[13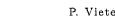] R. Hughes and S. Kantorovitz, *Spectral analysis of certain operator functions*, J. Operator Theory 22 (1989), 243–265.

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Purely non-atomic weak L^p spaces

by

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Abstract. Let (Ω, Σ, μ) be a purely non-atomic measure space, and let $1 < p < \infty$. If $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic, as a Banach space, to $L^{p,\infty}(\Omega', \Sigma', \mu')$ for some purely atomic measure space (Ω', Σ', μ'), then there is a measurable partition $\Omega = \Omega_1 \cup \Omega_2$ such that $(\Omega_1, \Sigma \cap \Omega_1, \mu|_{\Sigma \cap \Omega_1})$ is countably generated and σ-finite, and that $\mu(\sigma) = 0$ or ∞ for every measurable $\sigma \subseteq \Omega_2$. In particular, $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to $\ell^{p,\infty}$.

1. Introduction. In [3], the author proved that the spaces $L^{p,\infty}[0,1]$ and $L^{p,\infty}[0,\infty)$ are both isomorphic to the atomic space $\ell^{p,\infty}$. Subsequently, it was observed that if (Ω, Σ, μ) is countably generated and σ-finite, then $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to either $\ell^{p,\infty}$ or ℓ^∞ [4, Theorem 7]. In the present paper, we show that the isomorphism of atomic and non-atomic weak L^p spaces does not hold beyond the countably generated, σ-finite situation.

Before giving the precise statement of the main theorem, let us agree on some terminology. Throughout this paper, every measure space under discussion is assumed to be *non-trivial* in the sense that it contains a measurable subset of finite non-zero measure. A measurable subset σ of a measure space (Ω, Σ, μ) is an *atom* if $\mu(\sigma) > 0$, and either $\mu(\sigma') = 0$ or $\mu(\sigma \setminus \sigma') = 0$ for each measurable subset σ' of σ. A *purely non-atomic* measure space is one which contains no atoms. We say that a collection S of measurable sets *generates* a measure space (Ω, Σ, μ) if Σ is the smallest σ-algebra containing S as well as the μ-null sets. A measure space (Ω, Σ, μ) is *purely atomic* if it is generated by the collection of all of its atoms; it is *countably generated* if there is a sequence (σ_n) in Σ which generates (Ω, Σ, μ). For any measure space (Ω, Σ, μ), and $1 < p < \infty$, the *weak L^p space* $L^{p,\infty}(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions f such that

$$\|f\| = \sup_{c>0} c(\mu\{|f| > c\})^{1/p} < \infty.$$

It is well known that $\| \cdot \|$ is equivalent to a norm under which $L^{p,\infty}(\Omega, \Sigma, \mu)$ is a Banach space. However, since we are only concerned with isomorphic

questions, we will employ the quasi-norm $\| \cdot \|$ exclusively in our computations. The aim of this paper is to prove the following theorem.

THEOREM 1. *Let (Ω, Σ, μ) be a purely non-atomic measure space, and let $1 < p < \infty$. The following statements are equivalent:*

(1) *$L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to $L^{p,\infty}(\Omega', \Sigma', \mu')$ for some purely atomic measure space (Ω', Σ', μ').*

(2) *$L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to a subspace of $L^{p,\infty}(\Omega', \Sigma', \mu')$ for some purely atomic measure space (Ω', Σ', μ').*

(3) *There is a measurable partition $\Omega = \Omega_1 \cup \Omega_2$ such that $(\Omega_1, \Sigma \cap \Omega_1, \mu|_{\Sigma \cap \Omega_1})$ is countably generated and σ-finite, and that $\mu(\sigma) = 0$ or ∞ for every measurable $\sigma \subseteq \Omega_2$.*

(4) *$L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to $\ell^{p,\infty}$.*

It is interesting to note that with regard to (2), the weak L^p spaces behave in a way that is "in between" the behavior of the L^p spaces, $1 \leq p < \infty$, and L^∞. Indeed, if (Ω, Σ, μ) is purely non-atomic, then $L^p(\Omega, \Sigma, \mu)$ can never be embedded into an atomic L^p space ($1 \leq p < \infty$, $p \neq 2$). On the other hand, along with all Banach spaces, $L^\infty(\Omega, \Sigma, \mu)$ is isomorphic to a subspace of $\ell^\infty(J)$ for a sufficiently large index set J.

The other notation follows mainly that of [5, 6]. Banach spaces E and F are said to be *isomorphic* if they are linearly homeomorphic; E *embeds into* F if it is isomorphic to a subspace of F. If I is an arbitrary index set, and $(x_i)_{i \in I}$, $(y_i)_{i \in I}$ are indexed collections of elements in possibly different Banach spaces, we say that they are *equivalent* if there is a constant $0 < K < \infty$ such that

$$K^{-1} \Big\| \sum a_i x_i \Big\| \leq \Big\| \sum a_i y_i \Big\| \leq K \Big\| \sum a_i x_i \Big\|$$

for every collection $(a_i)_{i \in I}$ of scalars with finitely many non-zero terms. We will also have occasion to use terms and notation concerning vector lattices, for which the references are [6, 8]. In particular, two elements a, b of a vector lattice are said to be *disjoint* if $|a| \wedge |b| = 0$. A Banach lattice E satisfies an *upper p-estimate* if there is a constant $M < \infty$ such that

$$\Big\| \sum_{i=1}^n x_i \Big\| \leq M \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}$$

whenever $(x_i)_{i=1}^n$ is a pairwise disjoint sequence in E. It is trivial to check that every $L^{p,\infty}(\Omega, \Sigma, \mu)$ satisfies the upper p-estimate with constant 1. Finally, if A is an arbitrary set, we write $\mathcal{P}(A)$ for the power set of A, and A^c for its complement (with respect to some universal set).

2. Proof of the main theorem. Let us set the notation for the two types of measure spaces which will command a large part of our attention.

