Four characterizations of scalar-type operators with spectrum in a half-line

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Abstract. C₀-scalar-type spectrality criteria for operators A whose resolvent set contains the negative reals are provided. The criteria are given in terms of growth conditions on the resolvent of A and the semigroup generated by A. These criteria characterize scalar-type operators on the Banach space X if and only if X has no subspace isomorphic to the space of complex null-sequences.

0. Introduction. This paper is written in the spirit of Kantorovitz’ [14] work on the characterization of scalar-type spectral operators with spectrum in [0, ∞), and the reader is referred to that article for a brief introduction into the history and the importance of this subject.

A possibly unbounded linear operator A on a Banach space X is called a scalar-type spectral operator on [0, ∞) if there exists a strongly countably additive spectral measure E on the Borel subsets of [0, ∞) so that

\[ D(A) = \left\{ x \in X : \lim_{n \to \infty} \int_0^n t E(dt)x \text{ exists} \right\} \]

and

\[ Ax = \lim_{n \to \infty} \int_0^n t E(dt)x \quad \text{for} \ x \in D(A). \]

Let us denote by C₀[0, ∞) the space of complex-valued continuous functions on [0, ∞) which vanish at infinity. X always stands for a complex Banach space, X* for the dual space of X and U_X for the unit ball of X. The space of bounded linear operators on X is denoted by L(X). The main result of this paper is the following characterization of scalar-type spectral operators on [0, ∞).

Theorem 1. Let X be a Banach space which has no subspace isomorphic to the space c₀ of complex-valued null-sequences. Let A be a linear operator

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on $X$ whose resolvent set contains the half-line $(-\infty, 0)$. Then $A$ is spectral of scalar type on $[0, \infty)$ if and only if one of the following conditions is satisfied:

(i) $A$ is of $C_0$-scalar type on $[0, \infty)$, that means there exists a continuous algebra homomorphism $\Phi : C_0([0, \infty)) \to \text{L}(X)$ with $\Phi(g_s) = (s + A)^{-1}$ for $s > 0$, where $g_s(t) = 1/(s + t)$.

(ii) There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[
\sup_{k=1,2,\ldots} \int_0^\infty e^{k-1} |x^* A^k(t + A)^{-1} x| \, dt \leq M_1,
\]
where $c_k = (2k - 1)!/(k!(k - 2)!)$ if $k \geq 2$ and $c_1 = 1$.

(iii) There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[
\sup_{k=1,2,\ldots} \sum_{m=0}^k \binom{k}{m} |x^* A^m (1 + A)^{-k} x| \leq M_2.
\]

(iv) $A$ generates a strongly continuous semigroup $(U(t))_{t \geq 0}$ with $U(t)X \subseteq D(A)$ for every $t > 0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[
\sup_{k=1,2,\ldots} \frac{1}{(k - 1)!} \int_0^\infty e^{k-1} |x^* A^k U(t) x| \, dt \leq M_3.
\]

Note that the assumption $U(t)X \subseteq D(A)$ for all $t > 0$ implies $U(t)X \subseteq D(A^n)$ for every natural number $n$ (see e.g. [1], Proposition 1.1.10).

Dowson [10] showed that bounded scalar-type spectral operators in weakly complete Banach spaces can be characterized by criterion (i), and Kantorovitz and deLaubenfels [7] were the first to recognize that this equivalence, for possibly unbounded operators, is true in any Banach space not containing $c_0$. That (ii) is equivalent to $A$ being of scalar type on $[0, \infty)$, provided the underlying Banach space is reflexive, is due to Kantorovitz [14]. The proof of Kantorovitz’ criterion is based on Widder’s [18] characterization of Stieltjes transforms. The latter was also used by Rickers [15] to derive a description of scalar-type operators in quasi-complete locally convex spaces. Criteria (iii) and (iv) are related to deLaubenfels’ work on scalar-type spectral operators on Banach lattices [2] and on cyclic spaces [3]. The proof of these criteria uses respectively characterizations of complex-valued moment sequences and Laplace transforms. Some other characterizations of possibly unbounded operators on reflexive spaces being of scalar type on $[0, \infty)$ may be found in [9], [8] and [13].

In the first section of this paper we study $C_0$-scalar-type operators. In particular, it is shown that $A$ is of $C_0$-scalar type on $[0, \infty)$ if and only if $A$ satisfies condition (ii) or (iii) of Theorem 1, and if, in addition, $A$ generates a strongly continuous semigroup then also condition (iv) is equivalent to $A$ being of $C_0$-scalar type. In the second section we show that the classes of scalar-type operators on $X$ and of $C_0$-scalar-type operators on $X$ coincide in exactly those Banach spaces $X$ which have no subspace isomorphic to $c_0$.

Before we start our study of $C_0$-scalar-type operators let us present the following results of Widder’s which are the main tools for our investigations.

Let $k$ be a natural number. If $f : (0, \infty) \to C$ is infinitely often differentiable let
\[
S_k[f](t) = d_k(t)^{-k-1} \frac{d^{k+1}}{dt^{k+1}}(t^k f(t)) \quad \text{for} \quad 0 < t < \infty,
\]
where $d_k = 1/(k!(k - 2)!)$ if $k \geq 2$ and $d_1 = 1$, and let
\[
L_k[f](t) = (-1)^k k! f(k) \left( \frac{k}{t} \right)^{k+1} \quad \text{for} \quad 0 < t < \infty.
\]
The formal operators $S_k$ and $L_k$ are called the Widder inversion operators for the Stieltjes and Laplace transforms, respectively.

If $\mu = (\mu_n)_{n=0,1,\ldots}$ is a complex-valued sequence then
\[
\Lambda_k[\mu](m) = \left( \frac{k}{m} \right)^{-k-m} \Delta^{k-m}[\mu][m] \quad \text{for} \quad k, m = 0, 1, \ldots \quad \text{with} \quad m \leq k,
\]
where $\Delta$ denotes the difference operator $\Delta[\mu][m] = \mu_{m+1} - \mu_m$.

