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Four characterizations of scalar-type operators with spectrum in a half-line

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Abstract. C^0-scalar-type spectrality criterions for operators A whose resolvent set contains the negative reals are provided. The criterions are given in terms of growth conditions on the resolvent of A and the semigroup generated by A. These criterions characterize scalar-type operators on the Banach space X if and only if X has no subspace isomorphic to the space of complex null-sequences.

0. Introduction. This paper is written in the spirit of Kantorovitz' [14] work on the characterization of scalar-type spectral operators with spectrum in $[0,\infty)$, and the reader is referred to that article for a brief introduction into the history and the importance of this subject.

A possibly unbounded linear operator A on a Banach space X is called a *scalar-type spectral operator* on $[0,\infty)$ if there exists a strongly countably additive spectral measure E on the Borel subsets of $[0,\infty)$ so that

$$D(A)=\Big\{x\in X:\lim_{n\to\infty}\int_0^n t\,E(dt)x \text{ exists}\Big\}$$

and

$$Ax=\lim_{n\to\infty}\int_0^n t\,E(dt)x \quad \text{for } x\in D(A).$$

Let us denote by $C_0[0,\infty)$ the space of complex-valued continuous functions on $[0,\infty)$ which vanish at infinity. X always stands for a complex Banach space, X^* for the dual space of X and U_X for the unit ball of X. The space of bounded linear operators on X is denoted by $\mathbf{L}(X)$. The main result of this paper is the following characterization of scalar-type spectral operators on $[0,\infty)$.

THEOREM 1. *Let X be a Banach space which has no subspace isomorphic to the space c_0 of complex-valued null-sequences. Let A be a linear operator*

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on X whose resolvent set contains the half-line $(-\infty,0)$. Then A is spectral of scalar type on $[0,\infty)$ if and only if one of the following conditions is satisfied:

(i) *A is of C^0-scalar type on $[0,\infty)$; that means there exists a continuous algebra homomorphism $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ with $\Phi(\varrho_s) = (s+A)^{-1}$ for $s>0$, where $\varrho_s(t) = 1/(s+t)$.*

(ii) *There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=1,2,\ldots} c_k \int_0^\infty t^{k-1} |x^* A^k (t+A)^{-2k} x| \, dt \le M_1,$$

where $c_k = (2k-1)!/(k!(k-2)!)$ if $k \ge 2$ and $c_1 = 1$.

(iii) *There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=0,1,\ldots} \sum_{m=0}^{k} \binom{k}{m} |x^* A^m (1+A)^{-k} x| \le M_2.$$

(iv) *$-A$ generates a strongly continuous semigroup $(U(t))_{t\ge0}$ with $U(t)X \subseteq D(A)$ for every $t>0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=1,2,\ldots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t) x| \, dt \le M_3.$$

Note that the assumption $U(t)X \subseteq D(A)$ for all $t > 0$ implies $U(t)X \subseteq D(A^n)$ for every natural number n (see e.g. [1], Proposition 1.1.10).

Dowson [10] showed that bounded scalar-type spectral operators in weakly complete Banach spaces can be characterized by criterion (i), and Kantorovitz and deLaubenfels [7] were the first to recognize that this equivalence, for possibly unbounded operators, is true in any Banach space not containing c_0. That (ii) is equivalent to A being of scalar type on $[0,\infty)$, provided the underlying Banach space is reflexive, is due to Kantorovitz [14]. The proof of Kantorovitz' criterion is based on Widder's [18] characterization of Stieltjes transforms. The latter was also used by Ricker [15] to derive a description of scalar-type operators in quasi-complete locally convex spaces. Criterions (iii) and (iv) are related to deLaubenfels' work on scalar-type spectral operators on Banach lattices [2] and on cyclic spaces [3]. The proof of these criterions uses respectively characterizations of complex-valued moment sequences and Laplace transforms. Some other characterizations of possibly unbounded operators on reflexive spaces being of scalar type on $[0,\infty)$ may be found in [3], [6] and [13].

In the first section of this paper we study C_0-scalar-type operators. In particular, it is shown that A is of C_0-scalar type on $[0,\infty)$ if and only if A satisfies condition (ii) or (iii) of Theorem 1, and if, in addition, A generates a strongly continuous semigroup then also condition (iv) is equivalent to A being of C_0-scalar type. In the second section we show that the classes of scalar-type operators on X and of C_0-scalar-type operators on X coincide in exactly those Banach spaces X which have no subspace isomorphic to c_0.

Before we start our study of C_0-scalar-type operators let us present the following results of Widder's which are the main tools for our investigations.