By $\{-1, 1\}$ we will mean the two-point measure space each point of which is assigned a mass of $\frac{1}{2}$. If I is an arbitrary index set, $\{-1, 1\}^I$ is the product measure space of I copies of $\{-1, 1\}$. Now let $((\Omega_\alpha, \Sigma_\alpha, \mu_\alpha))_{\alpha \in A}$ be a collection of pairwise disjoint measure spaces. We define the measurable space (Ω, Σ) to be the set $\bigcup_{\alpha \in A} \Omega_\alpha$, endowed with the smallest σ-algebra Σ generated by $\bigcup_{\alpha \in A} \Sigma_\alpha$. For any $\sigma \in \Sigma$, define

$$\mu(\sigma) = \sum_{\alpha \in A} \mu_\alpha(\sigma \cap \Omega_\alpha).$$

The measure space (Ω, Σ, μ) is denoted by $\bigoplus_{\alpha \in A}(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$. Of particular interest will be $\bigoplus_{\alpha \in A} J_\alpha$, where each J_α is a copy of the measure space $[0, 1]$ with the Lebesgue measure.

THEOREM 2. *If I is an uncountable index set, $L^{p,\infty}(\{-1, 1\}^I)$ does not embed into $L^{p,\infty}(\Omega, \Sigma, \mu)$ for any purely atomic measure space (Ω, Σ, μ).*

THEOREM 3. *Let A be an arbitrary index set. For every $\alpha \in A$, let J_α be a copy of the measure space $[0, 1]$. If $L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha)$ embeds into $L^{p,\infty}(\Omega, \Sigma, \mu)$ for some purely atomic measure space (Ω, Σ, μ), then the set A is countable.*

The proofs of the crucial Theorems 2 and 3 will be the subject of the subsequent sections. To apply these theorems to the proof of the main theorem (Theorem 1) requires the use of certain known facts, which we now recall. Let (Ω, Σ, μ) and (Ω', Σ', μ') be measure spaces. Denote by Θ_μ and $\Theta_{\mu'}$ the μ- and μ'-null sets respectively. Then μ induces a function on the σ-complete Boolean algebra Σ/Θ_μ, which we denote again by μ. Similarly for μ'. We say that the measure spaces (Ω, Σ, μ) and (Ω', Σ', μ') are *isomorphic* if there exists a Boolean algebra isomorphism $\Phi : \Sigma/\Theta_\mu \to \Sigma'/\Theta_{\mu'}$ such that $\mu = \mu' \circ \Phi$. For notions and results regarding measure algebras, we refer to [2, §14]. The next fact, which can be found in [7], follows easily from the observation that the set of functions $f \in L^{p,\infty}(\Omega, \Sigma, \mu)$ of the form $f = \bigvee a_n \chi_{A_n}$, where $(a_n) \subseteq \mathbb{R}$, and (A_n) is a pairwise disjoint sequence in Σ, is dense in $L^{p,\infty}(\Omega, \Sigma, \mu)$.

THEOREM 4. *If (Ω, Σ, μ) and (Ω', Σ', μ') are isomorphic measure spaces, then the Banach spaces $L^{p,\infty}(\Omega, \Sigma, \mu)$ and $L^{p,\infty}(\Omega', \Sigma', \mu')$ are isometrically isomorphic.*

The next theorem is stated in the form in which we will use it. It is a consequence of Maharam's theorem on the classification of measure algebras; see [2, Theorems 14.7 and 14.8]. If (Ω, Σ, μ) is a measure space, and c is a positive number, we let $c\mu$ be the measure given by $(c\mu)(\sigma) = c\mu(\sigma)$ for all $\sigma \in \Sigma$. Clearly, the map $f \mapsto c^{-1/p}f$ is an isometric isomorphism from $L^{p,\infty}(\Omega, \Sigma, \mu)$ onto $L^{p,\infty}(\Omega, \Sigma, c\mu)$.

THEOREM 5 (Maharam). *Let (Ω, Σ, μ) be a purely non-atomic, finite measure space which is not countably generated. Then there is a measurable subset Ω' of Ω and an uncountable index set I such that (Ω', Σ', μ') is isomorphic to $\{-1, 1\}^I$, where $\Sigma' = \Sigma \cap \Omega'$, and $\mu' = (\mu(\Omega'))^{-1}\mu|_{\Sigma'}$. Consequently, $L^{p,\infty}(\Omega, \Sigma, \mu)$ has a subspace isometrically isomorphic to $L^{p,\infty}(\{-1, 1\}^I)$.*

For the following proof, recall that a Banach space E satisfies the *Dunford–Pettis property* if $\langle x_n, x'_n \rangle \to 0$ whenever (x_n) and (x'_n) are weakly null sequences in E and E' respectively. It is well known that ℓ^∞ satisfies the Dunford–Pettis property; see, e.g., [8, §II.9].

Proof of Theorem 1. Suppose (3) holds. Then $L^{p,\infty}(\Omega, \Sigma, \mu)$ is clearly isometrically isomorphic to $L^{p,\infty}(\Omega_1, \Sigma \cap \Omega_1, \mu|_{\Sigma \cap \Omega_1})$. By [4, Theorem 7], $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to either $\ell^{p,\infty}$ or ℓ^∞. However, since (Ω, Σ, μ) is purely non-atomic, we can easily verify that $L^{p,\infty}(\Omega, \Sigma, \mu)$ fails the Dunford–Pettis property. (Use Rademacher-like functions.) Hence it cannot be isomorphic to ℓ^∞. The implications (4)$\Rightarrow$(1)$\Rightarrow$(2) are trivial. Therefore, it remains to prove that (2)$\Rightarrow$(3). Using Zorn's Lemma, we obtain a (possibly empty) collection of measurable subsets $(\Omega_\alpha)_{\alpha \in A}$ of Ω which is maximal with respect to the following conditions: $\mu(\Omega_\alpha \cap \Omega_\beta) = 0$ if $\alpha \neq \beta$; $\mu(\Omega_\alpha) = 1$ for all $\alpha \in A$. For each $\alpha \in A$, let J_α be a copy of the measure space $[0, 1]$. Then J_α is isomorphic to a measure subalgebra of $(\Omega_\alpha, \Sigma \cap \Omega_\alpha, \mu|_{\Sigma \cap \Omega_\alpha})$. It follows that $\bigoplus_{\alpha \in A} J_\alpha$ is isomorphic to a measure subalgebra of (Ω, Σ, μ). Theorem 4 implies that $L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha)$ is isometrically isomorphic to a subspace of $L^{p,\infty}(\Omega, \Sigma, \mu)$, and hence, by the assumption (2), isomorphic to a subspace of an atomic weak L^p space. According to Theorem 3, A must be a countable set. By the maximality of $(\Omega_\alpha)_{\alpha \in A}$,