Let $I$ be a closed interval which is bounded from below by $b = \inf I$. A complex-valued function $\phi$ of bounded variation on $I$ is said to be normalized if $\phi(t) = (\phi(t^-) + \phi(t^+))/2$ for all $t$ in the interior of $I$ and if $\phi(b) = 0$. The total variation of $\phi$ is denoted by $\text{Var}(\phi)$.

**Theorem 2.** Let $f : (0, \infty) \to C$ be an infinitely often differentiable function and is a complex-valued sequence.

(i) (Widder inversion of the Stieltjes transform) There exists a unique complex-valued normalized function $\phi$ of bounded variation on $[0, \infty)$ with $f(s) = \int_0^\infty \frac{1}{s+t} \, d\phi(t)$ for $0 < s < \infty$ if and only if
\[
M = \sup_{k=0,1,\ldots} \int_0^\infty |S_k[f](t)| \, dt < \infty.
\]

In this case $M \leq \text{Var}(\phi) \leq M + |A|$, where $A = \lim_{t \to 0^+} tf(t)$.

(ii) (Widder inversion of the Laplace transform) There exists a unique complex-valued normalized function $\phi$ of bounded variation on $[0, \infty)$ with $f(s) = \int_0^\infty e^{-st} \, d\phi(t)$ for $0 < s < \infty$ if and only if
\[
M = \sup_{k=0,1,\ldots} \int_0^\infty |L_k[f](t)| \, dt < \infty.
\]

In this case $M \leq \text{Var}(\phi) \leq M + |B|$, where $B = \lim_{t \to \infty} f(t)$. 

(iii) (Widder characterization of moment sequences) There exists a unique complex-valued normalized function $\phi$ of bounded variation on $[0,1]$ with $\mu_n = \int_0^1 t^n \, d\phi(t)$ for $n = 0, 1, \ldots$ if and only if

$$M = \sup_{k=0,1,\ldots} \sum_{m=0}^{k} |A_k[A_j]| < \infty.$$ 

In this case $M \leq 2 \text{Var}(\phi) \leq 2M$.

For proofs of these statements we refer to Widder [18], Theorem VIII.8.16, Theorem VII.12a, and Theorem III.2b, respectively. Widder stated the results for real-valued functions and sequences, but the complex case is a trivial consequence. The estimates follow easily from the proofs in the cited references. Note that the existence of the limits $A$ in (i) and $B$ in (ii) follows from the conditions (1) and (2), respectively.

1. Characterization of $C_0$-scalar-type operators with spectrum in $[0,\infty)$. In the sequel linear mappings between (complex) vector spaces are called operators. Let $A$ be a bounded operator on $X$ and $D$ a bounded set of complex numbers. Following deLaubenbels [2] we say that $A$ is of $C_0$-scalar type on $D$ if there exists a continuous algebra homomorphism $\Phi$ defined on the space of continuous complex-valued functions $C(D)$ into $L(X)$ with $\Phi(I) = \text{Id}$ and $\Phi(\tau_n) = A$, where $\tau_n(z) = z^n$ for $n = 0, 1, \ldots$ and $z \in D$. We want to extend this definition to possibly unbounded operators with spectrum in the half-line $[0,\infty)$, for this reason we make the following observation.

LEMMA 3. If $A$ is a bounded operator on $X$ and if $D \subseteq C$ is closed and bounded then $A$ is of $C_0$-scalar type on $D$ if and only if $\sigma(A) \subseteq D$ and there exists a continuous algebra homomorphism $\Phi : C(D) \rightarrow L(X)$ with $\Phi(\tau_n) = (z + A)^{-1}$ for all $z \notin D$.

Proof. If $A$ is of $C_0$-scalar type on $D$ with the corresponding algebra homomorphism $\Phi$ then, since $D$ is closed, $\tau_n \in C(D)$ for all $z \notin D$, and

$$\Phi(\tau_n) = \Phi((z\tau_0 + \tau_1)\tau_n) = \Phi(\tau_n) = \text{Id},$$

and similarly $\Phi(\tau_n)(z + A) = \text{Id}$. Hence $-z$ is contained in the resolvent set of $A$ and $(z + A)^{-1} = \Phi(\tau_n)$. Conversely, if $\Phi : C(D) \rightarrow L(X)$ is a continuous algebra homomorphism with $\Phi(\tau_n) = (z + A)^{-1}$ for all $z \notin D$ then

$$\Phi(\tau_n)(z + A)^{-1} = \Phi(\tau_n) = (z + A)^{-1},$$

whence $\Phi(\tau_n) = \text{Id}$. Moreover,

$$(z + A)^{-1}(z + \Phi(\tau_1)) = \Phi(\tau_1 + \tau_0) = \Phi(\tau_0) = \text{Id},$$

whence $\Phi(\tau_1) = A$.

Motivated by this lemma we define $C_0$-scalar-type operators on $[0,\infty)$ as follows:

DEFINITION 4. A possibly unbounded operator $A$ on $X$ is of $C_0$-scalar type on $[0,\infty)$ if $\sigma(A) \subseteq [0,\infty)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0,\infty) \rightarrow L(X)$ with $\Phi(\tau_n) = (z + A)^{-1}$ for all $z \in C \setminus \{-\infty, 0\}$.

The next lemma shows that $A$ is of $C_0$-scalar type on $[0,\infty)$ if and only if $A$ admits a $C_0[0,\infty)$ functional calculus in the sense of deLaubenbels [4]. Moreover, if $A$ is densely defined, then $A$ is of $C_0$-scalar type in the sense of Definition 4 if and only if $A$ is $C_0$-scalar in the sense of deLaubenbels [2] (see Remark 7 below).

