n → ∞. Let f₁ = |ₗ₁|. If k ∈ ℕ, and 2ᵏ⁻¹ < j ≤ 2ᵏ, let fₖ = \sqrt{2ᵏ⁻¹} |ₗ₁|. Define n₁ = 1. Since x₁ = T(H₁ · r₁) ∈ m²,∞(Γ, w), there is a finite subset σ₁ of I such that \|x₁χ₀\| < 2⁻³. Now suppose that numbers nᵢ and finite sets σᵢ have been chosen for i ≤ j. Since T(fⱼ⁺¹ · rᵢ) → 0 weakly as n → ∞, and \(\bigcup_{i=1}^{j} σ_i\) is finite, there exists nⱼ⁺¹ > nⱼ so that \(\|xⱼ⁺¹χ₀\| < 2⁻²⁻⁴\), where xⱼ⁺¹ = T(fⱼ⁺¹ · rⱼ⁺⁺₁). Now we can choose a finite subset σⱼ⁺¹ of I, disjoint from \(\bigcup_{i=1}^{j} σ_i\), such that \(\|xⱼ⁺¹χ₀σⱼ⁺¹\| < 2⁻²⁻³\). Finally, let yⱼ = xⱼχ₀σⱼ for all j ∈ ℕ. Then \((yⱼ)\) is pairwise disjoint sequence, and hence is a basic sequence with basis constant 1. Moreover, 
\[\|yⱼ\| > \|xⱼ\| - 2⁻²⁻² ≥ \|fⱼ · rⱼ\| - 2⁻²⁻² > 1/2.\]

Also, \(\sum \|xⱼ - yⱼ\| < 1/4\). By Proposition 1.a.9 in [5], \((yⱼ)\) and \((xⱼ)\) are equivalent. But then \((fⱼ · rⱼ)\) is equivalent to a pairwise disjoint sequence in \(ℓ²,∞(Γ, w)\). However, it is easy to see that \((fⱼ · rⱼ)\) is equivalent to \((aⱼ hⱼ)\), where \(a₁ = 1\) and \(aⱼ = \sqrt{2ᵏ⁻¹}\) if \(2ᵏ⁻¹ < j ≤ 2ᵏ\). Hence we obtain an embedding \(S\) of \([hⱼ]\) into \(ℓ²,∞(Γ, w)\) such that \((S hⱼ)\) is a pairwise disjoint sequence. As \((hⱼ)\) is a basis of \(M₅,∞[₀, ₁]\), we have reached a contradiction to Proposition 8. This completes the proof of Theorem 3.

References


Distinguishing Jordan polynomials
by means of a single Jordan-algebra norm

by

A. MORENO GALINDO (Granada)

Abstract. For \(K = R\) or \(C\) we exhibit a Jordan-algebra norm \(| \cdot |\) on the simple associative algebra \(M_∞(K)\) with the property that Jordan polynomials over \(K\) are precisely those associative polynomials over \(K\) which act \(| \cdot |\)-continuously on \(M_∞(K)\). This analytic determination of Jordan polynomials improves the one recently obtained in [6].

1. Introduction. The Jordan product of a (real or complex) associative algebra is defined as the symmetrization of the associative product. Jordan polynomials are those (non-commutative) associative polynomials which can be expressed from the indeterminates by means of a finite process of taking sums, multiplications by scalars, and Jordan products. Clearly, every Jordan polynomial acts continuously on any associative algebra endowed with a Jordan-algebra norm. The question of the continuity of the action of particular non-Jordan associative polynomials (like the associative product \(xy\) or the tetrad \(xyzt + tzyx\)) on suitable associative algebras endowed with Jordan-algebra norms has received special attention in the literature, mainly because of its close relation to positive results and limits in the normed treatment of the Zel’manov prime theorem [15] for Jordan algebras. In this direction the interested reader can consult [14], [10], [11], [2], [8], [6], [7], [12], [13], [3], [4] and [9]. The introduction of [5], together with that of [9] already quoted, can also be interesting for a historical view of progresses in the above mentioned question. Among these progresses, we only emphasize here that every Jordan-algebra norm on a simple associative algebra with unit makes the associative product (and hence, every associative polynomial) continuous, and that the result need not remain true if the assumption of the existence of a unit is removed [3]. In fact, a first “monster” is built in [3] by providing a Jordan-algebra norm on the simple associative algebra \(M∞(K)\) (of all countably infinite matrices over \(K\) with a finite number of non-zero entries) and a \(K\)-linear involution * on
$M_\infty(K)$ such that the action of the tetrad (hence the associative product) on $H(M_\infty(K), \ast) := \{ A \in M_\infty(K) : A^* = A \}$ is discontinuous.