Let k be a natural number. If $f : (0,\infty) \to \mathbb{C}$ is infinitely often differentiable let

$$S_k[f](t) = d_k(-t)^{k-1} \frac{d^{2k-1}}{dt^{2k-1}} (t^k f(t)) \quad \text{for } 0 < t < \infty,$$

where $d_k = 1/(k!(k-2)!)$ if $k \ge 2$ and $d_1 = 1$, and let

$$L_k[f](t) = \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} \quad \text{for } 0 < t < \infty.$$

The formal operators S_k and L_k are called the *Widder inversion operators* for the Stieltjes and Laplace transforms, respectively.

If $\mu = (\mu_n)_{n=0,1,\ldots}$ is a complex-valued sequence then

$$\Lambda_k[\mu]_m = \binom{k}{m} (-1)^{k-m} \Delta^{k-m}[\mu]_m \quad \text{for } k, m = 0, 1, \ldots \text{ with } m \le k,$$

where Δ denotes the difference operator $\Delta[\mu]_n = \mu_{n+1} - \mu_n$.

Let I be a closed interval which is bounded from below by $b = \inf I$. A complex-valued function ϕ of bounded variation on I is said to be *normalized* if $\phi(t) = (\phi(t^-) + \phi(t^+))/2$ for all t in the interior of I and if $\phi(b) = 0$. The total variation of ϕ is denoted by $\mathrm{Var}(\phi)$.

THEOREM 2. *Let $f : (0,\infty) \to \mathbb{C}$ be an infinitely often differentiable function and μ a complex-valued sequence.*

(i) (Widder inversion of the Stieltjes transform) *There exists a unique complex-valued normalized function ϕ of bounded variation on $[0,\infty)$ with $f(s) = \int_0^\infty \frac{1}{s+t} \, d\phi(t)$ for $0 < s < \infty$ if and only if*

$$M = \sup_{k=1,2,\ldots} \int_0^\infty |S_k[f](t)| \, dt < \infty. \tag{1}$$

In this case $M \le \mathrm{Var}(\phi) \le M + |A|$, where $A = \lim_{t\to0^+} tf(t)$.

(ii) (Widder inversion of the Laplace transform) *There exists a unique complex-valued normalized function ϕ of bounded variation on $[0,\infty)$ with $f(s) = \int_0^\infty e^{-st} \, d\phi(t)$ for $0 < s < \infty$ if and only if*

$$M = \sup_{k=1,2,\ldots} \int_0^\infty |L_k[f](t)| \, dt < \infty. \tag{2}$$

In this case $M \le \mathrm{Var}(\phi) \le M + |B|$, where $B = \lim_{t\to\infty} f(t)$.

(iii) (Widder characterization of moment sequences) *There exists a unique complex-valued normalized function ϕ of bounded variation on $[0,1]$ with $\mu_n = \int_0^1 t^n\, d\phi(t)$ for $n = 0, 1, \ldots$ if and only if*

$$M = \sup_{k=0,1,\ldots} \sum_{m=0}^{k} |\Lambda_k[\mu]_m| < \infty.$$

In this case $M \le 2\operatorname{Var}(\phi) \le 2M$.

For proofs of these statements we refer to Widder [18], Theorem VIII.16, Theorem VII.12a, and Theorem III.2b, respectively. Widder stated the results for real-valued functions and sequences, but the complex case is a trivial consequence. The estimates follow easily from the proofs in the cited reference. Note that the existence of the limits A in (i) and B in (ii) follows from the conditions (1) and (2), respectively.

1. Characterization of C_0-scalar-type operators with spectrum in $[0,\infty)$. In the sequel linear mappings between (complex) vector spaces are called operators. Let A be a bounded operator on X and D a bounded set of complex numbers. Following deLaubenfels [2] we say that A is of C^0-*scalar type* on D if there exists a continuous algebra homomorphism Φ defined on the space of continuous complex-valued functions $C(D)$ into $\mathbf{L}(X)$ with $\Phi(\tau_0) = \mathrm{Id}$ and $\Phi(\tau_1) = A$, where $\tau_n(z) = z^n$ for $n = 0, 1, \ldots$ and $z \in D$. We want to extend this definition to possibly unbounded operators with spectrum in the half-line $[0,\infty)$. For this reason we make the following observation.

