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An open mapping theorem for analytic multifunctions

by

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Abstract. The paper gives sufficient conditions for projections of certain pseudoconcave sets to be open. More specifically, it is shown that the range of an analytic set-valued function whose values are simply connected planar continua is open, provided there does not exist a point which belongs to boundaries of all the fibers. The main tool is a theorem on existence of analytic discs in certain polynomially convex hulls, obtained earlier by the author.

Introduction and results. Analytic multifunctions are set-valued generalizations of analytic mappings (cf. [Ok], [Sł1]). They found applications mostly in functional analysis: in spectral theory, the structure of the Gelfand space of a uniform algebra and in the complex interpolation method for Banach spaces. However, they have also led to interesting new vistas in complex analysis, in relation with polynomially convex hulls and the corona problem. On the other hand, it is natural to ask which portions of the classical function theory extend to analytic multifunctions. While this approach did not always produce interesting questions, the problem of generalizing the classical open mapping theorem, posed by Ransford [Ra1], has proved to be rather intricate. This paper is devoted to the above problem.

Recall that an upper semicontinuous compact-valued correspondence (briefly, *multifunction*) $z \to K_z : G \to 2^{\mathbb{C}}$, $G \subset \mathbb{C}$ open, is called *analytic* if the set $U = \{(z,w) : z \in G,\ w \notin K_z\}$ is a pseudo-convex domain.

It is easy to see that a naive generalization of the open mapping theorem is false. Let K_z be the closed segment joining z to 1, for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then $z \to K_z : \mathbb{D} \to 2^{\mathbb{C}}$ is an analytic multifunction but its range is $R(K) := \bigcup\{K_z : z \in \mathbb{D}\} = \mathbb{D} \cup \{1\}$, and is clearly not an open set. The reason is that point 1 belongs to the boundaries of all the sets K_z, $z \in \mathbb{D}$. These observations are due to Ransford who explored the problem initially in [Ra1] and modified it as follows.

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Suppose $G \subset \mathbb{C}$ is open and connected, and $z \to K_z : G \to 2^{\mathbb{C}}$ is an analytic multifunction. Let $E := \{w \in \mathbb{C} : w \in \partial K_z \text{ for } z \in G\}$. Must the set $R(K) \setminus E$ be open, where $R(K) = \bigcup\{K_z : z \in G\}$?

Ransford gave an example showing that the answer is still negative if the fibers K_z are allowed to be multiply connected, and conjectured that the assumption of simple connectedness of the fibers should suffice. To our knowledge, this conjecture remains open. We now present a partial result in this direction, under assumptions of the connectedness of the fibers.

THEOREM 0.1. *Let $G \subset \mathbb{C}$ be open and connected and $z \to K_z : G \to 2^{\mathbb{C}}$ be an analytic multifunction. Define $E = \{w \in \mathbb{C} : w \in \partial K_z \text{ for } z \in G\}$ and $R(K) = \bigcup\{K_z : z \in G\}$. Assume that all the fibers K_z, $z \in G$, are connected and simply connected. Then $R(K) \setminus E$ is an open set.*

A special case of the this result, when all the fibers K_z are convex sets, was obtained by Baribou and Harbottle [BaHa].

For the proof of Theorem 0.1 we need a result on polynomially convex hulls and winding numbers which seems to be interesting in itself.

Let D be any plane disc and $X \subset \partial D \times \mathbb{C}$ a compact set with nonempty connected and simply connected fibers $X_\zeta = \{w \in \mathbb{C} : (\zeta, w) \in X\}$, for all $\zeta \in \partial D$. Let $a : \partial D \to \mathbb{C}$ be a continuous function such that $a(\zeta) \notin X_\zeta$ for $\zeta \in \partial D$. Informally speaking, the winding number of X relative to a, denoted by $n(X, a)$, is $n(f - a, 0)$, the index of $f - a$ with respect to 0, where $f : \partial D \to \mathbb{C}$ is any continuous function which is sufficiently close to X. The precise definition is given in Section 1.

We will denote by $A(\overline{D})$ the algebra of analytic functions on D with continuous extension to $\overline{D}$.

THEOREM 0.2. *Let $X \subset \partial D \times \mathbb{C}$ be a compact set with nonempty, connected and simply connected fibers X_ζ, $\zeta \in \partial D$. Let $g \in A(\overline{D})$. Assume that $g(\zeta) \notin X_\zeta$ for $\zeta \in \partial D$, and that $n(X, g|\partial D)$, the winding number of X relative to $g|\partial D$, is zero. Then the analytic disc $\operatorname{graph}(g) = \{(z, g(z)) : z \in \overline{D}\}$ is disjoint from the polynomial hull $\widehat{X}$ of X.*

The proof is based on our earlier result which we quote here for the convenience of the reader.