The question of the continuity of the action of general associative polynomials on associative algebras endowed with Jordan-algebra (semi-) norms has been first considered by R. Arens and M. Goldberger [1]. They prove that for “almost” every non-Jordan associative polynomial $p$ there exists a non-simple associative algebra (depending only on the degree of $p$ and the number of indeterminates involved in $p$) endowed with a Jordan-algebra seminorm making the action of $p$ discontinuous. Very recently, the Arens-Goldberger result has been significantly improved in [5], where it is shown that, for every non-Jordan associative polynomial $p$ over $K = \mathbb{R}$ or $\mathbb{C}$, there exists a Jordan-algebra norm $| \cdot |$ (depending only on the degree of $p$ and the number of indeterminates involved by $p$) on $M_\infty(K)$ such that the action of $p$ on $M_\infty(K)$ is $| \cdot |$-discontinuous. Moreover, the $| \cdot |$-discontinuity of the action of $p$ can be centered in $H(M_\infty(K), \ast)$ for a suitable $K$-linear involution $\ast$ on $M_\infty(K)$, which can be chosen of arbitrarily given type (hermitian or alternate).

In this paper we present the “absolute monster” for the analytical determination of Jordan polynomials. Precisely, for $K = \mathbb{R}$ or $\mathbb{C}$, we construct a Jordan-algebra norm on $M_\infty(K)$ making the action of any non-Jordan polynomial on $M_\infty(K)$ discontinuous. Moreover, our norm exhibits all additional pathologies of the norms built in [5]. For the most part, our arguments are more or less deep refinements of the ideas developed in [5]. However, we would like to emphasize, as a new auxiliary result of independent interest, the existence of a Jordan subalgebra $J$ of $M_\infty(K)$ such that any non-Jordan associative polynomial leaves is invariant. This property of $M_\infty(K)$ is shared in an obvious way by the free associative algebra on a countably infinite set of indeterminates, but this last algebra is not simple. It is also worth mentioning that, for a suitable (associative) algebra-norm $\| \cdot \|$ on $M_\infty(K)$, the Jordan subalgebra $J$ above becomes $\| \cdot \|$-closed.

2. The result. As we have said in the introduction, our work continues and refines the ideas developed in [5]. Therefore, in order to avoid repetition, we refer the reader to that paper for all standard concepts not explicitly explained here.

Given a field $F$, a natural number $n$, and $\epsilon = \pm 1$, we consider the involution $\ast$ on $M_{2n}(F)$ defined by $a^* := s^{-1}a'ts$, where $a'$ denotes the transpose of $a$ and $s := \text{diag}(q, n, q)$ with $q := (1, -1)$. If $\epsilon = 1$, then $\ast$ will be called the symmetric involution on $M_{2n}(F)$. In the case $\epsilon = -1$ we obtain the familiar symplectic involution. Both the symmetric and the symplectic involutions pass from matrix algebras of the form $M_{2n}(F)$ ($n \in \mathbb{N}$) to the algebra $M_\infty(F)$ (of all countably infinite matrices over $F$ with a finite number of non-zero entries) by regarding $M_\infty(F)$ as $\bigcup_{n \in \mathbb{N}} M_{2n}(F)$ in the most natural way.