LEMMA 5. An operator $A$ on $X$ is of $C_0$-scalar type on $[0,\infty)$ if and only if its resolvent set contains the half-line $(-\infty, 0)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0,\infty) \rightarrow L(X)$ with $\Phi(\tau_n) = (s + A)^{-1}$ for all $0 < s < \infty$.

Proof. That the first assertion implies the second is trivial. So let $\Phi : C_0[0,\infty) \rightarrow L(X)$ be a continuous algebra homomorphism with $\Phi(\tau_n) = (s + A)^{-1}$ for all $0 < s < \infty$. Let $z \in C \setminus \{-\infty, 0\}$. If $z \notin D(A)$ then

$$(1 + A)^{-1}\Phi(\tau_n)(z + A)x = \Phi(\tau_n)(1 + A)^{-1}(z + (1 + A))x$$

$$= \Phi(\tau_n)(z - 1)(\tau_n + 1)x$$

$$= \Phi(\tau_n)x = (1 + A)^{-1}x,$$

whence $\Phi(\tau_n)(z + A)x = x$. If $x \in X$ then

$$\Phi(\tau_n)x = \Phi(\tau_n)(1 + x)(z + A)^{-1}x$$

$$= (1 + A)^{-1}(z - 1)(\tau_n + 1)x \in D(A).$$

Applying $z + A = (z - 1) + (1 + A)$ to both sides of the last equation yields

$$(z + A)\Phi(\tau_n)x = (z - 1)\Phi(\tau_n)x + (1 - z)\Phi(\tau_n)x + z \in x = x.$$ 

Consequently, $x$ is contained in the resolvent set of $A$ and $\Phi(\tau_n)x = (z + A)^{-1}$.

If $I$ is a closed interval with inf $I = 0$ and if $\phi$ is a complex-valued function of bounded variation on $I$, then, for $t \geq 0$, $\lim_{\tau \to 0} \phi(t + \tau)$ is the variation of $\phi$ in the interval $[0,t]$ of $\phi$, provided $t$ is the right end point of $I$. If $I = [0,a]$, then $\lim_{\tau \to 0} \phi(t + \tau)$ is the variation of $\phi$ on the interval $[0,a]$. The function $\varphi : I \times I$ is bounded on variation on $I$, the variations of $\phi$ and of $\varphi$ are equal, and for every $F \in C_0(I)$,

$$\int_I f(t) \, d\phi(t) \leq \int_I |f(t)| \, d\varphi(t).$$

This is proven for example in [17].
THEOREM 6. Let $A$ be an operator on $X$ whose resolvent set contains the half-line $(-\infty, 0)$. Then $A$ is of $C^0$-scalar type on $[0, \infty)$ if and only if one of the following conditions (i) or (ii) is satisfied:

(i) There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[ \sup_{k=1,2,\ldots} \int_0^\infty \left| x^* A^k(t + A)^{-2k} x \right| dt \leq M_1, \]
where $c_k = (2k - 1)!/(k!(k - 2)!)$ if $k \geq 2$ and $c_1 = 1$.

(ii) There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[ \sup_{k=0,1,\ldots} \sum_{m=0}^k \left( \frac{k!}{m!} \right) |x^* A^m(1 + A)^{-k} x| \leq M_2. \]

If, in addition, $A$ is densely defined then $A$ is of $C^0$-scalar type on $[0, \infty)$ if and only if

(iii) $-A$ generates a $C_0$-semigroup $(U(t))_{t \geq 0}$ such that $U(t)X \subseteq D(A)$ for every $t > 0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,
\[ \sup_{k=1,2,\ldots} \frac{1}{(k - 1)!} \int_0^\infty \left| x^* A^k U(t) x \right| dt \leq M_3. \]

Proof. (i) Let $A$ be of $C^0$-scalar type on $[0, \infty)$ with the corresponding algebra homomorphism $\phi$. Given $x \in X$, $x^* \in X^*$ and $f \in C_0[0, \infty)$ we put $l^k[1, f](x) = x^* \Phi(f)x$. Then $l^k[1, f]$ is a continuous linear functional on $C_0[0, \infty)$ with $\|l^k[1, f]x^*\| \leq \|\Phi\| \cdot \|x\| \cdot \|l^k[1, f]x^*\|$. So we may apply the Riesz representation theorem to find a complex-valued function $\phi(x, x^*)$ of bounded variation on $[0, \infty)$ with $\|\phi(x, x^*)\| \leq M\|x\| \cdot \|x^*\|$, and so that $l^k[1, f](f) = \int_0^1 f(u) d\phi(x, x^*)(u)$ for all $f \in C_0[0, \infty)$. We then have
\[ t^{k-1} x^* A^k(t + A)^{-2k} x = t^{k-1} \int_0^\infty u^{k-1} (t + u)^{-2k} d\phi(x, x^*)(u). \]

Hence, by changing the order of integration and integrating by parts we get
\[ \int_0^\infty t^{k-1} |x^* A^k(t + A)^{-2k} x| dt \leq \int_0^\infty t^{k-1} u^{k-1} (t + u)^{-2k} d\Phi[\phi(x, x^*)](u) dt = 0 \]
\[ = \int_0^\infty s^{k-1} (s + 1)^{-2k} ds d\Phi[\phi(x, x^*)](u) = \frac{k - 1}{k} \text{Var}(\phi(x, x^*)) \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|. \]

Conversely, assume that (i) holds. For the proof that $A$ is of $C^0$-scalar type on $[0, \infty)$, we need Widder’s inversion theorem for the Stieltjes transform. Therefore, given $x \in X$ and $x^* \in X^*$ we calculate $S_k[x^* R x]$ where $R(t) = (t + A)^{-1}$. For every $k = 1, 2, \ldots$ we have
\[ \frac{d^{2k-1}}{dt^{2k-1}} \left( t^k(t + A)^{-1} \right) = \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{d^l}{dt^l} \left( t^k \frac{d^{2k-1-l}}{dt^{2k-1-l}} ((t + A)^{-1}) \right) \]
\[ = \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{((2k - 1) - l)!}{(k - l)!} t^{k-1}(-1)^{2k-1-l}((t + A)^{-1})^{2k-1-l} \]
\[ = (2k - 1)!(-1)^{k-1} \sum_{l=0}^{k} \binom{k}{l} (-t)^{k-1-l}((t + A)^l(t + A)^{-2k}) \]
whence
\[ S_k[x^* R x](t) = ct^{k-1} x^* A^k(t + A)^{-2k} x. \]