$$m \equiv \sup\left\{\mu(\sigma) : \right.$$

$$\left. \sigma \text{ is a measurable subset of } \Omega \setminus \bigcup_{\alpha \in A} \Omega_\alpha \text{ of finite measure}\right\} \leq 1.$$

It is easily seen that the supremum is attained, say, at Ω_0. Define $\Omega_1 = \Omega_0 \cup (\bigcup_{\alpha \in A} \Omega_\alpha)$. Since A is countable, $\Omega_1 \in \Sigma$. If Ω_α is not countably generated for some $\alpha \in A$, then Theorem 5 produces an uncountable index set I such that $L^{p,\infty}(\{-1, 1\}^I)$ is isometrically isomorphic to a subspace of $L^{p,\infty}(\Omega_\alpha)$, and thus isomorphic to a subspace of an atomic weak L^p space. This violates Theorem 2. Similarly, we see that Ω_0 is countably generated. Therefore, Ω_1 is countably generated; it is clearly σ-finite. If σ is a measurable subset of $\Omega_2 = \Omega \setminus \Omega_1$, and $0 < \mu(\sigma) < \infty$, then $m < \mu(\Omega_0 \cup \sigma) < \infty$, contrary to the choice of Ω_0. Hence $\mu(\sigma) = 0$ or ∞. ∎

3. The space $L^{p,\infty}(\{-1, 1\}^I)$. Let Γ be an arbitrary set, and let $w : \Gamma \to (0, \infty)$ be a weight function. We can define a measure μ on $\mathcal{P}(\Gamma)$ by $\mu(\sigma) = \sum_{\gamma \in \sigma} w(\gamma)$ for all $\sigma \subseteq \Gamma$. The resulting weak L^p space $L^{p,\infty}(\Gamma, \mathcal{P}(\Gamma), \mu)$ will be denoted by $\ell^{p,\infty}(\Gamma, w)$, or simply $\ell^{p,\infty}(\Gamma)$ if w is identically 1. It is easy to see that if (Ω, Σ, μ) is purely atomic, then $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isometrically isomorphic to $\ell^{p,\infty}(\Gamma, w)$ for some (Γ, w). If (Ω, Σ, μ) is a measure space, and $1 < p < \infty$, let $M^{p,\infty}(\Omega, \Sigma, \mu)$ be the closed subspace of $L^{p,\infty}(\Omega, \Sigma, \mu)$ generated by the functions χ_σ, where σ is a measurable set of finite measure. The corresponding subspace of $\ell^{p,\infty}(\Gamma, w)$ is denoted by $m^{p,\infty}(\Gamma, w)$. The proof of Theorem 2 for the case $p \neq 2$ is rather easy and is contained in Theorem 7. For the reader's convenience, we recall the following disjunctification result [4, Proposition 10].

PROPOSITION 6. *Let w be a weight function on a set Γ. Assume that A and B are subsets of $\ell^{p,\infty}(\Gamma, w)$ such that $|A| > \max\{|B|, \aleph_0\}$. Suppose also that there are constants $K < \infty$, $r > 1$ such that*

$$(1) \qquad \left\| \sum_{x \in F} \varepsilon_x x \right\| \leq K|F|^{1/r}$$

for all finite subsets F of A, and all $\varepsilon_x = \pm 1$. Then there exists $C \subseteq A$, $|C| = |A|$, such that the elements of C are pairwise disjoint, and $|b| \wedge |c| = 0$ whenever $b \in B$, $c \in C$.

Proof. First we show that if Γ' is a subset of Γ such that $|\Gamma'| < |A|$, then there exists $A' \subseteq A$, $|A'| = |A|$, such that $x\chi_{\Gamma'} = 0$ for all $x \in A'$. Indeed, let $A' = \{x \in A : x\chi_{\Gamma'} = 0\}$. For each $x \in A \setminus A'$, there is a $\gamma \in \Gamma'$ such that $x(\gamma) \neq 0$. Pick a choice function $f : A \setminus A' \to \Gamma'$ such that $x(f(x)) \neq 0$ for all $x \in A \setminus A'$. If $|A'| < |A|$, then $|A \setminus A'| = |A| > \aleph_0$. Hence there exist $C \subseteq A \setminus A'$ and $n \in \mathbb{N}$ such that $|C| = |A \setminus A'| = |A|$, and $|x(f(x))| \geq 1/n$ for all $x \in C$. Now $|C| = |A| > |\Gamma'| \geq |f(C)|$. Therefore, there is a $\gamma_0 \in f(C)$ such that $D = f^{-1}\{\gamma_0\} \cap C$ is infinite. Note that $x \in D$ implies $|x(\gamma_0)| \geq 1/n$. Now for any finite subset F of D,

$$\left\| \sum_{x \in F} \operatorname{sgn} x(\gamma_0) x \right\| \geq \sum_{x \in F} |x(\gamma_0)| \|\chi_{\{\gamma_0\}}\| \geq \frac{|F|}{n} w(\gamma_0)^{1/p}.$$

As D is infinite, this violates condition (1).