For a real or complex associative algebra $A$, a Jordan-algebra norm on $A$ is a norm $| \cdot |$ on the vector space of $A$ satisfying $|a b| \leq |a||b|$ for all $a, b$ in $A$, where $a b := \frac{1}{2}(a b + b a)$ is the Jordan product of $A$.

Now, we can state our main result:

**Theorem.** Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$, and denote by $\ast$ either the symmetric or the symplectic involution on $M_\infty(K)$. Then there exists a Jordan-algebra norm on $M_\infty(K)$ making discontinuous the action on $H(M_\infty(K), \ast)$ of every non-Jordan associative polynomial.

As in [5], the proof of the theorem relies on two results of independent interest (Propositions 1 and 2 below) refining the corresponding Propositions 1 and 2 of that paper.

Given an algebra $B$, we denote by $M_\infty(B)$ the algebra of all countably infinite matrices over $B$ with a finite number of non-zero entries. In the proof of the next proposition, for $n$ in $\mathbb{N}$, we will identify the algebra $M_n(B)$ of all $n \times n$ matrices over $B$ with the subalgebra of $M_\infty(B)$ of those matrices $(b_{ij})_{i,j \in \mathbb{N}}$ in $M_\infty(B)$ satisfying $b_{ij} = 0$ whenever either $i > n$ or $j > n$. If $B$ has an involution $\ast$, then $M_\infty(B)$ has a “standard” involution (also denoted by $\ast$) consisting in transposing a given matrix and applying the original involution to each entry.

**Proposition 1.** Let $(B, | \cdot |)$ be an associative normed algebra over $K$ ($= \mathbb{R}$ or $\mathbb{C}$), and $J$ be a closed Jordan subalgebra of $B$. Then there exists a Jordan-algebra norm $| \cdot |$ on $M_\infty(B)$ making discontinuous the action on $M_\infty(B)$ of every associative polynomial $p$ such that $J$ is not invariant under $p$. Moreover, if $B$ has an involution $\ast$, and if $J$ is contained in $H(B, \ast)$, then the norm $| \cdot |$ can be chosen in such a way that the action on $H(M_\infty(B), \ast)$ of every polynomial $p$ as above is $| \cdot |$-continuous.

**Proof.** The proof of this proposition involves only minor changes on that of [5, Proposition 1], hence we limit ourselves to provide a sketch of it, emphasizing only the required changes.

We consider the algebra norm $| \cdot |$ on $M_\infty(B)$ defined by

$$
| (b_{ij}) | := \sum_{i,j \in \mathbb{N}} |b_{ij}|
$$

for all $(b_{ij})$ in $M_\infty(B)$. Given a subspace $S$ of $M_\infty(B)$ and an element $\alpha$ in $M_{\infty}(B)$, we write $| \alpha + S | := \inf \{|a + \beta| : \beta \in S\}$. Also, we consider the identification $M_{\infty}(B) = M_{\infty}(K) \otimes_K B$. For $k$ in $\mathbb{N}$, we denote by $J_k$ the Jordan subalgebra of $M_\infty(B)$ given by $J_k := M_{k-1}(K) \otimes_K B + e_k \otimes J$, where $M_0(K) := 0$, and we denote by $e_k$ the element $(\lambda_{ij})_{i,j \in \mathbb{N}}$ in $M_\infty(K)$ given by $\lambda_{ij} = 0$ whenever $(i, j) \neq (k, k)$, and $\lambda_{kk} = 1$. 
Now the norm \(| \cdot |\) on the vector space of \(M_\infty(B)\) defined by
\[
|\alpha| := \|\alpha\| + \sum_{i=1}^{\infty} 2^i \|\alpha + J_i\|
\]
is a Jordan-algebra norm satisfying
\[
|e_k \otimes b| = (1 + 2^{k+1} + 2^{k+2} + \ldots + 2^{(k-1)b+1})\|b\| + 2^k \|b + J\|
\]
for all \(k \in \mathbb{N}\) and \(b \in B\).