THEOREM 0.3 [Sł2]. *Let X be as in Theorem 0.2. Assume that $\widehat{X} \neq X$. Then $\widehat{X} \setminus X$ is the union of the graphs of a family of bounded analytic functions $h \in H^\infty(D)$ such that $h^*(e^{i\theta})$, the nontangential boundary values, belong to $X_{e^{i\theta}}$, $\theta \in \mathbb{R}$. Furthermore, if $w_0 \in \partial(\widehat{X}_{z_0})$, the boundary of the fiber of $\widehat{X}$ over $z_0 \in D$, then there is a unique $h \in H^\infty$ with $\operatorname{graph}(h) \subset \widehat{X}$ and $h(z_0) = w_0$.*

We discuss the winding number in Section 1 and prove Theorems 0.1 and 0.2 in Section 2.

1. Properties of the winding number. The following is a generalization, to a class of sets, of a classical lifting theorem for paths. The proof, analogous to the standard one, is a consequence of the fact that the inverse image, under the map $w \to e^w : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, of a connected and simply connected compact subset X_ζ of $\mathbb{C} \setminus \{0\}$, is a union of simply connected components, each homeomorphic to X_ζ under the exponential map (the monodromy theorem). The details can be found in [Ra2].

PROPOSITION 1.1. *Let $D = D(a, R) = \{z \in \mathbb{C} : |z - a| < R\}$. Let $X \subset \partial D \times \mathbb{C}$ be a compact set with nonempty connected and simply connected fibers X_ζ, $\zeta \in \partial D$. Then*

(a) *there exist upper semicontinuous set-valued functions $t \to L(t) : [0, 2\pi] \to 2^{\mathbb{C}\setminus\{0\}}$ such that $L(t)$ are nonempty, connected and simply connected compact sets satisfying $\{e^w : w \in L(t)\} = X_{a+Re^{it}}$, $t \in [0, 2\pi]$;*

(b) *any two such L's ("lifts") are either identical or mutually disjoint;*

(c) *there is a unique integer $k = 0, \pm1, \pm2, \ldots$ such that for every lift L, $L(2\pi) = k + L(0)$.*

DEFINITION 1.2. Let X be as in Proposition 1.1.

(i) If $0 \notin X_\zeta$ for $\zeta \in \partial D$, define the *winding number* of X relative to zero to be the integer k from (c), and write $n(X, 0) = k$; cf. Ransford [Ra2].

(ii) If $g : \partial D \to \mathbb{C}$ is a continuous function such that $g(\zeta) \notin X_\zeta$ for $\zeta \in \partial D$, define $n(X, g) = n(Y, 0)$, where $Y = \bigcup_{\zeta \in \partial D}\{\zeta\} \times Y_\zeta$ and $Y_\zeta = X_\zeta - g(\zeta)$.

The next observation is an easy consequence of this definition and Proposition 1.1.

COROLLARY 1.3. *Let $X, Y, Z \subset \partial D \times \mathbb{C}$ be compact sets with nonempty, connected and simply connected fibers, and $g : \partial D \to \mathbb{C}$ be a continuous function. Assume that $X \cup Y \subset Z$ and $g(\zeta) \notin Z_\zeta$ for $\zeta \in \partial D$. Then $n(X, g) = n(Z, g) = n(Y, g)$.*

If $F \subset \mathbb{C}$ is a compact set and $r > 0$, we define $B(F, r) = \{z \in \mathbb{C} : \operatorname{dist}(z, F) < r\}$ and $\overline{B}(F, r) = \{z \in \mathbb{C} : \operatorname{dist}(z, F) \leq r\}$.

PROPOSITION 1.4. *Let $X \subset \partial D \times \mathbb{C}$ be a compact set with nonempty, connected and simply connected fibers. Let $g \in C(\partial D)$ ($=$ the class of all continuous functions on ∂D) satisfy $g(\zeta) \notin X_\zeta$, $\zeta \in \partial D$. Then there is a positive r_0 such that whenever $h \in C(\partial D)$ and $\sup_{\zeta \in \partial D} \operatorname{dist}(h(\zeta), X_\zeta) = r < r_0$, then $n(X, g) = n(h, g) := n(h - g, 0)$.*

Proof. The infinite interval $J = \{r \geq 0 : \text{there is } \zeta \in \partial D \text{ with } g(\zeta) \in \widehat{\overline{B}(X_\zeta, r)}\}$ (where $\widehat{F}$ denotes the polynomially convex hull of F) must contain $r_0 := \inf J$, by the compactness argument. On the other hand, $0 \notin J$ for $\bigcap_{r>0} \widehat{\overline{B}(X_\zeta, r)} = X_\zeta \not\ni g(\zeta)$. Thus, $r_0 > 0$.