Now for each $x \in \ell^{p,\infty}(\Gamma, w)$, let supp $x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Clearly $|\operatorname{supp} x| \leq \aleph_0$. Therefore, $|\bigcup_{x \in B} \operatorname{supp} x| \leq \max\{|B|, \aleph_0\} < |A|$. Let $\Gamma_1 = \bigcup_{x \in B} \operatorname{supp} x$. By the above, there is a subset A_1 of A, having the same cardinality as A, such that $x\chi_{\Gamma_1} = 0$ for all $x \in A_1$. It remains to choose a pairwise disjoint subset of A_1 of cardinality $|A|$. This will be done by induction. Choose x_0 arbitrarily in A_1. Now suppose a pairwise disjoint collection $(x_\varrho)_{\varrho < \beta}$ has been chosen up to some ordinal $\beta < |A| = |A_1|$. Since

$|A_1|$ is a cardinal, $|\beta| < |A_1|$. Hence $|\bigcup_{\varrho<\beta} \operatorname{supp} x_\varrho| \leq \max\{|\beta|, \aleph_0\} < |A_1|$. Let $\Gamma_2 = \bigcup_{\varrho<\beta} \operatorname{supp} x_\varrho$. Using the first part of the proof again, we find an $x_\beta \in A_1$ such that $x_\beta \chi_{\Gamma_2} = 0$. It is clear that the collection $(x_\varrho)_{\varrho \leq \beta}$ is pairwise disjoint. This completes the inductive argument. Consequently, we obtain a pairwise disjoint collection $C = (x_\varrho)_{\varrho < |A|}$ in A_1. As each $x \in C$ is disjoint from each $b \in B$, the proof is complete. ∎

THEOREM 7. *Let I and Γ be arbitrary sets such that I is uncountable. For any weight function w on Γ, and any $p \neq 2$, $1 < p < \infty$, $\ell^{p,\infty}(\Gamma, w)$ does not contain a subspace isomorphic to $\ell^2(I)$. Consequently, Theorem 2 holds if $p \neq 2$.*

Proof. For any set I, and any $i \in I$, let $\varepsilon_i : \{-1, 1\}^I \to \{-1, 1\}$ be the projection onto the ith coordinate. By Khinchin's inequality, $(\varepsilon_i)_{i \in I} \subseteq L^{p,\infty}(\{-1, 1\}^I)$ is equivalent to the unit vector basis of $\ell^2(I)$. Hence the first statement of the theorem implies the second. Now suppose $(x_i)_{i \in I}$ is a set of normalized elements of $\ell^{p,\infty}(\Gamma, w)$ which is equivalent to the unit vector basis of $\ell^2(I)$. If I is uncountable, apply Proposition 6 with $A = I$, $B = \emptyset$ to obtain an uncountable $C \subseteq I$ such that $(x_i)_{i \in C}$ are pairwise disjoint. Since $\ell^{p,\infty}(\Gamma, w)$ satisfies an upper p-estimate, there is a constant $0 < K < \infty$ such that

$$K^{-1}|F|^{1/2} \leq \left\| \sum_{i \in F} x_i \right\| \leq K|F|^{1/p}$$

for every finite subset F of C. We conclude that $1 < p < 2$. Denote by μ the measure associated with (Γ, w). For each $i \in C$, there is a rational number $c_i > 0$ such that $c_i(\mu\{|x_i| > c_i\})^{1/p} > 1/2$. By using an uncountable subset of C if necessary, we may assume that $c_i = c$, a constant, for all $i \in C$. For any finite subset F of C,

$$\mu\left\{ \left| \sum_{i \in F} x_i \right| > c \right\} = \sum_{i \in F} \mu\{|x_i| > c\} > (2c)^{-p}|F|.$$

Hence $\|\sum_{i \in F} x_i\| > \frac{1}{2}|F|^{1/p}$. Since $1 < p < 2$, and $(x_i)_{i \in C}$ is equivalent to the unit vector basis of $\ell^2(C)$, we have reached a contradiction. ∎

The proof of Theorem 2 for the case $p = 2$ is more involved. Let (h_n) denote the L^∞-normalized Haar functions on $[0, 1]$ (cf. [5, Definition 1.a.4]). Then by [6, Theorem 2.c.6], (h_n) is an unconditional basis of $M^{p,\infty}[0, 1]$. We first show that if $T : M^{2,\infty}[0, 1] \to L^{2,\infty}(\Omega, \Sigma, \mu)$ is an embedding, then (Th_n) cannot be pairwise disjoint.

PROPOSITION 8. *Suppose $T : M^{2,\infty}[0, 1] \to L^{2,\infty}(\Omega, \Sigma, \mu)$ is an embedding for some measure space (Ω, Σ, μ). Then (Th_n) cannot be a pairwise disjoint sequence.*

Proof. By Theorem 1.d.6(ii) in [6], there is a constant D such that

$$\left\| \sum a_n h_n \right\| \geq D^{-1} \left\| \left(\sum |a_n h_n|^2 \right)^{1/2} \right\|$$

for every sequence of scalars (a_n) which is eventually zero. Given $m \in \mathbb{N}$, let $a_{ij} = i + j - 1 \pmod{2^m}$, $1 \leq i, j \leq 2^m$. For $1 \leq j \leq 2^m$, define

$$g_j = \sum_{i=1}^{2^m} \left(\frac{2^m}{a_{ij}} \right)^{1/2} \chi_{I_i},$$

where $I_i = [(i-1)/2^m, i/2^m)$. If $1 \leq j \leq 2^m$, there exists

$$f_j = \sum_{n=2^{m+j-1}+1}^{2^{m+j}} b_n h_n \in \operatorname{span}\{h_n : 2^{m+j-1} < n \leq 2^{m+j}\}$$

