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On equivalence of K- and J-methods for $(n+1)$-tuples of Banach spaces

by

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Abstract. It is shown that the main results of the theory of real interpolation, i.e. the equivalence and reiteration theorems, can be extended from couples to a class of $(n+1)$-tuples of Banach spaces, which includes $(n+1)$-tuples of Banach function lattices, Sobolev and Besov spaces. As an application of our results, it is shown that Lions' problem on interpolation of subspaces and Semenov's problem on interpolation of subcouples have positive solutions when all spaces are Banach function lattices or their retracts. In general, these problems have negative solutions.

0. Introduction. One of the most important achievements of the real interpolation method for couples is Lions–Peetre's remarkable reiteration formula

$$(\overrightarrow{X}_{\theta_0,q_0},\overrightarrow{X}_{\theta_1,q_1})_{\theta,q}=\overrightarrow{X}_{(1-\theta)\theta_0+\theta\theta_1,q}\qquad(\theta_0\neq\theta_1).$$

The crucial point of the proof of this formula is the so-called equivalence theorem for K- and J-methods:

$$\overrightarrow{X}_{\theta,q;K}=\overrightarrow{X}_{\theta,q;J}.$$

Sparr [S] shows that if an analog of the equivalence theorem is valid for an $(n+1)$-tuple then an analog of Lions–Peetre's formula is also valid for this tuple. However, extension of the equivalence theorem to $(n+1)$-tuples for $n>1$ encountered considerable difficulties.

The first counterexamples to the equivalence theorem are due to A. Yoshikawa [Y] and G. Sparr [S], but these examples are "degenerate" in the sense that the intersection of all spaces X_i $(i=0,1,\ldots,n)$ of the tuple $\overrightarrow{X}=(X_0,X_1,\ldots,X_n)$ consists only of zero.

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DEFINITION. We shall say that the *Fundamental Lemma with the Calderón operator S is valid* for the $(n+1)$-tuple $\overrightarrow{X}$ if any element $x \in \sigma(\overrightarrow{X})$ can be represented as a series

$$x = \sum_{\mathbf{k}\in\mathbb{Z}^n} x_{\mathbf{k}}, \tag{1}$$

absolutely convergent in $\Sigma(\overrightarrow{X})$, where $x_{\mathbf{k}} \in \Delta(\overrightarrow{X})$ and

$$J(2^{\mathbf{k}}, x_{\mathbf{k}}; \overrightarrow{X}) \le C[SK(\cdot, x; \overrightarrow{X})](2^{\mathbf{k}}). \tag{2}$$

Here and below $2^{\mathbf{k}} = (2^{k_1}, \ldots, 2^{k_n})$, where $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, and $C > 0$ is a constant (different in different places) independent of x and $\mathbf{k}$.

Remark. An important difference between conditions (1)–(2) and the F-condition of Sparr [S] is the appearance of the operator S in (2) and of the operator S^2 in the definition of $\sigma(\overrightarrow{X})$.

LEMMA 1. *Let $\overrightarrow{X}$ be an $(n+1)$-tuple consisting of Banach function lattices X_i $(i = 0, 1, \ldots, n)$ on Ω. Then the Fundamental Lemma with the Calderón operator S is valid for $\overrightarrow{X}$.*

Proof. Without loss of generality we can assume that for any $\mathbf{k} \in \mathbb{Z}^n$ and $x \in \Sigma(\overrightarrow{X})$ there can be found nonoverlapping sets $A_j(\mathbf{k}) = A_j(\mathbf{k}, x)$, $j = 0, 1, \ldots, n$, such that

$$\begin{aligned} &\text{(i)} \quad \bigcup_{j=0}^{n} A_j(\mathbf{k}) = \Omega, \\ &\text{(ii)} \quad K(2^{\mathbf{k}}, x; \overrightarrow{X}) \approx \|x\chi_{A_0(\mathbf{k})}\|_{X_0} + \sum_{j=1}^{n} 2^{k_j} \|x\chi_{A_j(\mathbf{k})}\|_{X_j}. \end{aligned} \tag{3}$$

The notation "$\approx$" means that the constants of equivalence are independent of $x \in \Sigma(\overrightarrow{X})$ and $\mathbf{k} \in \mathbb{Z}^n$.

We shall construct the required decomposition (1) of $x \in \sigma(\overrightarrow{X})$ in several steps, using a special partition of unity.

Step 1: *Construction of a new family $\overline{A}_j(\mathbf{k})$ with the monotonicity property*. For $\mathbf{k} \in \mathbb{Z}^n$ we define

$$\Omega_0(\mathbf{k}) := \{\mathbf{s} \in \mathbb{Z}^n \mid 1 = \min(1, 2^{k_1-s_1}, \ldots, 2^{k_n-s_n})\},$$
$$\Omega_j(\mathbf{k}) := \{\mathbf{s} \in \mathbb{Z}^n \mid 2^{k_j-s_j} = \min(1, 2^{k_1-s_1}, \ldots, 2^{k_n-s_n})\}, \quad j = 1, \ldots, n.$$

Let

$$\overline{A}_j(\mathbf{k}) := \bigcup_{\mathbf{s}\in\Omega_j(\mathbf{k})} A_j(\mathbf{s}), \quad j = 1, \ldots, n, \tag{4}$$

$$\overline{A}_0(\mathbf{k}) := \Omega \setminus \bigcup_{j=1}^{n} \overline{A}_j(\mathbf{k}). \tag{5}$$

It is obvious that $\overline{A}_j(\mathbf{k}) \supset A_j(\mathbf{k})$, $j = 1, \ldots, n$, and

$$\overline{A}_0(\mathbf{k}) \subset A_0(\mathbf{k}). \tag{6}$$

Moreover, it follows from (3), (4), (6) and the inequality $SK(\cdot, x; \overrightarrow{X}) \ge K(\cdot, x; \overrightarrow{X})$ that

$$\|x\chi_{\overline{A}_0(\mathbf{k})}\|_{X_0} + \sum_{j=1}^{n} 2^{k_j} \|x\chi_{\overline{A}_j(\mathbf{k})}\|_{X_j} \le C[SK(\cdot, x; \overrightarrow{X})](2^{\mathbf{k}}).$$

Let $\Gamma_i : \mathbb{Z}^n \to \mathbb{Z}^n$ $(i = 1, \ldots, n)$ be the shift operator on the ith variable:

$$\Gamma_i(\mathbf{k}) = \Gamma_i(k_1, \ldots, k_n) := (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n).$$

Then monotonicity of the family $\{\overline{A}_j(\mathbf{k})\}$ $(j = 1, \ldots, n)$:

$$\overline{A}_j(\Gamma_i(\mathbf{k})) \subset \overline{A}_j(\mathbf{k}) \quad \text{for } j \ne i, \quad \overline{A}_j(\Gamma_j(\mathbf{k})) \supset \overline{A}_j(\mathbf{k}), \tag{7}$$

follows from the analogous monotonicity properties of the sets $\Omega_j(\mathbf{k})$, $j = 0, 1, \ldots, n$.