Let \(q = q(x_1, \ldots, x_s)\) be a homogeneous associative polynomial such that \(J\) is not invariant under \(q\), and let \(m\) denote the degree of \(q\). Then there exist \(x_1, \ldots, x_s \in J\) satisfying \(|q(x_1, \ldots, x_s) + J| > 0\), and we easily obtain
\[
|q(e_k \otimes x_1, e_k \otimes x_2, \ldots, e_k \otimes x_s)| \leq \max\{|e_k \otimes x_1|, |e_k \otimes x_2|, \ldots, |e_k \otimes x_s|\}^m \cdot 2^{kbk} \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m
\]
for \(k > m\), where
\[
|q(x_1, x_2, \ldots, x_s)| \leq \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m \cdot 2^{(k-1)b+1} \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m
\]
Therefore, for \(k > m\), we have
\[
|q(e_k \otimes x_1, e_k \otimes x_2, \ldots, e_k \otimes x_s)| \leq \max\{|e_k \otimes x_1|, |e_k \otimes x_2|, \ldots, |e_k \otimes x_s|\}^m \cdot 2^{(k-1)b+1} \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m
\]
\[
|q(x_1, x_2, \ldots, x_s)| \leq \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m \cdot 2^{(k-1)b+1} \max\{|x_1|^m, |x_2|^m, \ldots, |x_s|^m\}^m
\]
From [5, Lemma 1] we deduce that the action of \(q\) on \(M_\infty(B)\) is not \(| \cdot |\)-continuous at zero. The passing from homogeneous polynomials to general ones, as well as the remaining part of the proof, follow without changes the corresponding arguments in [5, Proposition 1].

**Proposition 2.** Let \(F\) be a field of characteristic not two, and let \(*\) denote either the symmetric or the symplectic involution on \(H_\infty(F)\). Then there exists a Jordan subalgebra \(J\) of \(M_\infty(F)\) contained in \(H_\infty(F)\) such that \(J\) is not invariant under any non-Jordan associative polynomial. Moreover, if \(F = \mathbb{R}\) or \(C\), and if we consider the algebra norm \(|(\mu_{ij})| := \sum_{(i,j) \in \mathbb{N}^2} |\mu_{ij}|\) on \(M_\infty(F)\), then the Jordan subalgebra \(J\) above can be chosen \(| \cdot |\)-closed.

**Proof.** For \(p\) in \(2\mathbb{N} \cup \{\infty\}\), let \(*\) denote the symmetric involution on \(M_p(F)\) (the argument for symplectic involutions is the same). According to [5, Proposition 2], for every natural number \(n\) there exists an even number \(a_n\) and a Jordan subalgebra \(J_n\) of \(M_{a_n}(F)\) contained in \(H(M_{a_n}(F))\) such that \(J_n\) is not invariant under any non-Jordan associative polynomial involving at most \(n\) indeterminates and of degree \(\leq n\).

For \(p, q \in \mathbb{N}\), denote by \(M_{p,q}(F)\) the vector space of all \(p \times q\)-matrices over \(F\) (so that \(M_p(F) = M_{p,0}(F)\)) and consider the algebra whose vector space is the abstract direct sum of the family \(\{M_{a_n}(F)\}_{n \in \mathbb{N}}\) and whose product is determined, for elements \(A_{n,m} \in M_{a_n}(F)\) and \(B_{n',m'} \in M_{a_{n'}}(F)\), by
\[
A_{n,m}B_{n',m'} = \begin{cases} A_{n,m}B_{n',m'} \in M_{a_n,a_{n'}} & \text{(the usual product)} \\ 0 & \text{otherwise} \end{cases}
\]
Then the algebra presented above is a copy of \(M_\infty(F)\), via the mapping
\[
\bigoplus_{n,m \in \mathbb{N}} A_{n,m} \mapsto \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots \\ A_{2,1} & A_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}
\]