Assume now $\sup_{\zeta\in\partial D} \operatorname{dist}(h(\zeta), X_\zeta) = r < r_0$, $h \in C(\partial D)$, and let $Y = \operatorname{graph}(h) = \{(\zeta, h(\zeta)) : \zeta \in \partial D\}$ and $Z = \bigcup_{\zeta\in\partial D}\{\zeta\} \times Z_\zeta$, where $Z_\zeta = \widehat{\overline{B}(X_\zeta, r)}$, $\zeta \in \partial D$. Then X, Y, Z are compact, have nonempty, connected and simply connected fibers and satisfy $X \cup Y \subset Z$ and $g(\zeta) \notin Z_\zeta$ for $\zeta \in \partial D$. By Corollary 1.3, $n(X, g) = n(Z, g) = n(Y, g) = n(h, g)$. ∎

In other words, the last proposition means that as long as $h \in C(\partial D)$ satisfies $\operatorname{dist}(h(\zeta), X_\zeta) \leq r < r_0$, then $n(h, g)$ does not depend on the choice of h. This offers an alternative way to define $n(h, g)$ in view of the following, well known fact.

PROPOSITION 1.5. *Let $X \subset \partial D\times\mathbb{C}$ be compact with nonempty, connected and simply connected fibers X_ζ, $\zeta \in \partial D$. Then, for every $\varepsilon > 0$, there is $h \in C(\partial D)$ such that $\operatorname{dist}(h(\zeta), X_\zeta) < \varepsilon$ for $\zeta \in \partial D$.*

2. Application of polynomial hulls

Proof of Theorem 0.2. Define $M = \{(z, g(z)) : z \in \overline{D}\}$. The statement is trivial when $\widehat{X} = X$. Consider now an arbitrary point $x_0 = (z_0, w_0) \in \widehat{X} \setminus X$. We have to show that $x_0 \notin M$. By Theorem 0.3 there is a function $h \in H^\infty(D)$ such that $h(z_0) = w_0$, and for every $\zeta \in \partial D$, $X_\zeta \supset \mathrm{Cl}(h, \zeta)$ (= the cluster of h at ζ). For $s \in (0, 1)$, let $h_s(z) = h(a + s(z - a))$ for $z \in \overline{D}$. Then $h_s \in A(\overline{D})$ for $s < 1$. Clearly, there is $s_0 \in (0, 1)$ such that for $s \in (s_0, 1)$, $\sup_{\zeta\in\partial D} \operatorname{dist}(h_s(\zeta), X_\zeta) = r < r_0$, where $r_0 > 0$ is as in Proposition 1.4, constructed for the pair X, $g|\partial D$. By Proposition 1.4, $n(X, g) = n(h_s, g)$, hence $n(h_s, g) = 0$ for $s_0 < s < 1$, that is, $h_s(z) - g(z) \neq 0$ for $z \in \overline{D}$, $s_0 < s < 1$. Since $h_s - g \to h - g$ uniformly on compact subsets of D, the Hurwitz theorem implies that either $h - g \equiv 0$ (which is impossible as $\mathrm{Cl}(h, \zeta) \subset X_\zeta \not\ni g(\zeta)$, $\zeta \in \partial D$), or $h(z) - g(z) \neq 0$ for all $z \in D$. For $z = z_0$ this means that $x_0 = (z_0, h(z_0)) \notin M$. ∎

Proof of Theorem 0.1. We have to show that given values $(b_n)_{n=1}^\infty \subset \mathbb{C}$ and $b \in \mathbb{C}$ such that $b_n \to b$ and $b_n \in \mathbb{C}\setminus K_z$ for all $z \in G$, $n = 1, 2, \ldots$, we have either $b \in \mathbb{C} \setminus K_z$ for all $z \in G$ or $b \in \partial K_z$ for all $z \in G$. We will prove a slightly more general fact.

ASSERTION. *Let $z \to K_z : G \to 2^{\mathbb{C}}$ be an analytic multifunction with connected and simply connected fibers K_z. Suppose that N_n, $n = 1, 2, \ldots$, and N are manifolds of the form $N_n = \{(z, g_n(z)) : z \in G\}$, $n = 1, 2, \ldots$, $N = \{(z, g(z)) : z \in G\}$, where $g_n, g : G \to \mathbb{C}$ are analytic functions. Define $U = G \times \mathbb{C} \setminus \bigcup_{z\in G}\{z\} \times K_z$. Assume that $N_n \subset U$ for $n = 1, 2, \ldots$, and $N_n \to N$ in the sense that $g_n(z) \to g(z)$ uniformly on compact subsets of G. Then either $N \subset U$ or $N \subset \bigcup_{z\in G}\{z\} \times \partial K_z \subset \partial U$.*

The desired conclusion about b will follow by setting in the Assertion $N_n = G \times \{b_n\}$ and $N = G \times \{b\}$.