Hence by Widder’s theorem $x^* R x$ is the Stieltjes transform of a unique normalized function $\phi(x, x^*)$ of bounded variation on $[0, \infty)$ with $\Var(l^k[1, f](f)) \leq \delta \|x\| \cdot \|x^*\| + A[\delta]$, where $A[\delta] = \lim_{\delta \to 0} \int_0^t \sigma^2 R(t) \, dt$. For every $t > 0$ the operator $i R(t)$ is bounded and
\[ |x^* R(t)x| = \int_0^t \frac{0}{\omega + \omega^2} d\phi(x, x^*)(u) \leq \delta \|x\| \cdot \|x^*\| \]
for every $x \in X$ and $x^* \in X^*$. Hence $\bar{A} = \sup_{0 < \infty} \|i R(t)\| < \infty$ by the uniform boundedness principle. Consequently,
\[ \Var(l^k[1, f](f)) \leq \delta \|x\| \cdot \|x^*\|, \]
where $\bar{M} = M + \bar{A}$. Now let $\Phi(x)$ be the linear functional on $X^*$ which assigns to every $x^* \in X^*$ the complex number $\int_0^1 l^k[1, f](f) d\phi(x, x^*)(t)$. Then $\Phi : C_0[0, \infty) \to L(X, X^*)$ is linear and we have the estimate
\[ \|\Phi(f)\| \leq \delta \|f\| \Var(l^k[1, f](f)) \leq \delta \|f\| \cdot \|x\| \cdot \|x^*\|. \]
So we infer $\|\Phi\| \leq \bar{M}$. Moreover, $\Phi(g_a) = (s + A)^{-1} \in L(X)$ because
\[ (\Phi(g_a) x, x^*) = \int_0^1 \frac{1}{\omega + \omega^2} d\phi(x, x^*)(t) = x^* (s + A)^{-1} x. \]

Since $\{g_s : 0 < s < \infty\}$ is a total subset of $C_0[0, \infty)$ we infer that $\Phi(f) \in L(X)$ for every $f \in C_0[0, \infty)$, that is, $\Phi : C_0[0, \infty) \to L(X)$. 

Scalar-type operators
We now claim that $\Phi$ is an algebra homomorphism. By the resolvent equality, for all $0 < r, s < \infty$ with $r \neq s$ we have
\[
\Phi(\theta_r \theta_s) = (r - s)^{-1} \Phi(\theta_s - \theta_r) = (r - s)^{-1}((s + A)^{-1} - (r + A)^{-1}) = (r + A)^{-1}(s + A)^{-1},
\]
and
\[
\Phi(\theta^2_s) = \lim_{h \to 0} \Phi(h^{-1}(\theta_s - \theta_{s+h})) = \lim_{h \to 0} h^{-1}((s + A)^{-1} - (s + h + A)^{-1}) = (s + A)^{-2}.
\]
So we may conclude $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C_0[0, \infty)$ because the set $\{ \theta_s : 0 < s \leq \infty \}$ is total in $C_0[0, \infty)$.

(ii) Let again $\Phi$ be the continuous algebra homomorphism corresponding to $A$. Given $x \in X$, $x^* \in X^*$ and $f \in C[0,1]$ put $f[x, x^*]f = x^*\Phi(f)x$. Then $f[x, x^*]$ is a continuous linear functional on $C[0,1]$ with $\|f[x, x^*]\| \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|$. So we may apply the Riesz representation theorem to get a complex-valued function $\phi[x, x^*]$ of bounded variation on $[0,1]$ which represents $f[x, x^*]$, and with $\text{Var}(\phi[x, x^*]) = \|f[x, x^*]\|$. Consequently,
\[
\sum_{m=0}^{k} \binom{k}{m} x^* A^m (1 + A)^{-k} x = \sum_{m=0}^{k} \binom{k}{m} \int_0^1 |t^m(1 + t)^{-k} \, d(\phi[x, x^*])(t)|
\leq \int_0^1 \sum_{m=0}^{k} \binom{k}{m} t^m(1 + t)^{-k} \, d\text{Var}(\phi[x, x^*])(t)
\leq \text{Var}(\phi[x, x^*]) \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|.
\]
Thus, if $A$ is of $C^0$-scalar type on $[0, \infty)$, then (ii) follows.