Step 2: *Construction of elements from the intersection*. For any point $2^{\mathbf{k}} = (2^{k_1}, \ldots, 2^{k_n}) \in \mathbb{R}^n_+$ we define

$$B(\mathbf{k}) := \overline{A}_0(\mathbf{k}) \setminus \bigcup_{i=1}^{n} \overline{A}_0(\Gamma_i(\mathbf{k})). \tag{8}$$

Taking into account the definition (5) of $\overline{A}_0(\mathbf{k})$ and monotonicity (7) of $\overline{A}_i(\mathbf{k})$, $i = 1, \ldots, n$, we obtain

$$B(\mathbf{k}) = \bigcap_{i=1}^{n} \Big[\overline{A}_i(\Gamma_i(\mathbf{k})) \setminus \bigcup_{j=1}^{n} \overline{A}_j(\mathbf{k})\Big]. \tag{9}$$

Since $|x|\chi_{B(\mathbf{k})} \le |x|\chi_{\overline{A}_0(\mathbf{k})}$ and $|x|\chi_{B(\mathbf{k})} \le |x|\chi_{\overline{A}_i(\Gamma_i(\mathbf{k}))}$, $i = 1, \ldots, n$, it follows that $x\chi_{B(\mathbf{k})} \in \Delta(\overrightarrow{X})$ and for $i = 1, \ldots, n$ we have

$$2^{k_i} \|x\chi_{B(\mathbf{k})}\|_{X_i} \le 2^{k_i} \|x\chi_{\overline{A}_i(\Gamma_i(\mathbf{k}))}\|_{X_i} \le C[SK(\cdot, x; \overrightarrow{X})](2^{\Gamma_i(\mathbf{k})}).$$

Since $SK(\cdot, x; \overrightarrow{X})$ is a nondecreasing function, we obtain

$$\begin{aligned} &\|x\chi_{B(\mathbf{k})}\|_{X_0} \le C[SK(\cdot, x; \overrightarrow{X})](2^{\mathbf{k}}), \\ &2^{k_i} \|x\chi_{B(\mathbf{k})}\|_{X_i} \le C[SK(\cdot, x; \overrightarrow{X})](2^{\mathbf{k}}), \quad i = 1, \ldots, n. \end{aligned} \tag{10}$$

Let

$$y_{\mathbf{k}} := |x|\chi_{B(\mathbf{k})}, \quad k \in \mathbb{Z}^n. \tag{11}$$

Then it follows from (10) that

$$J(2^{\mathbf{k}}, y_{\mathbf{k}}; \overrightarrow{X}) \le C[SK(\cdot\,, x; \overrightarrow{X})](2^{\mathbf{k}}). \tag{12}$$

Step 3: *Construction of the required decomposition.* Since $x \in \sigma(\overrightarrow{X})$, from (9) we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} \|y_{\mathbf{k}}\|_{\Sigma(\vec{X})} \le \sum_{j=0}^{n} \sum_{\mathbf{k}\in\Omega_j(1)} \|y_{\mathbf{k}}\|_{X_j}$$
$$\le \sum_{\mathbf{k}\in\Omega_0(1)} C[SK(\cdot\,, x; \overrightarrow{X})](2^{\mathbf{k}}) + \sum_{j=1}^{n} \sum_{\mathbf{k}\in\Omega_j(1)} C\frac{[SK(\cdot\,, x,; \overrightarrow{X})](2^{k})}{2^{k_j}}$$
$$\le C[S^2K(\cdot\,, x; \overrightarrow{X})](1) < \infty.$$

Thus the series $\sum_{\mathbf{k}\in\mathbb{Z}^n} y_{\mathbf{k}}$ is absolutely convergent in $\Sigma(\overrightarrow{X})$ and therefore (see, for example, Corollary 2 of Theorem 1, Chapter 2 of [KPS]) it is pointwise convergent almost everywhere.

Below we shall show that the inequality

$$|x| \le \sum_{\mathbf{k}\in\mathbb{Z}^n} y_{\mathbf{k}} \tag{13}$$

holds almost everywhere.

If (13) is correct then $\operatorname{supp} x \subset \operatorname{supp} \sum_{\mathbf{k}\in\mathbb{Z}^n} y_{\mathbf{k}}$ and the series composed of the elements

$$x_{\mathbf{k}} = x\frac{y_{\mathbf{k}}}{\sum_{\mathbf{k}\in\mathbb{Z}^n} y_{\mathbf{k}}}$$

pointwise converges to x almost everywhere. It follows from (13) that

$$\|x_{\mathbf{k}}\|_{\Sigma(\vec{X})} \le \|y_{\mathbf{k}}\|_{\Sigma(\vec{X})}.$$

So the series $\sum_{\mathbf{k}\in\mathbb{Z}^n} x_{\mathbf{k}}$ is absolutely convergent in $\Sigma(\overrightarrow{X})$, and since it pointwise converges to x, its sum will be equal to x.

It follows from (13) and (12) that

$$J(2^{\mathbf{k}}, x_{\mathbf{k}}; \overrightarrow{X}) \le J(2^{\mathbf{k}}, y_{\mathbf{k}}; \overrightarrow{X}) \le C[SK(\cdot\,, x; \overrightarrow{X})](2^{\mathbf{k}}).$$

Therefore the elements $x_{\mathbf{k}}$ satisfy all the requirements of the lemma. So, we only need to prove the inequality (13).

