma_{-z} f)(x) = \int_\Gamma \phi_j(y) \widehat{f}(z+y) \gamma_y(x)\, dy.$$

From this it follows that

$$|T_{\phi_j}(\gamma_{-z} f)(x)| \le B\|\widehat{f}\|_1, \quad \forall z \in \Gamma,$$

uniformly in $j = 1, \ldots, J$ and in $x \in G$. Also (2.11) shows that

$$|T_{\phi_j}(\gamma_{-z_1} f)(x) - T_{\phi_j}(\gamma_{-z_2} f)(x)| \le B\|\widehat{f}(z_1+\cdot) - \widehat{f}(z_2+\cdot)\|_1.$$

Since the map $z \mapsto f(z+\cdot)$ is uniformly continuous from Γ to $L^1(\Gamma)$ ([Ru], 1.1.5), we deduce that $z \mapsto T_{\phi_j}(\gamma_{-z} f)(x)$ is uniformly continuous "uniformly in $j = 1, \ldots, J$ and in $x \in G$". Thus the same is true for $F_{j,x}(z) = |T_{\phi_j}(\gamma_{-z} f)(x)|^2$. That is, for all $\varepsilon > 0$ there exists a neighborhood V_ε of $0 \in \Gamma$ such that

$$z_1 - z_2 \in V_\varepsilon \Rightarrow |F_{j,x}(z_1) - F_{j,x}(z_2)| < \varepsilon, \ \forall j = 1, \ldots, J, \ \forall x \in G.$$

For all $n \in \mathbb{N}$ let then $W_n, U_n \subset \Gamma$ be open neighborhoods of zero such that $W_n - W_n \subset U_n$ and

$$z_1 - z_2 \in U_n \Rightarrow |F_{j,x}(z_1) - F_{j,x}(z_2)| < 1/n, \ \forall j = 1, \ldots, J, \ \forall x \in G.$$

Cover K with a finite number of sets $\{y_i + W_n\}_{i=1}^{J_n}$, $\{y_i\}_i \subset K$. Then the sets we are looking for are (after discarding the empty ones): $V_1^n = y_1 + W_n$ and, for $i = 2, \ldots, J_n$, $V_i^n = (y_i + W_n) \setminus (\bigcup_{k=1}^{i-1} V_k^n)$. $\blacksquare$

3. Further results. The maximal function associated with a family $\{T_j\}_j$ of linear maps is in fact the $q = \infty$ version of the more general operators given by

$$T_*^q f(x) = \Big(\sum_j |T_j f(x)|^q \Big)^{1/q}.$$

Let us introduce some notation. For each $q \in (0,\infty]$ and each family $\Phi = \{\phi_j\}_{j=1}^\infty \subset M_1^{(w)}(\Gamma)$ let T_Φ be the vector-valued operator given by

$$T_\Phi f(x) = \{T_{\phi_j} f(x)\}_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}}, \quad f \in (L^1 \cap L^2)(G),$$

and let

$$T_\Phi^q f(x) = \|T_\Phi f\|_{l^q},$$

where for each sequence $a = \{a_j\}_j$ of complex numbers $\|a\|_{l^q}^q = \sum_j |a_j|^q$. Also for $\mu \in M(\Gamma)$ we denote by $\mu * \Phi$ the family of weak type (1,1) multipliers $\{\mu * \phi_j\}_j$ (see Thm. 1.2).

In these terms what Theorem 1.2 says is that the weak type $(1,1)$ boundedness of T^{∞}_{Φ} implies that of $T^{\infty}_{\mu*\Phi}$. The next theorem tells us that this is a special case (when $q=\infty$) of an analogous statement which is valid for every $q\in(0,\infty]$.

THEOREM 3.1. *Let $\Phi=\{\phi_j\}_{j=1}^{\infty}\subset M_1^{(w)}(\Gamma)$ and $\mu\in M(\Gamma)$. Suppose that for some $q\in(0,\infty]$ there exists a positive constant B such that*

$$\|T^q_{\Phi}f\|_{1,\infty}\le B\|f\|_1,\qquad \forall f\in(L^1\cap L^2)(G).$$

Then

$$\|T^q_{\mu*\Phi}f\|_{1,\infty}\le CB\|\mu\|\|f\|_1,\quad \forall f\in L^1(G),$$

where C is the absolute constant in (2.10) if $q\ge 1$ and

$$C=\inf_{0<p\le q}(1-p)^{-1/p}A_p^{-1}\qquad \text{if } q<1.$$

Proof. The proof is essentially the same as that of Theorem 1.2. Only small changes must be made. Once more we may assume that the family $\{\phi_j\}_j$ is finite and $f\in SL^1(G)$. We proceed as before but we choose $p\in(0,1)$ such that $p\le q$. From (2.7) it follows using Minkowski's inequality that

$$\begin{aligned}
&T^q_{\mu*\Phi}f(x)\\
&\le\frac{\|\mu\|^{1/2}}{A_p}\liminf_n\Big\|\Big\{\Big(\int_{D^{I_n}}\Big|T_{\phi_j}\Big(\sum_{i=1}^{I_n}|\mu|(V_i^n)^{1/2}\varepsilon_i\gamma_{-z_i^n}f\Big)(x)\Big|^p d\varepsilon\Big)^{1/p}\Big\}_{j=1}^{J}\Big\|_{l^q}\\
&=\frac{\|\mu\|^{1/2}}{A_p}\liminf_n\Big\|\Big\{\int_{D^{I_n}}|T_{\phi_j}(\ldots)(x)|^p\,d\varepsilon\Big\}_{j=1}^{J}\Big\|_{l^{q/p}}^{1/p}\\
&\le\frac{\|\mu\|^{1/2}}{A_p}\liminf_n\Big(\int_{D^{I_n}}\|\{T_{\phi_j}(\ldots)(x)\}_{j=1}^{J}\|_{l^q}^p\,d\varepsilon\Big)^{1/p}\\
&=\frac{\|\mu\|^{1/2}}{A_p}\liminf_n\Big(\int_{D^{I_n}}T^q_{\Phi}(\ldots)(x)^p\,d\varepsilon\Big)^{1/p}.
\end{aligned}$$