Assume now that (ii) is satisfied. We show first that this assumption implies that $(1 + A)^{-1}$ is of $C$-scalar type on $[0, 1]$. To this end put $\mu_k = (1 + A)^{-n-k}$. Then by induction it follows that $\Delta^k \mu_k = (-1)^k A^k (1 + A)^{-k(n-k)}$ for $k, n = 0, 1, \ldots$, whence
\[
A_k[\theta] = \binom{k}{m} A^k(1 + A)^{-k} \quad \text{for } k, m = 0, 1, \ldots \text{ with } k \geq m.
\]
Hence (ii) implies, for every $x \in X$ and $x^* \in X^*$,
\[
\sup_{k=0, 1, \ldots} \sum_{m=0}^{k} |A_k[x^* \mu x]| \leq M \|x\| \cdot \|x^*\|.
\]
So we may conclude from Widder's Theorem 2(iii) that there exists a unique complex-valued normalized function $\phi[x, x^*]$ of bounded variation on $[0, 1]$ with $\text{Var}(\phi[x, x^*]) \leq M \|x\| \cdot \|x^*\|$ and so that
\[
x^* \mu_k x = \int_0^1 t^k \, d(\phi[x, x^*])(t).
\]
Since
\[
\int_0^1 \|f(t)\| \Phi(x^* \phi[x, x^*])(t) \leq \|f\| \cdot \text{Var}(\phi[x, x^*]) \leq M \|f\| \cdot \|x\| \cdot \|x^*\|,
\]
and since $\phi[x, x^*]$ depends linearly on $x$ and $x^*$, we can define a continuous linear operator $\Psi : C[0,1] \to L(X, X^*)$ by
\[
(\Psi(f)x, x^*) = \int_0^1 f(t) \, d\phi[x, x^*](t).
\]
We then have, for every $n = 0, 1, \ldots$,
\[
(\Psi(\tau_n)x, x^*) = \int_0^1 t^n \, d\phi[x, x^*](t) = x^* (1 + A)^{-n} x,
\]
whence $\Psi(\tau_n) = (1 + A)^{-n}$ is in $L(X)$. Since the set $\{\tau_n : n = 0, 1, \ldots\}$ is total in $C[0,1]$ we conclude that $\Psi$ is an operator from $C[0,1]$ into $L(X)$. Moreover, for every $l, n = 0, 1, \ldots$,
\[
\Psi(\tau_l \tau_n) = \Psi(\tau_{l+n}) = (1 + A)^{-(l+n)} = (1 + A)^{-l}(1 + A)^{-n} = \Psi(\tau_l)\Psi(\tau_n).
\]
This equation shows that the bounded operator $\Psi$ is, in addition, an algebra homomorphism, because the set $\{\tau_n : n = 0, 1, \ldots\}$ is total in $C[0,1]$. Note that $\Psi(\tau_0) = (1 + A)^{-1} = \text{Id}$ and $\Psi(\tau_1) = (1 + A)^{-1}$, so that $(1 + A)^{-1}$ is of $C^0$-scalar type on $[0, 1]$.

To show that $A$ is of $C^0$-scalar type on $[0, \infty)$ put $\Gamma f(t) = f(1/t - 1)$ for $0 < t \leq 1$ and $\Gamma f(0) = 0$. Then $\Gamma : C_0[0, \infty) \to C_{0}([1,0])$ is a continuous algebra homomorphism. Hence $\phi = \Psi \circ \Gamma : C_0[0, \infty) \to L(X)$ is also a continuous algebra homomorphism, and for every $0 < s < \infty$ we have
\[
\Phi(\theta_s) = \Psi(\Gamma(\theta_s)) = \Psi(\tau_1 + (s - 1)\tau_1)^{-1} = (1 + A)^{-(1 + (s - 1)A)} = (1 + A)^{-1} = (s + A)^{-1}.
\]
It follows, by Lemma 5, that $A$ is of $C^0$-scalar type on $[0, \infty)$.

(iii) Let $A$ be of $C^0$-scalar type on $[0, \infty)$ with the corresponding algebra homomorphism $\Phi$ and assume $A$ to be densely defined. Since
\[
\|\lambda (1 + A)^{-1}\| = \|\Phi(\lambda \theta_s)\| \leq \|\Phi\| \cdot \|\lambda \theta_s\| \leq \|\Phi\|
\]
for all $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| \leq 3\pi/4$, Theorem 3.3 of [12] tells us that $-A$ generates an analytic semigroup. In particular, $D(A^n) \subseteq U(t)X$ for all $t > 0$ and
\[
U(t)^n(t) = (-1)^n A^n U(t).
\]
We denote by \( \varepsilon_t \in C_0[0, \infty) \), \( t > 0 \), the function \( \varepsilon_t(u) = e^{-ut} \). Put \( V(t) = \Phi(\varepsilon_t) \) for \( t > 0 \) and let \( V(0) = \text{Id} \). We next show that \( U = V \).

Given \( x \in X \) and \( x^* \in X^* \) let \( \phi(x, x^*) \) be the complex-valued normalized function of bounded variation on \([0, \infty)\) which represents the linear functional \( x^*\Phi x \in C_0[0, \infty) \). Then for \( s > 0 \) we have

\[
\int_0^\infty e^{-st} x^* V(t)x \, dt = \int_0^\infty e^{-st} x^* \Phi(\varepsilon_t)x \, dt
\]

\[= \int_0^\infty \int_0^{\infty} e^{-u} \, \phi(x, x^*)(u) \, dt \, du\]

\[= \int_0^\infty \int_0^{\infty} e^{-u} \phi(x, x^*)(u) \, dt \, du\]

\[= \int_0^\infty \frac{1}{s+u} \, \phi(x, x^*)(u) \, du\]

\[= x^* \Phi(s)x = x^*(s + A)^{-1}x.
\]

Hence the function \( s \mapsto (s + A)^{-1} \) is the Laplace transform of \( V \). Since \(-A\) generates the \( C_0\)-semigroup \( U \) we know, by [12], p. 17, that \( s \mapsto (s + A)^{-1} \) is also the Laplace transform of \( U \). By the injectivity of the Laplace transform, \( V(t) = U(t) \) follows for all \( t > 0 \), since \( U \) and \( V \) are strongly continuous in \((0, \infty)\). In particular, \( U^{(n)}(t) = (-1)^n A^n \varepsilon_t \varepsilon_t \) for all \( t > 0 \).

Now, by Widder’s Theorem 2(ii) the inequality

\[
\sup_{k=1,2,\ldots,(k-1)!} \int_0^\infty \frac{1}{k} \, |x^* A^k U(t)x| \, dt \leq \| \Phi \| \cdot \| x^* \| \cdot \| x \|
\]

follows, because \( L_k[U](t) = \frac{1}{k!} A^k U(k^{-1} t)(k^{-1} t)^{k+1} \). We have thus proved that \( \Phi \) is a densely defined operator of \( C_0\)-scalar type on \([0, \infty)\).