It follows from the definition (11) of the elements $y_{\mathbf{k}}$ that (13) holds almost everywhere on $\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}$. Hence it is enough to prove that the set

$$\{\omega \in \Omega \mid x(\omega) \ne 0\} \setminus \bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}$$

has measure zero. For this it is enough to show that

$$\|x - x\chi_{\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}}\|_{\Sigma(\vec{X})} = 0.$$

Let us take an arbitrary $\varepsilon > 0$ and prove that

$$\|x - x\chi_{\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}}\|_{\Sigma(\vec{X})} < \varepsilon.$$

From the definition of the set $\bar{A}_0(\mathbf{m})$ ($\mathbf{m} = (m, \ldots, m) \in \mathbb{Z}^n$), using the fact that $x \in \sigma(\overrightarrow{X})$, we deduce that for sufficiently large m we have

$$\|x - x\chi_{\bar{A}_0(\mathbf{m})}\|_{\Sigma(\vec{X})} \le \sum_{i=1}^{n} \|x\chi_{\bar{A}_i(\mathbf{m})}\|_{X_i} \le C\frac{[SK(\cdot\,, x, \overrightarrow{X})](2^{\mathbf{m}})}{2^m} < \frac{\varepsilon}{2}.$$

From this and from the fact that

$$\|x - x\chi_{\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}}\|_{\Sigma(\vec{X})} \le \|x - x\chi_{\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}\cap A_0(\mathbf{m})}\|_{\Sigma(\vec{X})}$$

it is clear that in order to prove (13) it is enough to show that

$$\|x\chi_{\bar{A}_0(\mathbf{m})} - x\chi_{\bar{A}_0(\mathbf{m})\cap\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}}\|_{\Sigma(\vec{X})} < \frac{\varepsilon}{2}. \tag{14}$$

To prove this we shall consider the sets

$$\Omega_{m,l} = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \mid l < k_i \le m,\ i = 1, \ldots, n\}$$

and

$$\Omega_{m,l}^{j} = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \mid k_j = l,\ l < k_i \le m,\ i \ne j\}, \qquad j = 1, \ldots, n.$$

It follows from the definition (8) of $B(\mathbf{k})$ that

$$\bar{A}_0(\mathbf{k}) \subset B(\mathbf{k}) \cup \bigcup_{i=1}^{n} \bar{A}_0(\Gamma_i(\mathbf{k})). \tag{15}$$

In particular,

$$\bar{A}_0(\mathbf{m}) \subset B(\mathbf{m}) \cup \bigcup_{i=1}^{n} \bar{A}_0(\Gamma_i(\mathbf{m}))$$

and

$$\bar{A}_0(\Gamma_i(\mathbf{m})) \subset B(\Gamma_i(\mathbf{m})) \cup \bigcup_{j=1}^{n} \bar{A}_0(\Gamma_j\Gamma_i(\mathbf{m})).$$

Therefore

$$\bar{A}_0(\mathbf{m}) \subset B(\mathbf{m}) \cup \bigcup_{i=1}^{n} B(\Gamma_i(\mathbf{m})) \cup \bigcup_{i,j=1}^{n} \bar{A}_0(\Gamma_j\Gamma_i(\mathbf{m})).$$

Repeatedly using the embedding (15), we continue this process of replacing the sets $\bar{A}_0(\mathbf{s})$ for $\mathbf{s} \in \Omega_{m,l}$. Then we obtain

$$\bar{A}_0(\mathbf{m}) \subset \bigcup_{\mathbf{k}\in\Omega_{m,l}} B(\mathbf{k}) \cup \bigcup_{j=1}^{n} \bigcup_{\mathbf{s}\in\Omega^j_{m,l}} \bar{A}_0(\mathbf{s}).$$

Hence, taking into account (6) and (3), we have

$$\|x\chi_{\bar{A}_0(\mathbf{m})} - x\chi_{\bar{A}_0(\mathbf{m})\cap\bigcup_{\mathbf{k}\in\mathbb{Z}^n} B_{\mathbf{k}}}\|_{\Sigma(\vec{X})} \le \sum_{j=1}^{n} \sum_{\mathbf{s}\in\Omega^j_{m,l}} \|x\chi_{\bar{A}_0(\mathbf{s})}\|_{\Sigma(\vec{X})}$$

$$\le C\sum_{j=1}^{n} \sum_{\mathbf{s}\in\Omega^j_{m,l}} K(2^{\mathbf{s}}, x; \vec{X}).$$

We also note that

$$\sum_{\mathbf{s}\in\Omega^j_{m,l}} K(2^{\mathbf{s}}, x; \vec{X}) \le C[SK(\cdot, x; \vec{X})](2^{\Gamma_j^{m-l}(\mathbf{m})})$$

and it follows from $x \in \sigma(\vec{X})$ that for fixed m, as $l \to -\infty$,

$$[SK(\cdot, x; \vec{X})](2^{\Gamma_j^{m-l}(\mathbf{m})}) \to 0.$$

Now, (14) directly follows from this. ∎

Similarly to Proposition 5.2 of [S], it is easy to obtain:

Corollary 1. *Let the $(n+1)$-tuple $\vec{X}$ be a retract (or partial retract) of an $(n+1)$-tuple of Banach function lattices. Then the Fundamental Lemma with the Calderón operator S holds for $\vec{X}$.*

It is well known (see [BL]) that the $(n+1)$-tuple of Sobolev spaces

$$\vec{W}^{\mathbf{k}}_{\mathbf{p}} = (W^{k_0}_{p_0}, W^{k_1}_{p_1}, \ldots, W^{k_n}_{p_n}), \quad 1 < p_i < \infty,\ k_i \ge 0\ (i = 0, 1, \ldots, n),$$

is a retract of the tuple $(L_{p_0}(l_2^{k_0}), L_{p_1}(l_2^{k_1}), \ldots, L_{p_n}(l_2^{k_n}))$. Therefore we deduce from Corollary 1 that the Fundamental Lemma with the operator S is valid for the tuple $\vec{W}^{\mathbf{k}}_{\mathbf{p}}$.

Similarly, since the $(n+1)$-tuple of Besov spaces

$$\vec{B}^{\sigma,\mathbf{q}}_{\mathbf{p}} = (B^{\sigma_0,q_0}_{p_0}, B^{\sigma_1,q_1}_{p_1}, \ldots, B^{\sigma_n,q_n}_{p_n}), \quad 1 < p_i < \infty,\ 1 \le q_i \le \infty,\ \sigma_i > 0,$$

is a retract (see [BL]) of the tuple $(l^{\sigma_0}_{q_0}(L_{p_0}), l^{\sigma_1}_{q_1}(L_{p_1}), \ldots, l^{\sigma_n}_{q_n}(L_{p_n}))$, the Fundamental Lemma with the Calderón operator S is also valid for $\vec{B}^{\sigma,\mathbf{q}}_{\mathbf{p}}$.