From this again we obtain, by Fubini's theorem and Jensen's inequality,

$$\|T^q_{\mu*\Phi}f\|_{L^p(E)}\le\frac{\|\mu\|^{1/2}}{A_p}\liminf_n\int_{D^{I_n}}\Big\|T^q_{\Phi}\Big(\sum_{i=1}^{I_n}|\mu|(V_i^n)^{1/2}\varepsilon_i\gamma_{-z_i^n}f\Big)(\cdot)\Big\|_{L^p(E)}d\varepsilon.$$

The remainder of the proof is exactly as in Theorem 1.2 using the same argument here for T^q_{Φ} as the one employed there for T^*. ■

As an application of Theorem 1.2 to the theory of singular integrals we give a Hilbert transform perturbation theorem. We omit the proof which is an easy application of (1.2) (and the obviously analogous result for strong

type (p,p) multipliers, $p>1$) to the theory of singular integrals (see for example [GR]).

THEOREM 3.2. *Suppose that P is a locally integrable 2π-periodic function on $\mathbb{R}$ such that its Fourier coefficients as a function on $[-\pi,\pi]$ satisfy*

$$\sum_{k\in\mathbb{Z}}|\widehat{P}(k)|<\infty.$$

Then the maximal operator

$$H_P^*f(x)=\sup_{\varepsilon>0}\int_{y>\varepsilon}\frac{P(y)}{y}f(x-y)\,dy,\qquad x\in\mathbb{R},$$

is of weak type $(1,1)$ and strong type (p,p), $1<p<\infty$. It follows also that the limit

$$H_Pf(x)=\lim_{\varepsilon\to 0}\int_{y>\varepsilon}\frac{P(y)}{y}f(x-y)\,dy$$

exists for a.e. $x\in\mathbb{R}$ for all $f\in L^p(\mathbb{R}),1\le p<\infty$, and the map H_P is of weak type $(1,1)$ and strong type (p,p), $1<p<\infty$.

Remark 3.3. (1) The above theorem is still valid if we put $P=\widehat{\mu}$ for a measure $\mu\in M(\mathbb{R})$. In fact, the case considered corresponds to a totally atomic μ.

(2) Also we may substitute the one-dimensional Hilbert transform kernel $1/x$ by an n-dimensional standard kernel $K(x)$ (see [GR]). Obviously P must now be the Fourier transform of a complex measure on $\mathbb{R}^n$.

Also Theorem 3.1 can be applied to known results to obtain new vector inequalities. As an example, we have the next theorem which follows immediately from (3.1) and [GR, Theorem V.5.3].

THEOREM 3.4. *Let $k\in L^1(\mathbb{R}^n)$ be such that $\widehat{k}(0)=0$ and assume that for some $\alpha>0$,*

$$|k(x)|\le C|x|^{-n-\alpha},\qquad x\in\mathbb{R}^n,$$

and

$$\int|k(x+h)-k(x)|\,dx\le C|h|^{\alpha},\qquad h\in\mathbb{R}^n.$$

For $\mu\in M(\mathbb{R}^n)$ put

$$K_j(x)=2^{-jn}k(2^{-j}x)\widehat{\mu}(x),\qquad x\in\mathbb{R}^n,\ j\in\mathbb{Z}.$$

Then the operator

$$Gf(x)=\Big(\sum_{j\in\mathbb{Z}}|K_j*f(x)|^2\Big)^{1/2},\qquad x\in\mathbb{R}^n,$$

is of weak type $(1,1)$.

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Semi-Browder operators and perturbations

by

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Abstract. An operator in a Banach space is called upper (resp. lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (resp. descent). An operator in a Banach space is called semi-Browder if it is upper semi-Browder or lower semi-Browder. We prove the stability of the semi-Browder operators under commuting Riesz operator perturbations. As a corollary we get some results of Grabiner [6], Kaashoek and Lay [8], Lay [11], Rakočević [15] and Schechter [16].

Let X be an infinite-dimensional complex Banach space and denote the set of bounded (resp. compact) linear operators on X by $B(X)$ (resp. $K(X)$). For T in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Set $N^\infty(T) = \bigcup_n N(T^n)$, $R^\infty(T) = \bigcap_n R(T^n)$, $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an *index* $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. It is well known that the index is a continuous function on the set of semi-Fredholm operators. Let $\Phi_+(X)$ (resp. $\Phi_-(X)$) denote the set of *upper* (*resp. lower*) *semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). It is well known that $\Phi_+(X)$ and $\Phi_-(X)$ are open semigroups in $B(X)$ (see [1], [7]). Recall that $a(T)$ (resp. $d(T)$), the *ascent* (*resp. descent*) of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ (resp. $R(T^n) = R(T^{n+1})$). If no such n exists, then $a(T) = \infty$ (resp. $d(T) = \infty$). An operator T is called *upper semi-Browder* if $T \in \Phi_+(X)$ and $a(T) < \infty$; T is called *lower semi-Browder* if $T \in \Phi_-(X)$ and $d(T) < \infty$ [7, Definition 7.9.1]. Let $\mathcal{B}_+(X)$ (resp. $\mathcal{B}_-(X)$) denote the set of upper (resp. lower) semi-Browder operators. An operator in a Banach space is called *semi-Browder* if it is upper semi-Browder or lower semi-Browder. Semi-Browder

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