Suppose the Assertion fails. Then the set $F := \{z \in G : g(z) \in K_z\} = \{z \in G : g(z) \in \partial K_z\}$, which is relatively closed in G, is nonempty and different from G. Choose $a \in (\partial F) \cap G$ and $r > 0$ such that $\overline{D}(a, r) \subset G$ and

$$\text{(2.1)} \qquad \text{there is } \zeta_0 \in (\partial D) \setminus F, \quad \text{where } D = D(a, r).$$

Let $X = \operatorname{graph}(K) \cap (\partial D \times \mathbb{C}) = \bigcup_{\zeta\in\partial D}\{\zeta\} \times K_\zeta$ and $M_n = \{(z, g_n(z)) : z \in \overline{D}\}$, $n = 1, 2, \ldots$, $M = \{(z, g(z)) : z \in \overline{D}\}$, i.e. $M_n = N_n \cap (\overline{D} \times \mathbb{C})$, $n = 1, 2, \ldots$, $M = N \cap (\overline{D} \times \mathbb{C})$.

Observe now that X has an extension to $\overline{D}$, namely $z \to K_z : \overline{D} \to 2^{\mathbb{C}}$, which is an upper semicontinuous multifunction with connected and simply connected fibers disjoint from $g_n|\overline{D}$. Hence, it is clear that $n(X, g_n|\partial D) = 0$, $n = 1, 2, \ldots$ (One can apply Propositions 1.4 and 1.5 to prove this in detail.) By Theorem 0.2,

$$\text{(2.2)} \qquad \widehat{X} \cap M_n = \emptyset, \quad n = 1, 2, \ldots$$

Define $b_0 = g(a)$. Then $(a, b_0) \in \operatorname{graph}(K|\overline{D}) \subset \widehat{X}$, by [Sł1], and since $g_n(a) \to g(a)$, $b_0 \in \partial(\widehat{X})$ by (2.2). Using Theorem 0.3, we obtain a unique function $h \in H^\infty(D)$ such that $h(a) = b_0$ and $(z, h(z)) \in \widehat{X}$ for $z \in D$. In view of (2.2), $h(z) - g_n(z) \neq 0$ for all $z \in D$. On the other hand, $(h - g_n)(a) = b_0 - g_n(a) \to 0$. By the Hurwitz theorem, $(h - g)(z) \equiv 0$, i.e. $g(z) \in \partial(\widehat{X}_z)$ for $z \in D$. Hence, $g(\zeta_0) \in \widehat{X} = K_{\zeta_0}$, which contradicts (2.1). ∎

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Weak type $(1,1)$ multipliers on LCA groups

by

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Abstract. In [ABB] Asmar, Berkson and Bourgain prove that for a sequence $\{\phi_j\}_{j=1}^{\infty}$ of weak type $(1,1)$ multipliers in $\mathbb{R}^n$ and a function $k \in L^1(\mathbb{R}^n)$ the weak type $(1,1)$ constant of the maximal operator associated with $\{k * \phi_j\}_j$ is controlled by that of the maximal operator associated with $\{\phi_j\}_j$. In [ABG] this theorem is extended to LCA groups with an extra hypothesis: the multipliers must be continuous. In this paper we prove a more general version of this last result without assuming the continuity of the multipliers. The proof arises after simplifying the one in [ABB] which becomes then extensible to LCA groups.

1. Introduction. Let G be a locally compact Hausdorff abelian group (LCA group) and Γ the dual group of G. For each function $\phi \in L^\infty(\Gamma)$ we denote by T_ϕ the associated bounded operator in $L^2(G)$ given by $\widehat{T_\phi f} = \phi\widehat{f}$. We say that $\phi \in L^\infty(\Gamma)$ is a *weak type* $(1,1)$ *multiplier* in Γ (and we write $\phi \in M_1^{(w)}(\Gamma)$) if T_ϕ is of weak type $(1,1)$ on $(L^1 \cap L^2)(G)$. In this case T_ϕ extends to a bounded operator from $L^1(G)$ into $L^{1,\infty}(G)$. If $\{\phi_j\}_j$ is a sequence in $M_1^{(w)}(\Gamma)$ we denote by $N_1^{(w)}(\{\phi_j\}_j)$ the weak type $(1,1)$ norm (possibly ∞) of the maximal operator

$$T^* f = \sup_j |T_{\phi_j} f|, \quad f \in L^1(G). \tag{1.1}$$

Also we shall denote by $(M(\Gamma), \|\cdot\|)$ the Banach space of complex regular (necessarily finite) measures on Γ. The main result in this paper (proved in §2) is stated as follows.

THEOREM 1.2. *Let $\mu \in M(\Gamma)$ and $\{\phi_j\}_{j=1}^{\infty} \subset M_1^{(w)}(\Gamma)$. Then*

$$\{\mu * \phi_j\}_{j=1}^{\infty} \subset M_1^{(w)}(\Gamma)$$

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