Moreover, in that identification, the restriction of the symmetric involution on \(M_\infty(F)\) to each diagonal summand \(M_{a_n,a_n}(F)\) is nothing but the symmetric involution on that summand. Putting \(J := \bigoplus_{n,m \in \mathbb{N}} K_{n,m}\) with \(K_{n,n} = J_n\) and \(K_{n,m} = 0\) if \(n \neq m\), it follows that \(J\) is a Jordan subalgebra of \(M_\infty(F)\) contained in \(H(M_\infty(F))\). Also \(J\) is not invariant under any non-Jordan associative polynomial. Indeed, if \(p(x_1, \ldots, x_n)\) is a non-Jordan associative polynomial of degree \(g\), then, for \(n := \max(a_n, g)\), we have \(p(J_n) \subseteq M_{a_n}(F)\) and \(p(J_n) \not\subseteq J_n\), and therefore \(p(J) \not\subseteq J\).

Now assume \(F = \mathbb{R}\) or \(C\) and let \(|\cdot|\) be the norm on \(M_\infty(F)\) given in the statement of the proposition. Let \(\{X_k\}\) be a sequence in \(J\) convergent to some element \(A\) in \(M_\infty(F)\). Then there exists \(N \in \mathbb{N}\) such that \(A \in \bigoplus_{n,m \leq N} M_{a_n,a_n}(F)\). Since the natural projection \(H_\infty(F)\) onto \(\bigoplus_{n,m \leq N} M_{a_n,a_n}(F)\) is \(|\cdot|\)-continuous, \(\{p(X_k)\}\) converges to \(A\). Since \(\{p(J)\}\) is finite-dimensional, it follows that \(A \in H_\infty(F)\). Therefore \(J\) is \(|\cdot|\)-closed in \(M_\infty(F)\).

Now we are ready to conclude the proof of our main result.

**Proof of the theorem.** Applying Proposition 1 with \(B = M_\infty(K)\), \(|\cdot|\) equal to the algebra norm on \(M_\infty(K)\) given in the statement of Proposition 2, and \(J\) equal to the closed Jordan subalgebra of \(M_\infty(K)\) provided also by Proposition 2, we obtain a Jordan-algebra norm \(|\cdot|\) on \(M_\infty(M_\infty(K))\) making the action on \(H(M_\infty(M_\infty(K)), *)\) of every non-Jordan associative polynomial discontinuous. Now, the proof is concluded by realizing that the algebras with involution \((M_\infty(M_\infty(K)), *)\) and \((M_\infty(K), *)\) are isomorphic. Indeed, regarding \(M_\infty(M_\infty(K))\) as \(M_\infty(K) \otimes_K M_\infty(K)\), the standard involution on \(M_\infty(M_\infty(K))\) relative to either the symmetric or the symplectic involution \(*\) on \(M_\infty(K)\) becomes \(t \otimes \ast\), where \(t\) denotes transposition. In other words,
\[
(M_\infty(M_\infty(K)), *) \simeq (M_\infty(K), t) \otimes_K (M_\infty(K), *).
\]
But it is easy to find isomorphisms

\[(M_\infty(K), *) \simeq (M_\infty(K), t) \otimes_K (M_2(K), *)\]

and

\[(M_\infty(K), t) \simeq (M_\infty(K), t) \otimes_K (M_\infty(K), t)\]

It follows that

\[(M_\infty(M_\infty(K)), *) \simeq (M_\infty(K), t) \otimes_K (M_\infty(K), t) \otimes_K (M_\infty(K), t) \simeq (M_\infty(K), t) \otimes_K (M_\infty(K), t) \otimes_K (M_\infty(K), t) \simeq (M_\infty(K), t) \simeq (M_\infty(K), t) \simeq (M_\infty(K), t).\]

Acknowledgements. I would like to thank M. Cabrera García and A. Rodríguez Palacios for their advice and encouragement.

References