such that $|f_j| = g_j$. Note that $(f_j)_{j=1}^{2^m}$ is a normalized sequence in $M^{2,\infty}[0, 1]$. If $T : M^{2,\infty}[0, 1] \to L^{2,\infty}(\Omega, \Sigma, \mu)$ is a bounded linear operator such that (Th_n) is pairwise disjoint, then $(Tf_j)_{j=1}^{2^m}$ is a pairwise disjoint sequence which is bounded in norm by $\|T\|$. Hence, using the upper 2-estimate in $L^{2,\infty}(\Omega, \Sigma, \mu)$, we get $\|\sum_{j=1}^{2^m} Tf_j\| \leq 2^{m/2}\|T\|$. On the other hand,

$$\left\| \sum_{j=1}^{2^m} f_j \right\| = \left\| \sum_{j=1}^{2^m} \sum_{n=2^{m+j-1}+1}^{2^{m+j}} b_n h_n \right\| \geq D^{-1} \left\| \left(\sum_{j=1}^{2^m} \sum_{n=2^{m+j-1}+1}^{2^{m+j}} |b_n h_n|^2 \right)^{1/2} \right\|$$

$$= D^{-1} \left\| \left(\sum_{j=1}^{2^m} |f_j|^2 \right)^{1/2} \right\| = D^{-1} \left\| \left(\sum_{j=1}^{2^m} |g_j|^2 \right)^{1/2} \right\|$$

$$= D^{-1} \left(2^m \sum_{j=1}^{2^m} \frac{1}{j} \right)^{1/2} \|\chi_{[0,1)}\| = D^{-1} \left(2^m \sum_{j=1}^{2^m} \frac{1}{j} \right)^{1/2}.$$

Since m is arbitrary, T cannot be an embedding. ∎

We now complete the proof of Theorem 2 for the case $p = 2$. Suppose I is uncountable, and $T : L^{2,\infty}(\{-1, 1\}^I) \to \ell^{2,\infty}(\Gamma, w)$ is a bounded linear operator. We will construct a sequence $(g_n) \subseteq L^{2,\infty}(\{-1, 1\}^I)$ which is equivalent to the Haar basis $(h_n) \subseteq L^{2,\infty}[0, 1]$, and such that (Tg_n) is a pairwise disjoint sequence in $\ell^{2,\infty}(\Gamma, w)$. An appeal to Proposition 8 will then yield the desired result that T is not an embedding. Let the functions $(\varepsilon_i) \subseteq L^{2,\infty}(\{-1, 1\}^I)$ be the same as those which appeared in the proof of Theorem 7. For any finite subset F of I, and any $\{-1, 1\}$-valued sequence $(b_i)_{i \in F}$, the family

$$\left(\left(\prod_{i \in F} \chi_{\{\varepsilon_i = b_i\}} \right) \varepsilon_j \right)_{j \in I \setminus F}$$

is easily seen to be equivalent to the unit vector basis of $\ell^2(I \setminus F)$. Applying Proposition 6, we see that for any F and $(b_i)_{i \in F}$ as above, and any countable $S \subseteq \Gamma$, there exists $j \in I \setminus F$ such that $(T((\prod_{i \in F} \chi_{\{\varepsilon_i = b_i\}})\varepsilon_j))\chi_S = 0$. For any $x \in \ell^{2,\infty}(\Gamma, w)$, we let its *support* be the set $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Any element in $\ell^{2,\infty}(\Gamma, w)$ has countable support. Let g_1 be the identically 1 function on $\{-1, 1\}^I$. Then there exists $i_2 \in I$ such that $\operatorname{supp} T\varepsilon_{i_2} \cap \operatorname{supp} Tg_1 = \emptyset$. Let $g_2 = \varepsilon_{i_2}$. Now $\operatorname{supp} Tg_1 \cup \operatorname{supp} Tg_2$ is countable. Therefore, one can find $i_3 \neq i_2$ such that $\operatorname{supp} T((\chi_{\{\varepsilon_{i_2} = -1\}})\varepsilon_{i_3})$ is disjoint from $\operatorname{supp} Tg_1 \cup \operatorname{supp} Tg_2$. Define $g_3 = (\chi_{\{\varepsilon_{i_2} = -1\}})\varepsilon_{i_3}$. Next define $g_4 = (\chi_{\{\varepsilon_{i_2} = 1\}})\varepsilon_{i_4}$, where i_4 is chosen so that it is distinct from i_2, i_3, and $\operatorname{supp} Tg_4$ is disjoint from $\bigcup_{n=1}^3 \operatorname{supp} Tg_n$. Continuing in this way, we obtain the desired sequence (g_n).

4. The space $L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha)$. In this section, we present the proof of Theorem 3. Let A be an uncountable set, and let w be a weight function defined on a set Γ. Suppose $T : M^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha) \to \ell^{p,\infty}(\Gamma, w)$ is a bounded linear operator. The first step is to show that the range of T is mostly contained in $m^{p,\infty}(\Gamma, w)$. This will require the following technical lemma.

LEMMA 9. *Let $k \in \mathbb{N}$ be given, and let $\delta, c_1, c_2, \ldots, c_k$ be strictly positive numbers. Suppose $l \in \mathbb{N}$ is so large that*

$$l\left(\frac{\min(c_1, \ldots, c_k)}{\max(c_1, \ldots, c_k)}\right)^p \geq 1.$$

Let (Ω, Σ, μ) be any measure space, and let $f_1, \ldots, f_l$ be pairwise disjoint functions in $L^{p,\infty}(\Omega, \Sigma, \mu)$ such that $|f_j| \geq \sum_{m=1}^k c_m \chi_{\sigma(m,j)}$, where, for each j, $\sigma(1,j), \ldots, \sigma(k,j)$ are pairwise disjoint sets in Σ such that $\mu(\sigma(m,j)) > (\delta/c_m)^p$, $1 \leq m \leq k$. Then

$$\left\|\sum_{j=1}^l j^{-1/p} f_j\right\| \geq \left(\frac{k}{2}\right)^{1/p}\delta.$$