Remark. In [A1] it was proved that the reiteration theorem is valid for an $(n+1)$-tuple of Banach function lattices if there exist sets $A_j(t,x)$ $(j = 0, \ldots, n)$ which are nonoverlapping and have some monotonicity properties. Later on the first author constructed the required sets for triples of Banach function lattices. The proof was rather combinatorial and all attempts to generalize it even to 4-tuples failed. In this paper we construct the sets $A_j(t,x)$ $(j = 0, \ldots, n)$ satisfying weaker conditions than in [A1] and show that they are sufficient. We rejected the idea of monotonicity of the set $A_0(t,x)$ and we do not need the sets $A_i(t,x)$ $(i > 0)$ to be nonoverlapping.

We would also like to note that Lemma 1 is not the first example of an interpolation result which is true for Banach function lattices and not valid in general. See, for example, [M]. This and some other results were reasons for the hope that the reiteration theorem is true for $(n+1)$-tuples of Banach fuction lattices.

2. The equivalence and reiteration theorems. It was shown above that the class of $(n+1)$-tuples for which the Fundamental Lemma with the Calderón operator S holds, is sufficiently wide and includes function spaces which are important for applications. In this section we show that for such $(n+1)$-tuples the equivalence and reiteration theorems are valid.

Definition. A Banach function lattice Φ on $\mathbb{R}^n_+$ with the measure $\frac{d\mathbf{t}}{\mathbf{t}} = \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}$ will be called a *parameter of the real method* if the Calderón operator S is bounded in Φ.

The most important example of parameters of the real method is the lattice $\Phi_{\theta,q}$. In this lattice the norm is defined by the formula

$$\|f\|_{\theta,q} = \Big[\int_{\mathbb{R}^n_+} (\mathbf{t}^{-\theta} f(\mathbf{t}))^q \,\frac{d\mathbf{t}}{\mathbf{t}}\Big]^{1/q},$$

where $\mathbf{t}^{-\theta} = t_1^{-\theta_1} \ldots t_n^{-\theta_n}$, $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_i > 0$, $\sum_{i=1}^n \theta_i < 1$ and $q \in [1, \infty]$.

Similarly to the case $\Phi = \Phi_{\theta,q}$ which was considered by Sparr [S], we define the interpolation spaces $K_\Phi(\vec{X})$ and $J_\Phi(\vec{X})$ by the norms

$$\|x\|_{K_\Phi(\vec{X})} = \|K(\cdot, x; \vec{X})\|_\Phi,$$

$$\|x\|_{J_\Phi(\vec{X})} = \inf\Big\{\|J(\cdot, u(\cdot); \vec{X})\|_\Phi : x = \int_{\mathbb{R}^n_+} u(s)\,\frac{ds}{s}\Big\}.$$

Theorem 1 (The Equivalence Theorem). *Let $\vec{X} = (X_0, \ldots, X_n)$ be an $(n+1)$-tuple of Banach spaces for which the Fundamental Lemma with the Calderón operator S is valid. Then for any parameter Φ of the real method* *we have*

$$K_\Phi(\overrightarrow{X}) = J_\Phi(\overrightarrow{X}).$$

Proof. The embedding $J_\Phi(\overrightarrow{X}) \hookrightarrow K_\Phi(\overrightarrow{X})$ immediately follows from the definitions of the norms of these spaces and the fact that the Calderón operator S is bounded in Φ.

The opposite embedding follows from the fact that the Fundamental Lemma with the operator S is valid for the $(n+1)$-tuple $\overrightarrow{X}$. Let $x \in K_\Phi(\overrightarrow{X})$. This means that $K(\cdot,x;\overrightarrow{X}) \in \Phi$. Hence $S^2K(\cdot,x;\overrightarrow{X}) \in \Phi$, i.e. $x \in \sigma(\overrightarrow{X})$. Therefore a decomposition of x into a series (1) satisfying the estimates (2) is possible.

Let

$$Q_{\mathbf{k}} = \{\mathbf{s} = (s_1,\ldots,s_n) \mid 2^{k_i} \le s_i < 2^{k_i+1},\ i = 1,\ldots,n\}, \quad \mathbf{k} \in \mathbb{Z}^n.$$

We define

$$u(\mathbf{s}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} (\ln 2)^{-n} x_{\mathbf{k}} \chi_{Q_{\mathbf{k}}}(\mathbf{s}),$$

where $x_{\mathbf{k}}$ is a summand in the decomposition (1) of x. Hence

$$x = \int_{\mathbb{R}^n_+} u(\mathbf{s})\,\frac{d\mathbf{s}}{\mathbf{s}}$$

and for any $\mathbf{s} \in \mathbb{R}^n_+$ from (2) and concavity of the K-functional we have

$$J(\mathbf{s}, u(\mathbf{s}); \overrightarrow{X}) \le C[SK(\cdot,x,\overrightarrow{X})](\mathbf{s}) \tag{16}$$

with the constant $C > 0$ independent of $\mathbf{s}$ and $x \in \sigma(\overrightarrow{X})$.

Applying $\|\cdot\|_\Phi$ to both sides of (16) we deduce from the boundedness of the operator S in Φ that $K_\Phi(\overrightarrow{X}) \hookrightarrow J_\Phi(\overrightarrow{X})$, and this completes the proof of the theorem. ■

Remark. It should be noted (see [BK]) that in the case of couples the equivalence theorem $K_\Phi = J_\Phi$ holds if and only if the operator S is bounded in Φ.

Existence of a wide class of $(n+1)$-tuples for which the equivalence theorem is valid leads to the necessity of distinguishing the class of "right" $(n+1)$-tuples of Banach spaces.

Definition. We shall call an $(n+1)$-tuple of Banach spaces an *LP-tuple* (*Lions–Peetre tuple*) if

$$K_\Phi(\overrightarrow{X}) = J_\Phi(\overrightarrow{X})$$

holds for any parameter Φ of the real method.