Conversely, assume that \( A \) is densely defined and (iii) holds. Since \( T(t)X \) is contained in \( D(A) \) for every \( t > 0 \) it follows, by [1], Proposition 1.1.10, that \( U \) is infinitely often differentiable in \( t > 0 \). \( U(t)x \subseteq D(A^n) \), and \( U^{(n)}(t) = (-1)^n A^n U(t) \). Hence condition (iii) is equivalent to

\[
\sup_{k=1,2,\ldots,(k-1)!} \int_0^\infty |x^* L_k[U](t)x| \, dt \leq M \| x^* \| \cdot \| x \|
\]

So Widder’s theorem implies that for every \( x \in X \) and \( x^* \in X^* \) there exists a unique complex-valued normalized function \( \phi(x, x^*) \) on \([0, \infty)\) with \( \text{Var}(\phi(x, x^*)) \leq M \| x^* \| \cdot \| x \| + |B(x, x^*)| \), where \( B(x, x^*) = \lim_{t \to \infty} x^* U(t)x \), and so that

\[
x^* U(t)x = \int_0^\infty e^{-st} \, \phi(x, x^*)(u) \, du \quad \text{for all } 0 < t < \infty.
\]

We deduce from this equation that \( \sup_{t \geq 0} |x^* U(t)x| \leq \text{Var}(\phi(x, x^*)) \). Thus the uniform boundedness principle implies \( B = \sup_{t \geq 0} \| U(t) \| < \infty \) and we infer that \( \text{Var}(\phi(x, x^*)) \leq M \| x \| \cdot \| x^* \| \), where \( M = M + B \). It follows that \( \Phi : C_0[0, \infty) \to \mathbf{L}(X, X^*) \) defined by \( \Phi(f)x, x^* = \int_0^\infty f(t) \, \phi(x, x^*)(t) \) is a continuous linear operator with \( \Phi(\varepsilon_t) = U(t) \in \mathbf{L}(X) \). Since the set \( \{ \varepsilon_t : 0 < t < \infty \} \) is total in \( C_0[0, \infty) \) we infer \( \Phi f \in \mathbf{L}(X) \) for all \( f \in C_0[0, \infty) \).

From \( \Phi(\varepsilon_t) = U(t) \), and from the fact that \( U \) is generated by \(-A\), we deduce

\[
x^* (s + A)^{-1}x = \int_0^\infty e^{-st} x^* U(t)x \, dt = \int_0^\infty e^{-st} \int_0^\infty e^{-u} \phi(x, x^*)(u) \, du \, dt\]

\[= \int_0^\infty \frac{1}{s+u} \phi(x, x^*)(u) \, du = x^* \Phi(s)x.
\]

Hence \( \Phi(s)x = (s + A)^{-1}x \). That \( \Phi \) is an algebra homomorphism follows as in the proof of (i). Consequently, \( A \) is of \( C_0\)-scalar type on \([0, \infty)\).

**Remark 7.** (i) If \( A \) is a bounded operator with spectrum contained in some interval \([m, M]\) then \( A \) is of \( C_0\)-scalar type on \([m, M]\) if and only if \( \| p(A) \| < \infty \), where the supremum is taken over all polynomials \( p \) on \([m, M]\) with \( \sup_{m \leq t \leq M} |p(t)| \leq 1 \). This follows immediately from the observation that the polynomials are dense in \( C([m, M]) \).

Following an idea of Schäfer [16] we can give a second characterization of bounded operators being of \( C_0\)-scalar type on \([m, M]\), which uses Widder’s characterization of moment sequences. It is clear that \( A \) is of \( C_0\)-scalar type on \([m, M]\) if and only if \( B = (M - m)^{-1}(m - A) \) is of \( C_0\)-scalar type on \([0, 1]\). Now consider the operator-valued sequence \( \mu_n = B^n, n = 0, 1, 2, \ldots \). Then \( A \approx \lambda \approx (B^n/m)^{1-B} \) for \( m, k = 0, 1, 2, \ldots \). Now applying Widder’s characterization of moment sequences it can be seen as in the proof of Theorem 6(ii) that \( B \) is of \( C_0\)-scalar type on \([0, 1]\) if and only if there exists \( M > 0 \) such that

\[
\sup_{k=1,2,\ldots,(k-1)!} \sum_{m=0}^{k} \left( \frac{k}{m} \right) |x| B^m (1-B)^{k-m} \, x = M \| x \| \cdot \| x^* \| \quad \text{for all } x \in X, x^* \in X^*.
\]

(ii) deLaubenbklens [2] defined an operator \( A \) to be \( C_0\)-scalar on \([0, \infty)\) if \(-A\) generates a uniformly bounded semigroup and \((1 + A)^{-1}\) is of \( C_0\)-scalar type on \([0, 1]\). From the proof of Theorem 6(iii) we see that a densely defined operator \( A \) with \((-\infty, 0) \subseteq \sigma(A) \) is of \( C_0\)-scalar type on \([0, \infty)\) in our sense if and only if \( A \) is \( C_0\)-scalar on \([0, \infty)\) in the sense of deLaubenbklens.
We note that not every operator of $C_0$-scalar type on $[0, \infty)$ generates a $C_0$-semigroup. Consider the operator $A$ on $C_0[0, \infty)$ with $D(A) = \{v \in C_0[0, \infty) : \lim_{t \to 0^+} v(t)/t \text{ exists} \}$ and $A v(t) = v(t)/t$ for $t > 0$ and $A v(0) = \lim_{t \to 0^+} v(t)/t$. Since $(x + A)^{-1} v(t) = (t/(tx + 1)) v(t)$ we see that every $x \in \mathbb{C} \setminus [0, \infty)$ is contained in the resolvent set of $A$. Indeed, $\sigma(A) = [0, \infty)$. Moreover, $A$ is of $C_0$-scalar type on $[0, \infty)$, because the operator $\Phi : C_0[0, \infty) \to \mathbb{L}(X)$ defined by $\Phi(f) v(t) = f(1/t) v(t)$ if $t > 0$, and $\Phi(f) v(0) = 0$, is a continuous algebra homomorphism with $\Phi(g v)(t) = (t/(t+1)) v(t)$. But $A$ does not generate a $C_0$-semigroup, because $A$ is not densely defined.