P r o o f. We may assume without loss of generality that $c_1 \geq \ldots \geq c_k > 0$. Then $l(c_k/c_1)^p \geq 1$. For $1 \leq m \leq k$, let i_m be the largest integer in $\mathbb{N}$ not greater than $l(c_m/c_1)^p$. Note that $1 \leq i_m \leq l$. For any $\varepsilon < c_1 l^{-1/p}$,

$$\left\{\left|\sum_{j=1}^l j^{-1/p} f_j\right| > \varepsilon\right\}$$

$$\supseteq \bigcup\{\sigma(m,j) : c_m j^{-1/p} > \varepsilon\} \supseteq \bigcup\{\sigma(m,j) : c_m j^{-1/p} \geq c_1 l^{-1/p}\}$$

$$= \bigcup\{\sigma(m,j) : j \leq l(c_m/c_1)^p\} = \bigcup_{m=1}^k \bigcup_{j=1}^{i_m} \sigma(m,j).$$

Thus

$$\mu\left\{\left|\sum_{j=1}^l j^{-1/p} f_j\right| > \varepsilon\right\} \geq \sum_{m=1}^k \sum_{j=1}^{i_m} \mu(\sigma(m,j))$$

$$> \sum_{m=1}^k \sum_{j=1}^{i_m} (\delta/c_m)^p = \sum_{m=1}^k i_m (\delta/c_m)^p.$$

Now $i_m \geq 1$ implies $i_m \geq (1 + i_m)/2 \geq 2^{-1} l(c_m/c_1)^p$. Hence

$$\mu\left\{\left|\sum_{j=1}^l j^{-1/p} f_j\right| > \varepsilon\right\} \geq \sum_{m=1}^k 2^{-1} l(c_m/c_1)^p (\delta/c_m)^p = (lk/2)(\delta/c_1)^p.$$

Therefore,

$$\left\|\sum_{j=1}^l j^{-1/p} f_j\right\| \geq \varepsilon\left(\mu\left\{\left|\sum_{j=1}^l j^{-1/p} f_j\right| > \varepsilon\right\}\right)^{1/p} \geq (\varepsilon\delta/c_1)(lk/2)^{1/p}.$$

Taking the supremum over all $\varepsilon < c_1 l^{-1/p}$ yields the desired result. ∎

PROPOSITION 10. *Let A be an index set, and let $T : m^{p,\infty}(A) \to \ell^{p,\infty}(\Gamma, w)$ be a bounded linear operator for some (Γ, w). Then $T\chi_{\{\alpha\}} \in m^{p,\infty}(\Gamma, w)$ for all but countably many $\alpha \in A$.*

P r o o f. Let $f_\alpha = T\chi_{\{\alpha\}}$, and assume $f_\alpha \notin m^{p,\infty}(\Gamma, w)$ for uncountably many α. Applying Proposition 6, we may assume that the f_α's are pairwise disjoint. Choose an uncountable $A_0 \subseteq A$, and $\delta > 0$, such that $d(f_\alpha, m^{p,\infty}(\Gamma, w)) > \delta$ for all $\alpha \in A_0$. For each $\alpha \in A_0$, there is a rational $r > 0$ such that $\mu\{|f_\alpha| > r\} > (\delta/r)^p$, where μ is the measure associated with (Γ, w). Hence we can find an uncountable $A_1 \subseteq A_0$, and $c_1 > 0$, such that $\mu\{|f_\alpha| > c_1\} > (\delta/c_1)^p$ for every $\alpha \in A_1$. For all $\alpha \in A_1$, choose a finite set $\sigma(1, \alpha) \subseteq \{|f_\alpha| > c_1\}$ such that $\mu(\sigma(1, \alpha)) > (\delta/c_1)^p$. Now $\|f_\alpha - f_\alpha \chi_{\sigma(1,\alpha)}\| > \delta$ for all $\alpha \in A_1$. Arguing as before, we find an uncountable $A_2 \subseteq A_1$, and $c_2 > 0$, such that

$$\mu\{|f_\alpha - f_\alpha \chi_{\sigma(1,\alpha)}| > c_2\} > (\delta/c_2)^p$$

for all $\alpha \in A_2$. Hence, for each $\alpha \in A_2$, there exists a finite set $\sigma(2, \alpha) \subseteq \{|f_\alpha| > c_2\}$, disjoint from $\sigma(1, \alpha)$, such that $\mu(\sigma(2, \alpha)) > (\delta/c_2)^p$. Continue inductively to obtain a decreasing sequence of uncountable subsets (A_m) of A, a positive sequence (c_m), and finite subsets $\sigma(m, \alpha) \subseteq \{|f_\alpha| > c_m\}$ for all $\alpha \in A_m$, such that $\mu(\sigma(m, \alpha)) > (\delta/c_m)^p$, and $\sigma(m, \alpha) \cap \sigma(n, \alpha) = \emptyset$ if $\alpha \in A_m \cap A_n$ and $m \neq n$. Now let $k \in \mathbb{N}$ be given. Choose l so large that

$$l\left(\frac{\min(c_1, \ldots, c_k)}{\max(c_1, \ldots, c_k)}\right)^p \geq 1.$$

Lemma 9 implies that $\|\sum_{j=1}^l j^{-1/p} f_j\| \geq (k/2)^{1/p}\delta$ if $\alpha_1, \ldots, \alpha_l$ are distinct elements of A_k. This violates the boundedness of T since k is arbitrary. ∎

COROLLARY 11. *Let A be an index set. For each $\alpha \in A$, $n \in \mathbb{N}$, and $1 \le j \le 2^n$, let $f_{n,j,\alpha}$ be the characteristic function of the subinterval $[(j-1)/2^n, j/2^n)$ in J_α. If $T : M^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha) \to \ell^{p,\infty}(\Gamma, w)$ is a bounded linear operator, then all but countably many members of $\{Tf_{n,j,\alpha} : \alpha \in A, n \in \mathbb{N}, 1 \le j \le 2^n\}$ belong to $m^{p,\infty}(\Gamma, w)$.*