An important property of LP-tuples is:

Theorem 2 (The Reiteration Theorem). *Let $\overrightarrow{X}$ be an LP $(n+1)$-tuple of Banach spaces. Then for arbitrary parameters $\Phi_0,\Phi_1,\ldots,\Phi_m$ and Φ of the real method we have*

$$K_\Phi(K_{\Phi_0}(\overrightarrow{X}), K_{\Phi_1}(\overrightarrow{X}),\ldots,K_{\Phi_m}(\overrightarrow{X})) = K_\Psi(\overrightarrow{X}),$$

where $\Psi = K_\Phi(\Phi_0,\Phi_1,\ldots,\Phi_m)$.

Proof. Let $x \in \sum_{i=0}^m K_{\Phi_i}(\overrightarrow{X})$. Take a "nearly best" decomposition of this element, i.e. $x = \sum_{i=0}^m x_i$ and

$$\begin{aligned}\|x_0\|_{K_{\Phi_0}(\vec{X})} &+ \sum_{i=1}^m t_i\|x_i\|_{K_{\Phi_i}(\vec{X})}\\ &\le (1+\varepsilon)K(t,x;K_{\Phi_0}(\overrightarrow{X}),K_{\Phi_1}(\overrightarrow{X}),\ldots,K_{\Phi_m}(\overrightarrow{X})).\end{aligned}$$

We denote by ϕ the function $K(\cdot\, x;\overrightarrow{X})$ and by ϕ_i the function $K(\cdot,x_i;\overrightarrow{X})$ $(i = 0,1,\ldots,m)$. We note that $\phi_i \in \Phi_i$ $(i = 0,1,\ldots,m)$ and $\phi \le \sum_{i=0}^m \phi_i$. Since

$$\phi = \sum_{i=0}^m \phi_i \frac{\phi}{\sum_{i=0}^m \phi_i},$$

we have

$$\begin{aligned}K(t,\phi;\Phi_0,\Phi_1,\ldots,\Phi_m) &\le \|\phi_0\|_{\Phi_0} + \sum_{i=1}^m t_i\|\phi_i\|_{\Phi_i}\\ &= \|x_0\|_{K_{\Phi_0}(\vec{X})} + \sum_{i=1}^m t_i\|x_i\|_{K_{\Phi_i}(\vec{X})}\\ &\le (1+\varepsilon)K(t,x;K_{\Phi_0}(\overrightarrow{X}),\ldots,K_{\Phi_m}(\overrightarrow{X})).\end{aligned}$$

Applying the norm of the space Φ to both sides, we obtain

$$K_\Phi(K_{\Phi_0}(\overrightarrow{X}),\ldots,K_{\Phi_m}(\overrightarrow{X})) \hookrightarrow K_\Psi(\overrightarrow{X}).$$

To prove the opposite embedding we note that since $\overrightarrow{X}$ is an LP-tuple, it is enough to prove

$$J_\Psi(\overrightarrow{X}) \hookrightarrow K_\Phi(J_{\Phi_0}(\overrightarrow{X}),\ldots,J_{\Phi_m}(\overrightarrow{X})). \tag{17}$$

Let $x \in J_\Psi(\overrightarrow{X})$. Then there exists a decomposition of x as

$$x = \int_{\mathbb{R}^n_+} u(\mathbf{s})\,\frac{d\mathbf{s}}{\mathbf{s}} \quad (\text{convergence in } \Sigma(\overrightarrow{X}))$$

such that for $g(s) = J(s, u(s); \overrightarrow{X})$ we have

$$\|g\|_{\Psi} \le (1+\varepsilon)\|x\|_{J_{\Psi}(\vec{X})}. \tag{18}$$

Let the operator $A : (\Phi_0, \Phi_1, \ldots, \Phi_m) \to (J_{\Phi_0}(\overrightarrow{X}), J_{\Phi_1}(\overrightarrow{X}), \ldots, J_{\Phi_m}(\overrightarrow{X}))$ be defined by the formula

$$A\phi = \int_{\mathbb{R}_+^n} \phi(s) \frac{u(s)}{J(s, u(s); \overrightarrow{X})} \frac{ds}{s}.$$

It is clear that $\|A\|_{\Phi_i \to J_{\Phi_i}(\vec{X})} \le 1$ and $Ag = x$. Therefore

$$K(\cdot, x; J_{\Phi_0}(\overrightarrow{X}), \ldots, J_{\Phi_m}(\overrightarrow{X})) \le K(\cdot, g; \Phi_0, \ldots, \Phi_m). \tag{19}$$

Since $\Psi = K_{\Phi}(\Phi_0, \ldots, \Phi_m)$, applying the norm of the space Φ to both sides of (19) and taking into consideration (18) we obtain

$$\begin{aligned}\|K(\cdot, x; J_{\Phi_0}(\overrightarrow{X}), \ldots, J_{\Phi_m}(\overrightarrow{X}))\|_{\Phi} &\le \|K(\cdot, g; \Phi_0, \ldots, \Phi_m)\|_{\Phi} \\ &= \|g\|_{\Psi} \le (1+\varepsilon)\|x\|_{J_{\Psi}}(\overrightarrow{X}).\end{aligned}$$

This proves the required embedding (17). ■

As shown in Section 1, the Fundamental Lemma with the Calderón operator S holds for tuples consisting of Besov $B_p^{\sigma,q}$ or Sobolev W_p^k spaces. Therefore by Theorem 1 such tuples are LP-tuples, and the reiteration theorem is valid for them.

Remark. It should be noted (see [BK]) that in the case of couples the reiteration theorem holds without assuming that the operator S is bounded in parameters. However, from [A4] (see also [A2]–[A5]) it follows that without restrictions on parameters the reiteration theorem is not valid even for the triple $(L_1, L_1(1/t_1), L_1(1/t_2))$.

3. Some applications to interpolation of couples. Below it will be shown that Lions' and Semenov's problems have positive solutions in the case of Banach function lattices.

We need two remarkable theorems of Sparr [S]. For simplicity we shall give them in the particular case we need.