(iii) The proof of Theorem 6 shows that for densely defined operators the condition (iii) in this theorem is equivalent to

$$\sup_{k=1,2,\ldots} \{ (k-1)! \} \int_0^\infty |e^{A t} U(t)| \, dt \leq M_4.$$  

2. Characterization of scalar-type operators. In this section we prove the main theorem. The key for the proof is the relationship between vector measures and operators on spaces of continuous functions. For a deeper discussion of this relationship we refer the reader to the monograph of Diestel and Uhl [8]. Here we only recall some basic facts concerning the representation of operators on spaces of continuous functions.

If $\Sigma$ is a $\sigma$-algebra and $F : \Sigma \to X$ has the property

$$F\left( \bigcup_{k=1}^\infty B_k \right) = \sum_{k=1}^\infty F(B_k)$$

whenever $(B_k)$ is a sequence of pairwise disjoint members of $\Sigma$, then $F$ is called a countably additive vector measure on $\Sigma$. If $\Sigma$ is the Borel $\sigma$-algebra on a locally compact Hausdorff space then we call $F$ weakly regular if $x^* \circ F$ is a regular complex Borel measure for every $x^* \in X^*$.

The following theorem would be a combination of results presented in [8] if we considered operators on $C(K)$, where $K$ is a compact Hausdorff space. But for our investigation of scalar-type operators we have to study operators on $C_0[0,\infty)$.

**Theorem 8.** If $T : C_0[0,\infty) \to X$ is a bounded operator, then $T$ is weakly compact if and only if $T$ can be represented by a unique weakly regular countably additive vector measure $F$ on $[0,\infty)$, that is, $Tf = \int_0^\infty f(t) F(dt)$ for $f \in C_0[0,\infty)$. In particular, if $X$ has no subspace isomorphic to $c_0$, then every bounded operator from $C_0[0,\infty)$ into $X$ can be represented by a unique weakly regular countably additive vector measure $F$.

**Proof.** In this proof we identify the dual space of $C_0[0,\infty)$ with the space of regular Borel measures on $[0,\infty)$ endowed with the variation norm. This is possible by the Riesz representation theorem. Thus every Borel function $g$ on $[0,\infty)$ can be identified with a member in the second dual of $C_0[0,\infty)$ by putting $\langle g, \nu \rangle = \int_0^\infty \nu(t) g(t) \, dt$ for all regular Borel measures $\nu$. In particular, this identification is possible for characteristic functions $\chi_E$ of Borel sets $E$.

If $T : C_0[0,\infty) \to X$ is weakly compact then the second dual $T^{**}$ of $T$ takes all its values in $X$. Given a Borel set $B$ let $F(B) = T^{**}(\chi_B)$. Then for every $x^* \in X^*$ we have

$$x^* F(B) = \langle \chi_B, T^{**} x^* \rangle = T^{**}(x^*)$$

Here we identify $T^{**} x^* \in C_0[0,\infty)^*$ with a regular Borel measure on $[0,\infty)$. Hence $F$ is weakly regular. Moreover, it follows that $F$ is weakly countably additive, and by the Orlicz–Pettis theorem [8], Corollary I.4.4, we infer that $F$ is countably additive. Since for all $x^* \in X^*$,

$$x^* T f = T^{**} (x^*) = \int_0^\infty f(t) x^* F(\, dt),$$

we see that $T$ is represented by $F$.

Conversely, if there exists a countably additive vector measure $F$ on $[0,\infty)$ which represents $T$, then define $T_n f = \int_0^n f(t) F(dt)$ for every natural number $n$. The operators $T_n$ from $C_0[0,\infty)$ into $X$ are weakly compact, by the Bartle–Dunford–Schwartz Theorem [8], Theorem VI.2.5, and they converge in norm to $T$. Since the operator ideal of weakly compact operators is closed with respect to the operator norm, the weak compactness of $T$ is established.

Now assume that $c_0$ is not contained in $X$ and let $T$ be a bounded operator from $C_0[0,\infty)$ into $X$. To prove the second part of the theorem, by the first it is sufficient to establish the weak compactness of $T$. We extend $T$ continuously to a operator $T_1 : C([0,\infty) \to X$ by $T_1(f) = T(f - f_\infty)$, where $f_\infty(t) = \lim_{s \to \infty} f(s)$ for all $0 \leq t < \infty$. Define $\Gamma : C([0,1] \to C_0[0,\infty)$ by $\Gamma f(t) = f(t/(1+t))$ and let $S = T_0 \Gamma : C([0,1] \to X$. Since $c_0$ is not contained in $X$ the operator $\Gamma$ is weakly compact by [8], Theorem VI.2.15. Hence $T_1$ is also weakly compact because $\Gamma$ is continuously invertible. Now the weak compactness of $T_1$ follows from the equality of the ranges of $T$ and $T_1$. $

If $\Phi : C_0[0,\infty) \to \mathbb{L}(X)$ is a bounded operator, let $\Phi_0 f = \Phi(f) x$ for $x \in X$ and $f \in C_0[0,\infty)$. We call the operators $\Phi_0 : C_0[0,\infty) \to X$ the components of $\Phi$. 


Theorem 9. Let $A$ be a linear operator on $X$ with $(-\infty, 0) \subseteq \sigma(A)$. Then $A$ is of scalar type on $[0, \infty)$ if and only if $A$ is of $C^0$-scalar type on $[0, \infty)$ and the components $\Phi(x)$ of the corresponding algebra homomorphism $\Phi$ are weakly compact for every $x \in X$. In particular, if $X$ has no subspace isomorphic to $C_0$, then $A$ is of scalar type on $[0, \infty)$ if and only if $A$ is of $C^0$-scalar type on $[0, \infty)$.