Proof. If n is fixed, the collection $\{f_{n,j,\alpha} : \alpha \in A, 1 \le j \le 2^n\}$ is equivalent to the unit vector basis in $m^{p,\infty}(A \times \{1, \ldots, 2^n\})$. Apply Proposition 10 to complete the proof. ∎

If A is uncountable, and $T : L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha) \to \ell^{p,\infty}(\Gamma, w)$ is an embedding, then it follows from Corollary 11 that there exists $\alpha_0 \in A$ such that $T(M^{p,\infty}(J_{\alpha_0})) \subseteq m^{p,\infty}(\Gamma, w)$, where we identify $M^{p,\infty}(J_{\alpha_0})$ with a subspace of $L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha)$ in the obvious way. Hence $M^{p,\infty}[0,1]$ embeds into $m^{p,\infty}(\Gamma, w)$. The proof of Theorem 3 is completed by showing that this is impossible. Once again, we find it necessary to distinguish between the cases $p \ne 2$ and $p = 2$. If $p \ne 2$, we use a Kadec–Pełczyński type argument [1] to show that ℓ^2 does not embed into $m^{p,\infty}(\Gamma, w)$. For $p = 2$, we resort once again to Proposition 8. If f is a real-valued function and $1 < M < \infty$, let $(f)_M = f\chi_{\{M^{-1} < |f| < M\}}$.

LEMMA 12. *Let (Ω, Σ, μ) be any measure space, and suppose $1 < p < \infty$. If (f_n) is a pairwise disjoint sequence in the unit ball of $L^{p,\infty}(\Omega, \Sigma, \mu)$, and (M_n) is a real sequence such that $1 < M_n \le 2^{-1/p} M_{n+1}$ for all $n \in \mathbb{N}$, define $g_1 = (f_1)_{M_1}$ and*

$$g_{n+1} = (f_{n+1})_{M_{n+1}} - (f_{n+1})_{M_n}.$$

Then $\sup_k \|\sum_{n=1}^k g_n\| \le 4$.

Proof. Let $g = $ pointwise-$\sum g_n$ and $M_0 = 1/M_1$. If $M_{k-1} \le c < M_k$ for some $k \in \mathbb{N}$, then

$$\mu\{|g| > c\} = \sum_{n=k}^\infty \mu\{|g_n| > c\}$$

$$= \mu\{|g_k| > c\} + \sum_{n=k+1}^\infty \mu\{|g_n| \ge M_{n-1}\}$$

$$\le c^{-p} + \sum_{n=k+1}^\infty M_{n-1}^{-p} \le c^{-p} + 2M_k^{-p} \le 3c^{-p}.$$

On the other hand, if $M_{k+1}^{-1} \le c < M_k^{-1}$ for some $k \in \mathbb{N}$, then

$$\mu\{|g| > c\} = \sum_{n=1}^k \mu\{|g_n| > M_n^{-1}\} + \mu\{|g_{k+1}| > c\} + \sum_{n=k+2}^\infty \mu\{|g_n| \ge M_{n-1}\}$$

$$\le \sum_{n=1}^k M_n^p + c^{-p} + \sum_{n=k+2}^\infty M_{n-1}^{-p}$$

$$\le 2M_k^p + c^{-p} + M_k^{-p} \le 2c^{-p} + c^{-p} + 1 \le 4c^{-p}.$$

Hence $g \in L^{p,\infty}(\Omega, \Sigma, \mu)$, and $\|g\| \le 4$. ∎

THEOREM 13. *For any (Γ, w), and $1 < p < \infty$, $p \ne 2$, there is no embedding of ℓ^2 into $m^{p,\infty}(\Gamma, w)$.*

Proof. Suppose, on the contrary, that $m^{p,\infty}(\Gamma, w)$ contains a sequence (f_n) equivalent to the unit vector basis of ℓ^2. Since each f_n has countable support, we may assume that Γ is countable. Then clearly $(\chi_{\{\gamma\}})_{\gamma \in \Gamma}$ is an unconditional basis of $m^{p,\infty}(\Gamma, w)$. Since (f_n) is a weakly null sequence, we may apply the Bessaga–Pełczyński selection principle [5, Proposition 1.a.12] to it. Thus, we may assume without loss of generality that (f_n) is pairwise disjoint. Since $m^{p,\infty}(\Gamma, w)$ satisfies an upper p-estimate, this is possible only if $1 < p < 2$. Now suppose there exists $1 < M < \infty$ such that $\limsup_n \|(f_n)_M\| > 0$. We may assume that there exists $\varepsilon > 0$ such that $\|(f_n)_M\| > \varepsilon$ for all n. For each n, choose $c_n \in [M^{-1}, M]$ such that $c_n(\mu\{|f_n| > c_n\})^{1/p} > \varepsilon$, where μ is the measure associated with (Γ, w). Using the compactness of $[M^{-1}, M]$, and going to a subsequence if necessary, we may assume the existence of a $c \in [M^{-1}, M]$ such that $c(\mu\{|f_n| > c\})^{1/p} > \varepsilon$ for all n. Then

$$\left\|\sum_{n=1}^k f_n\right\| \ge c\left(\sum_{n=1}^k \mu\{|f_n| > c\}\right)^{1/p} \ge \varepsilon k^{1/p}$$

for all $k \in \mathbb{N}$, a contradiction. Therefore, it must be that $\lim_n \|(f_n)_M\| = 0$ for all $1 < M < \infty$. Note that $\lim_{M \to \infty} \|f - (f)_M\| = 0$ for every $f \in m^{p,\infty}(\Gamma, w)$. By a standard perturbation argument, we obtain a subsequence of (f_n), denoted again by (f_n), and a real sequence (M_n) satisfying $1 < M_n \le 2^{-1/p} M_{n+1}$ for all $n \in \mathbb{N}$, such that (f_{n+1}) is equivalent to $((f_{n+1})_{M_{n+1}} - (f_{n+1})_{M_n})$. Lemma 12, however, shows that (f_{n+1}) cannot be equivalent to the unit vector basis of ℓ^2. ∎