Let $\overrightarrow{U} = (U_0, U_1, U_2)$ be a triple of Banach spaces. Let

$$\overline{H} = \Big\{\overline{\lambda} = (\lambda_0, \lambda_1, \lambda_2) \;\Big|\; \lambda_i \ge 0 \ (i = 0, 1, 2),\ \sum_{i=0}^{2} \lambda_i = 1\Big\}$$

and

$$H = \Big\{\overline{\lambda} = (\lambda_0, \lambda_1, \lambda_2) \;\Big|\; \lambda_i > 0 \ (i = 0, 1, 2),\ \sum_{i=0}^{2} \lambda_i = 1\Big\}.$$

For $\overline{\lambda} \in H$ and $q \in [1, \infty]$ we denote by $\overrightarrow{U}_{\overline{\lambda},q}$ the space defined by the norm

$$\|x\|_{\overline{\lambda},q} = \Big\{\int_0^\infty \int_0^\infty \{t_1^{-\lambda_1} t_2^{-\lambda_2} K(t_1, t_2; x, \overrightarrow{U})\}^q \frac{dt_1}{t_1} \frac{dt_2}{t_2}\Big\}^{1/q} \tag{20}$$

with the usual changes for $q = \infty$.

In the case $\overline{\lambda} \in \overline{H} \setminus H$ we denote by $\overrightarrow{U}_{\overline{\lambda},q}$ the spaces

$$\overrightarrow{U}_{(1,0,0),q} = U_0, \quad \overrightarrow{U}_{(0,1,0),q} = U_1, \quad \overrightarrow{U}_{(0,0,1),q} = U_2$$

and

$$\overrightarrow{U}_{(\lambda_0,\lambda_1,0),q} = (U_0, U_1)_{\lambda_1,q}, \quad \overrightarrow{U}_{(0,\lambda_1,\lambda_2),q} = (U_1, U_2)_{\lambda_2,q},$$
$$\overrightarrow{U}_{(\lambda_0,0,\lambda_2),q} = (U_1, U_2)_{\lambda_2,q}.$$

Definition. We say that *the power equivalence theorem is valid for $\overrightarrow{U}$* if

$$\overrightarrow{U}_{\overline{\lambda},q;K} = \overrightarrow{U}_{\overline{\lambda},q;J} \tag{21}$$

for all $\overline{\lambda} \in H$ and $q \in [1, \infty]$.

Of course if $\overrightarrow{U}$ is an LP-triple, for example if U_i $(i = 0, 1, 2)$ are Banach function lattices, then (21) is valid.

Theorem A ([S]). *If the power equivalence theorem is valid for the triple $\overrightarrow{U} = (U_0, U_1, U_2)$ and the vectors $\overline{\lambda}^0, \overline{\lambda}^1, \overline{\lambda}^2 \in \overline{H}$ are linearly independent then*

(a) *the power equivalence theorem is true for the triple $(\overrightarrow{U}_{\overline{\lambda}^0,q_0}, \overrightarrow{U}_{\overline{\lambda}^1,q_1}, \overrightarrow{U}_{\overline{\lambda}^2,q_2})$,*

(b) *an analog of the Lions–Peetre reiteration formula holds for $\overline{\lambda} = (\lambda_0, \lambda_1, \lambda_2) \in H$:*

$$(\overrightarrow{U}_{\overline{\lambda}^0,q_0}, \overrightarrow{U}_{\overline{\lambda}^1,q_1}, \overrightarrow{U}_{\overline{\lambda}^2,q_2})_{\overline{\lambda},q} = \overrightarrow{U}_{\lambda_0\overline{\lambda}^0+\lambda_1\overline{\lambda}^1+\lambda_2\overline{\lambda}^2,q}. \tag{22}$$

Remark. It is important that $\overline{\lambda}$ belongs to H and not to $\overline{H} \setminus H$, because the formula

$$(\overrightarrow{U}_{\overline{\lambda}^0,q_0}, \overrightarrow{U}_{\overline{\lambda}^1,q_1})_{\theta,q} = \overrightarrow{U}_{(1-\theta)\overline{\lambda}^0+\theta\overline{\lambda}^1,q} \tag{23}$$

does not hold in general (see [S]).

The next theorem indicates one particular case in which the formula (23) is true.

THEOREM B ([S]). *If the power equivalence theorem is true for the triple* $\overrightarrow{U}=(U_0,U_1,U_2)$ *then for all* $\theta,\mu\in(0,1)$ *we have*

$$((U_0,U_2)_{\theta,q},(U_1,U_2)_{\theta,q})_{\mu,q}=\overrightarrow{U}_{((1-\theta)(1-\mu),(1-\theta)\mu,\theta),q}. \tag{24}$$

Remark. It would be interesting to investigate conditions for formula (23) to be valid.

We shall also need

THEOREM C ([JNP]). *If the couple* $\overrightarrow{Z}=(Z_0,Z_1)$ *is such that* $\overrightarrow{Z}_{\theta_0,q_0}=\overrightarrow{Z}_{\theta_1,q_1}$ *for some* $\theta_0\neq\theta_1$ *then* $Z_0\cap Z_1$ *is closed in* Z_0+Z_1 *and* $Z_{\theta,q}=Z_0\cap Z_1$ *for all* $\theta\in(0,1)$ *and* $q\in[1,\infty]$.

Now we are ready to prove

THEOREM 3. *Suppose that the power equivalence theorem is valid for the triple* $\overrightarrow{U}=(U_0,U_1,U_2)$. *Then*

(a) *if* $(U_0,U_2)_{\theta_0,q_0}=(U_1,U_2)_{\theta_0,q_0}$ *for some* $\theta_0\in(0,1)$ *and* $q_0\in[1,\infty]$ *then*

$$(U_0,U_2)_{\theta,q}=(U_1,U_2)_{\theta,q} \tag{25}$$

for all $\theta\in(0,1)$ *and* $q\in[1,\infty]$,

(b) *if* $U_0\subset U_1$ *is a closed subspace of* U_1 *then* $(U_0,U_2)_{\theta,q}$ *is a closed subspace of* $(U_1,U_2)_{\theta,q}$ *for all* $\theta\in(0,1)$ *and* $q\in[1,\infty]$.