Remark 10. For bounded $A$, the first half of this theorem is shown in [10], Theorem 4.6.24, and the second half first appeared in [9]. For possibly unbounded $A$ the second half of Theorem 9 is an immediate consequence of [6], Theorem 3.3 and Corollary 3.5.

Proof of Theorem 9. If $A$ is a scalar-type operator with spectral measure $E$ on $[0, \infty)$ then it is well known that $\Phi f = \int_0^\infty f(t) E(dt)$ defines a continuous algebra homomorphism $\Phi : C_0[0, \infty) \to L(X)$ with $\Phi(1) = (s + A)^{-1}$. This is proven for example in [11]. Moreover, for every $x \in X$ the $X$-valued measure $E[x]$ with $E[x](B) = E(B)x$ is countably additive by assumption. Since $\Phi(x) f = \int_0^\infty f(t) E[x](dt)$ it follows, by Theorem 8, that $\Phi(x)$ is weakly compact.

Conversely, assume $A$ is of $C^0$-scalar type on $[0, \infty)$ with the corresponding algebra homomorphism $\Phi$. If $\Phi$ has weakly compact components or if $X$ has no subspace isomorphic to $C_0$ then, by Theorem 8, for every $x \in X$ there exists a countably additive $X$-valued measure $E[x]$ on $[0, \infty)$ with $\Phi(x) f = \int_0^\infty f(t) E[x](dt)$. Given a Borel measurable set $B$ in $[0, \infty)$ we now define $E(B) : X \to X$ by $E(B)x = E[x](B)$. Then $E(B)$ is linear and, by [8], Proposition I.1.11 and Theorem I.1.13,

$$\|E(B)\| = \sup_{\|x\| \leq 1} \|E[x](B)\| \leq \sup_{\|x\| \leq 1} \|\Phi(x)\| \leq \|\Phi\|.$$  

Hence $E$ is an operator-valued, strongly countably additive bounded measure defined on the Borel subsets of $[0, \infty)$ with

$$(s + A)^{-1}x = \int_0^\infty \frac{1}{s + t} E(dt)x.$$

Moreover, we have $\|(t + A)^{-1}\| = \|\Phi(t 1)\| \leq \|\Phi\|$.

Now proceed as in Kantorovitz' proof of Theorem 1.1 in [14] to show that $E$ is a spectral measure for $A$ with

$$D(A) = \{ x \in X : \lim_{n \to \infty} \int_0^n t E(dt)x \text{ exists} \}$$

and $Ax = \lim_{n \to \infty} \int_0^n t E(dt)x$ for $x \in D(A)$.

The proof of the main theorem now follows from Theorem 6 combined with Theorem 9.
Purely non-atomic weak \(L^p\) spaces

by

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Abstract. Let \((\Omega, \Sigma, \mu)\) be a purely non-atomic measure space, and let \(1 < p < \infty\). If \(L^{p,\infty}(\Omega, \Sigma, \mu)\) is isomorphic, as a Banach space, to \(L^{p,\infty}(\Omega', \Sigma', \mu')\) for some purely atomic measure space \((\Omega', \Sigma', \mu')\), then there is a measurable partition \(\Omega = \Omega_1 \cup \Omega_2\) such that \((\Omega_1, \Sigma \cap \Omega_1, \mu \cap \Omega_1)\) is countably generated and \(\sigma\)-finite, and that \(\mu(\sigma) = 0\) or \(\infty\) for every measurable \(\sigma \subseteq \Omega_2\). In particular, \(L^{p,\infty}(\Omega, \Sigma, \mu)\) is isomorphic to \(E^{p,\infty}\).

1. Introduction. In [3], the author proved that the spaces \(L^{p,\infty}[0, 1]\) and \(L^{p,\infty}[0, \infty)\) are both isomorphic to the atomic space \(E^{p,\infty}\). Subsequently, it was observed that if \((\Omega, \Sigma, \mu)\) is countably generated and \(\sigma\)-finite, then \(L^{p,\infty}(\Omega, \Sigma, \mu)\) is isomorphic to either \(E^{p,\infty}\) or \(E^{\infty}\) [4, Theorem 7]. In the present paper, we show that the isomorphism of atomic and non-atomic weak \(L^p\) spaces does not hold beyond the countably generated, \(\sigma\)-finite situation.

Before giving the precise statement of the main theorem, let us agree on some terminology. Throughout this paper, every measure space under discussion is assumed to be non-trivial in the sense that it contains a measurable subset of finite non-zero measure. A measurable subset \(\sigma\) of a measure space \((\Omega, \Sigma, \mu)\) is an atom if \(\mu(\sigma) > 0\), and either \(\mu(\sigma') = 0\) or \(\mu(\sigma \setminus \sigma') = 0\) for each measurable subset \(\sigma'\) of \(\sigma\). A purely non-atomic measure space is one which contains no atoms. We say that a collection \(S\) of measurable sets generates a measure space \((\Omega, \Sigma, \mu)\) if \(\Sigma\) is the smallest \(\sigma\)-algebra containing \(S\) as well as the \(\mu\)-null sets. A measure space \((\Omega, \Sigma, \mu)\) is purely atomic if it is generated by the collection of all of its atoms; it is countably generated if there is a sequence \(\{\sigma_n\}\) in \(\Sigma\) which generates \((\Omega, \Sigma, \mu)\). For any measure space \((\Omega, \Sigma, \mu)\), and \(1 < p < \infty\), the weak \(L^p\) space \(L^{p,\infty}(\Omega, \Sigma, \mu)\) is the space of all (equivalence classes of) \(\Sigma\)-measurable functions \(f\) such that

\[
\|f\| = \sup_{c > 0} c(\mu(|f| > c))^{1/p} < \infty.
\]

It is well known that \(\| \cdot \|\) is equivalent to a norm under which \(L^{p,\infty}(\Omega, \Sigma, \mu)\) is a Banach space. However, since we are only concerned with isomorphic