We now give the proof of Theorem 3. Assume that for some uncountable set A, $L^{p,\infty}(\bigoplus_{\alpha \in A} J_\alpha)$ embeds into $\ell^{p,\infty}(\Gamma, w)$ for some (Γ, w). As in the discussion following Corollary 11, $M^{p,\infty}[0,1]$ embeds into $m^{p,\infty}(\Gamma, w)$. Since the sequence of Rademacher functions in $M^{p,\infty}[0,1]$ is equivalent to the unit vector basis of ℓ^2, Theorem 13 implies that this is impossible unless $p = 2$. Now let $T : M^{2,\infty}[0,1] \to m^{2,\infty}(\Gamma, w)$ be an embedding. Without loss of generality, assume that $\|Tf\| \ge \|f\|$ for all $f \in M^{2,\infty}[0,1]$. Denote by (r_n), respectively (h_n), the sequence of Rademacher functions, respectively Haar functions, on $[0,1]$. Note that for all $f \in M^{2,\infty}[0,1]$, $f \cdot r_n \to 0$ weakly as

$n \to \infty$. Let $f_1 = |h_1|$. If $k \in \mathbb{N}$, and $2^{k-1} < j \leq 2^k$, let $f_j = \sqrt{2^{k-1}}|h_j|$. Define $n_1 = 1$. Since $x_1 = T(h_1 \cdot r_{n_1}) \in m^{2,\infty}(\Gamma, w)$, there is a finite subset σ_1 of Γ such that $\|x_1 \chi_{\sigma_1^c}\| < 2^{-3}$. Now suppose that numbers n_i and finite sets σ_i have been chosen for $i \leq j$. Since $T(f_{j+1} \cdot r_n) \to 0$ weakly as $n \to \infty$, and $\bigcup_{i=1}^{j} \sigma_i$ is finite, there exists $n_{j+1} > n_j$ so that $\|x_{j+1} \chi_{\bigcup_{i=1}^{j} \sigma_i}\| < 2^{-j-4}$, where $x_{j+1} = T(f_{j+1} \cdot r_{n_{j+1}})$. Now we can choose a finite subset σ_{j+1} of Γ, disjoint from $\bigcup_{i=1}^{j} \sigma_i$, such that $\|x_{j+1}\chi_{\sigma_{j+1}^c}\| < 2^{-j-3}$. Finally, let $y_j = x_j \chi_{\sigma_j}$ for all $j \in \mathbb{N}$. Then (y_j) is pairwise disjoint sequence, and hence is a basic sequence with basis constant 1. Moreover,

$$\|y_j\| > \|x_j\| - 2^{-j-2} \geq \|f_j \cdot r_{n_j}\| - 2^{-j-2} > 1/2.$$

Also, $\sum \|x_j - y_j\| < 1/4$. By Proposition 1.a.9 in [5], (y_j) and (x_j) are equivalent. But then $(f_j \cdot r_{n_j})$ is equivalent to a pairwise disjoint sequence in $\ell^{2,\infty}(\Gamma, w)$. However, it is easy to see that $(f_j \cdot r_{n_j})$ is equivalent to $(a_j h_j)$, where $a_1 = 1$ and $a_j = \sqrt{2^{k-1}}$ if $2^{k-1} < j \leq 2^k$. Hence we obtain an embedding S of $[(h_j)]$ into $\ell^{2,\infty}(\Gamma, w)$ such that (Sh_j) is a pairwise disjoint sequence. As (h_j) is a basis of $M^{2,\infty}[0,1]$, we have reached a contradiction to Proposition 8. This completes the proof of Theorem 3.

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Distinguishing Jordan polynomials by means of a single Jordan-algebra norm

by

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Abstract. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we exhibit a Jordan-algebra norm $|\cdot|$ on the simple associative algebra $M_\infty(\mathbb{K})$ with the property that Jordan polynomials over $\mathbb{K}$ are precisely those associative polynomials over $\mathbb{K}$ which act $|\cdot|$-continuously on $M_\infty(\mathbb{K})$. This analytic determination of Jordan polynomials improves the one recently obtained in [5].

1. Introduction. The Jordan product of a (real or complex) associative algebra is defined as the symmetrization of the associative product. Jordan polynomials are those (non-commutative) associative polynomials which can be expressed from the indeterminates by means of a finite process of taking sums, multiplications by scalars, and Jordan products. Clearly, every Jordan polynomial acts continuously on any associative algebra endowed with a Jordan-algebra norm. The question of the continuity of the action of particular non-Jordan associative polynomials (like the associative product $\mathbf{xy}$ or the tetrad $\mathbf{xyzt} + \mathbf{tzyx}$) on suitable associative algebras endowed with Jordan-algebra norms has received special attention in the literature, mainly because of its close relation to positive results and limits in the normed treatment of the Zel'manov prime theorem [15] for Jordan algebras. In this direction the interested reader can consult [14], [10], [11], [2], [8], [6], [7], [12], [13], [3], [4] and [9]. The introduction of [5], together with that of [9] already quoted, can also be interesting for a historical view of progresses in the above mentioned question. Among these progresses, we only emphasize here that every Jordan-algebra norm on a simple associative algebra with unit makes the associative product (and hence, every associative polynomial) continuous, and that the result need not remain true if the assumption of the existence of a unit is removed [3]. In fact, a first "monster" is built in [3] by providing a Jordan-algebra norm on the simple associative algebra $M_\infty(\mathbb{K})$ (of all countably infinite matrices over $\mathbb{K}$ with a finite number of non-zero entries) and a $\mathbb{K}$-linear involution $*$ on