Proof. (a) Define $X_0=(U_0,U_2)_{\theta_0,q_0}$ and $X_1=(U_1,U_2)_{\theta_0,q_0}$. Then from Theorem A for any $\overline{\lambda}\in H$ we have

$$V_{\overline{\lambda},q}=(X_0,X_1,U_0)_{\overline{\lambda},q}=\overrightarrow{U}_{(\lambda_0(1-\theta_0)+\lambda_2,\lambda_1(1-\theta_0),(1-\lambda_2)\theta_0),q} \tag{26}$$

and

$$\widetilde{V}_{\overline{\lambda},q}=(X_0,X_1,U_1)_{\overline{\lambda},q}=\overrightarrow{U}_{(\lambda_0(1-\theta_0),\lambda_1(1-\theta_0)+\lambda_2,(1-\lambda_2)\theta_0),q}. \tag{27}$$

As $\lambda_0+\lambda_1=1-\lambda_2$ there exists $\gamma\in(0,1)$ such that $\lambda_0=(1-\lambda_2)(1-\gamma)$ and $\lambda_1=(1-\lambda_2)\gamma$. So if we apply Theorem B to the triple (X_0,X_1,U_0) we have

$$V_{\overline{\lambda},q}=((X_0,U_0)_{\lambda_2,q},(X_1,U_0)_{\lambda_2,q})_{\gamma,q}.$$

As $X_0=X_1$ by assumption, from the last formula it follows that

$$V_{\overline{\lambda},q}=(X_0,U_0)_{\lambda_2,q}=((U_0,U_2)_{\theta_0,q_0},U_0)_{\lambda_2,q}=(U_0,U_2)_{(1-\lambda_2)\theta_0,q}.$$

And, analogously,

$$\widetilde{V}_{\overline{\lambda},q}=(U_1,U_2)_{(1-\lambda_2)\theta_0,q}.$$

So we see that for fixed $\lambda_2\in(0,1)$ the family $V_{\overline{\lambda},q}$ consists of spaces equal to $(U_0,U_2)_{(1-\lambda_2)\theta_0,q}$ and the family $\widetilde{V}_{\overline{\lambda},q}$ of spaces equal to $(U_1,U_2)_{(1-\lambda_2)\theta_0,q}$.

On the other hand, from Theorem B and (26)–(27) it follows that for fixed $\lambda_2\in(0,1)$ spaces from $V_{\overline{\lambda},q}$ and $\widetilde{V}_{\overline{\lambda},q}$ are θ,q interpolation spaces of the couple $((U_0,U_2)_{(1-\lambda_2)\theta_0,q},(U_1,U_2)_{(1-\lambda_2)\theta_0,q})$.

Applying Theorem C we obtain $(U_0,U_2)_{(1-\lambda_2)\theta_0,q}=(U_1,U_2)_{(1-\lambda_2)\theta_0,q}$.

So for $\theta<\theta_0$ and $q\in[1,\infty]$ we have

$$(U_0,U_2)_{\theta,q}=(U_1,U_2)_{\theta,q}. \tag{28}$$

If we interpolate the spaces from (28) with U_2 and use the reiteration theorem for couples then we obtain (28) also for $\theta\ge\theta_0$:

$$\begin{aligned}(U_0,U_2)_{(1-\mu)\theta+\mu,q}&=((U_0,U_2)_{\theta,q},U_2)_{\mu,q}\\&=((U_1,U_2)_{\theta,q},U_2)_{\mu,q}=(U_1,U_2)_{(1-\mu)\theta+\mu,q}.\end{aligned}$$

(b) Since $(U_0,U_1)_{\theta,q}=U_0$, from Theorem B it follows that for $\overline{\lambda}\in H$ the space

$$V_{\overline{\lambda},q}=(U_0,(U_0,U_1)_{\theta,q},U_2)_{\overline{\lambda},q}=U_{(\lambda_0+\lambda_1(1-\theta),\lambda_1\theta,\lambda_2),q} \tag{29}$$

is equal to the space $(U_0,U_2)_{\lambda_2,q}$.

Applying Theorem B to the triple $\overrightarrow{U}=(U_0,U_1,U_2)$ we see that the space on the right hand side of (29) can be obtained by interpolation from the couple $((U_0,U_2)_{\lambda_2,q},(U_1,U_2)_{\lambda_2,q})$.

As the family $V_{\overline{\lambda},q}$ for fixed λ_2 consists of equal spaces, from Theorem C and the embedding $U_0\subset U_1$ it follows that the spaces $V_{\overline{\lambda},q}=(U_0,U_2)_{\lambda_2,q}$ are closed in $(U_1,U_2)_{\lambda_2,q}$. ∎

From the next result it follows that Lions' problem (see [L]) has a positive solution for Banach function lattices.

COROLLARY 2. *Let* X_0, X_1, Y_0, Y_1 *be Banach function lattices and* X_i *be closed subspaces of* Y_i $(i=0,1)$. *Then for all* $\theta\in(0,1)$ *and* $q\in[1,\infty]$ *the space* $(X_0,X_1)_{\theta,q}$ *is closed in* $(Y_0,Y_1)_{\theta,q}$.

Proof. First consider the triple (X_0,Y_0,X_1). As X_0 is a closed subspace of Y_0, from Theorem 3(b) it follows that $(X_0,X_1)_{\theta,q}$ is a closed subspace of $(Y_0,X_1)_{\theta,q}$. Analogously if we consider the triple (X_1,Y_1,Y_0) then we find that $(X_1,Y_0)_{\theta,q}$ is a closed subspace of $(Y_1,Y_0)_{\theta,q}$ for all $\theta\in(0,1)$ and $q\in[1,\infty]$. From this and the general equality $(Z_0,Z_1)_{\theta,q}=(Z_1,Z_0)_{1-\theta,q}$ we obtain the result. ∎

The next proposition shows that Semenov's problem also has a positive solution for Banach function lattices.

COROLLARY 3. *Let* X_0, X_1, Y_0, Y_1 *be Banach function lattices and* $X_i\hookrightarrow Y_i$ $(i=0,1)$. *If* $(X_0,X_1)_{\theta_0,q_0}=(Y_0,Y_1)_{\theta_0,q_0}$ *for some* $\theta_0\in(0,1)$ *and* $q_0\in[1,\infty]$ *then* $(X_0,X_1)_{\theta,q}=(Y_0,Y_1)_{\theta,q}$ *for all* $\theta\in(0,1)$ *and* $q\in[1,\infty]$.

Proof. First consider the triple (X_0, Y_0, X_1). The equality $(X_0, X_1)_{\theta_0,q_0} = (Y_0, Y_1)_{\theta_0,q_0}$ and the embeddings $X_i \hookrightarrow Y_i$ $(i = 0, 1)$ imply that

$$(X_0, X_1)_{\theta_0,q_0} \hookrightarrow (Y_0, X_1)_{\theta_0,q_0} \hookrightarrow (Y_0, Y_1)_{\theta_0,q_0} = (X_0, X_1)_{\theta_0,q_0}.$$

So $(X_0, X_1)_{\theta_0,q_0} = (Y_0, X_1)_{\theta_0,q_0}$. Now Theorem 3(a) shows that $(X_0, X_1)_{\theta,q} = (Y_0, X_1)_{\theta,q}$ for all $\theta \in (0, 1)$ and $q \in [1, \infty]$.

Analogous consideration of the triple (X_1, Y_1, Y_0) leads to the equality $(X_1, Y_0)_{\theta,q} = (Y_1, Y_0)_{\theta,q}$ for all $\theta \in (0, 1)$ and $q \in [1, \infty]$. From this and $(Y_0, X_1)_{\theta,q} = (X_1, Y_0)_{1-\theta,q}$, the required statement follows. ∎

The next result which is usually called the uniqueness theorem for the real method spaces was obtained in [CN] under some additional assumptions (Fatou and some other properties).

COROLLARY 4. *Let X_0, X_1, Y_0, Y_1 be Banach function lattices. If*

$$(X_0, X_1)_{\theta_i,q_i} = (Y_0, Y_1)_{\theta_i,q_i} \quad (i = 0, 1)$$

for some $\theta_0, \theta_1 \in (0, 1)$, $\theta_0 \neq \theta_1$ and $q_0, q_1 \in [1, \infty]$, then

$$(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$$

for all $\theta \in (0, 1)$ and $q \in [1, \infty]$.

Proof. Without loss of generality we can assume that $\theta_0 < \theta_1$. If $\lambda \in (0, 1)$ is such that $(1 - \theta_0)(1 - \lambda) = 1 - \theta_1$ then

$$(X_0, X_1)_{\theta_1,q_1} = ((X_0, X_1)_{\theta_0,q_0}, X_1)_{\lambda,q_1} \tag{30}$$

and

$$(Y_0, Y_1)_{\theta_1,q_1} = ((Y_0, Y_1)_{\theta_0,q_0}, Y_1)_{\lambda,q_1} = ((X_0, X_1)_{\theta_0,q_0}, Y_1)_{\lambda,q_1} \tag{31}$$

Consider the triple $((X_0, X_1)_{\theta_0,q_0}, X_1, Y_1)$. From the equality $(X_0, X_1)_{\theta_1,q_1} = (Y_0, Y_1)_{\theta_1,q_1}$ and (30)–(31) it follows that

$$((X_0, X_1)_{\theta_0,q_0}, X_1)_{\lambda,q_1} = ((X_0, X_1)_{\theta_0,q_0}, Y_1)_{\lambda,q_1}.$$

Applying Theorem 3(a) we obtain

$$((X_0, X_1)_{\theta_0,q_0}, X_1)_{\mu,q} = ((X_0, X_1)_{\theta_0,q_0}, Y_1)_{\mu,q} = ((Y_0, Y_1)_{\theta_0,q_0}, Y_1)_{\mu,q}$$

for all $\mu \in (0, 1)$ and $q \in [1, \infty]$. From this and the reiteration theorem for couples it follows that for $\theta > \theta_0$ we have

$$(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}.$$

The case $\theta < \theta_1$ is considered in an analogous way. ∎

Theorem 3 allows us to give a simple example of a triple for which the power equivalence theorem is not valid. In [W] it was shown that there exists a subspace M of codimension 1 in L_1 such that $(M, L_\infty)_{\theta,q} = (L_1, L_\infty)_{\theta,q}$ for $\theta > 1/2$ and $(M, L_\infty)_{1/2,2}$ is not closed in $(L_1, L_\infty)_{1/2,2}$. These properties imply

COROLLARY 5. *The power equivalence theorem is not valid for the triple (M, L_1, L_∞).*

Proof. Otherwise from Theorem 3(b) it would follow that $(M, L_\infty)_{\theta,q}$ is a closed subspace of $(L_1, L_\infty)_{\theta,q}$ for all $\theta \in (0, 1)$ and $q \in [1, \infty]$. But this is not the case for $\theta = 1/2$ and $q = 2$. ∎

Remark. The same example shows that Semenov's problem has a negative answer for general couples.

A counterexample to Lions' problem can also be found in [W].

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An open mapping theorem for analytic multifunctions

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Abstract. The paper gives sufficient conditions for projections of certain pseudoconcave sets to be open. More specifically, it is shown that the range of an analytic set-valued function whose values are simply connected planar continua is open, provided there does not exist a point which belongs to boundaries of all the fibers. The main tool is a theorem on existence of analytic discs in certain polynomially convex hulls, obtained earlier by the author.

Introduction and results. Analytic multifunctions are set-valued generalizations of analytic mappings (cf. [Ok], [Sł1]). They found applications mostly in functional analysis: in spectral theory, the structure of the Gelfand space of a uniform algebra and in the complex interpolation method for Banach spaces. However, they have also led to interesting new vistas in complex analysis, in relation with polynomially convex hulls and the corona problem. On the other hand, it is natural to ask which portions of the classical function theory extend to analytic multifunctions. While this approach did not always produce interesting questions, the problem of generalizing the classical open mapping theorem, posed by Ransford [Ra1], has proved to be rather intricate. This paper is devoted to the above problem.

Recall that an upper semicontinuous compact-valued correspondence (briefly, *multifunction*) $z \to K_z : G \to 2^{\mathbb{C}}$, $G \subset \mathbb{C}$ open, is called *analytic* if the set $U = \{(z,w) : z \in G,\ w \notin K_z\}$ is a pseudo-convex domain.

It is easy to see that a naive generalization of the open mapping theorem is false. Let K_z be the closed segment joining z to 1, for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then $z \to K_z : \mathbb{D} \to 2^{\mathbb{C}}$ is an analytic multifunction but its range is $R(K) := \bigcup\{K_z : z \in \mathbb{D}\} = \mathbb{D} \cup \{1\}$, and is clearly not an open set. The reason is that point 1 belongs to the boundaries of all the sets K_z, $z \in \mathbb{D}$. These observations are due to Ransford who explored the problem initially in [Ra1] and modified it as follows.

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