LEMMA 3. *If A is a bounded operator on X and if $D \subseteq \mathbb{C}$ is closed and bounded then A is of C^0-scalar type on D if and only if $\sigma(A) \subseteq D$ and there exists a continuous algebra homomorphism $\Phi : C(D) \to \mathbf{L}(X)$ with $\Phi(\varrho_z) = (z+A)^{-1}$ for all $z \notin -D$.*

Proof. If A is of C^0-scalar type on D with the corresponding algebra homomorphism Φ then, since D is closed, $\varrho_z \in C(D)$ for all $z \notin -D$, and

$$(z+A)\Phi(\varrho_z) = \Phi((z\tau_0 + \tau_1)\varrho_z) = \Phi(\tau_0) = \mathrm{Id},$$

and similarly $\Phi(\varrho_z)(z+A) = \mathrm{Id}$. Hence $-z$ is contained in the resolvent set of A and $(z+A)^{-1} = \Phi(\varrho_z)$. Conversely, if $\Phi : C(D) \to \mathbf{L}(X)$ is a continuous algebra homomorphism with $\Phi(\varrho_z) = (z+A)^{-1}$ for all $z \notin -D$ then

$$\Phi(\tau_0)(z+A)^{-1} = \Phi(\varrho_z) = (z+A)^{-1},$$

whence $\Phi(\tau_0) = \mathrm{Id}$. Moreover,

$$(z+A)^{-1}(z + \Phi(\tau_1)) = \Phi(\varrho_z(z\tau_0 + \tau_1)) = \Phi(\tau_0) = \mathrm{Id},$$

whence $\Phi(\tau_1) = A$. ∎

Motivated by this lemma we define C^0-scalar-type operators on $[0,\infty)$ as follows:

DEFINITION 4. A possibly unbounded operator A on X is of C^0-*scalar type* on $[0,\infty)$ if $\sigma(A) \subseteq [0,\infty)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ with $\Phi(\varrho_z) = (z+A)^{-1}$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

The next lemma shows that A is of C^0-scalar type on $[0,\infty)$ if and only if A admits a $C_0[0,\infty)$ functional calculus in the sense of deLaubenfels [4]. Moreover, if A is densely defined, then A is of C^0-scalar type in the sense of Definition 4 if and only if A is C^0-scalar in the sense of deLaubenfels [2] (see Remark 7 below).

LEMMA 5. *An operator A on X is of C^0-scalar type on $[0,\infty)$ if and only if its resolvent set contains the half-line $(-\infty, 0)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ with $\Phi(\varrho_s) = (s+A)^{-1}$ for all $0 < s < \infty$.*

Proof. That the first assertion implies the second is trivial. So let $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ be a continuous algebra homomorphism with $\Phi(\varrho_s) = (s+A)^{-1}$ for all $0 < s < \infty$. Let $z \in \mathbb{C} \setminus (-\infty, 0]$. If $x \in D(A)$ then

$$\begin{aligned}(1+A)^{-1}\Phi(\varrho_z)(z+A)x &= \Phi(\varrho_z)(1+A)^{-1}((z-1) + (1+A))x \\ &= \Phi(\varrho_z((z-1)\varrho_1 + 1))x \\ &= \Phi(\varrho_1)x = (1+A)^{-1}x,\end{aligned}$$

whence $\Phi(\varrho_z)(z+A)x = x$. If $x \in X$ then

$$\begin{aligned}\Phi(\varrho_z)x &= \Phi(\varrho_z)((1-z) + (z+A))(1+A)^{-1}x \\ &= (1+A)^{-1}((1-z)\Phi(\varrho_z)x + x) \in D(A).\end{aligned}$$

Applying $z + A = (z-1) + (1+A)$ to both sides of the last equation yields

$$(z+A)\Phi(\varrho_z)x = (z-1)\Phi(\varrho_z) + (1-z)\Phi(\varrho_z) + x = x.$$

Consequently, z is contained in the resolvent set of A and $\Phi(\varrho_z) = (z+A)^{-1}$. ∎

If I is a closed interval with $\inf I = 0$ and if ϕ is a complex-valued function of bounded variation on I, then, for $t \ge 0$, let $\operatorname{Var}_\phi(t)$ be the variation of ϕ in the interval $[0,t)$, provided t is not the right end point of I. If $I = [0,a]$, then $\operatorname{Var}_\phi(a)$ is the variation of ϕ on the whole interval $[0,a]$. The function $\operatorname{Var}_\phi : I \to \mathbb{R}$ is of bounded variation on I, the variations of ϕ and of Var_ϕ are equal, and for every $f \in C_0(I)$,

$$\int_I f(t)\, d\phi(t) \le \int_I |f(t)|\, d\operatorname{Var}_\phi(t).$$

This is proven for example in [17].

THEOREM 6. *Let A be an operator on X whose resolvent set contains the half-line $(-\infty, 0)$. Then A is of C^0-scalar type on $[0,\infty)$ if and only if one of the following conditions* (i) *or* (ii) *is satisfied*:

(i) *There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=1,2,\ldots} c_k \int_0^\infty t^{k-1} |x^* A^k (t+A)^{-2k} x| \, dt \le M_1,$$

where $c_k = (2k-1)!/(k!(k-2)!)$ if $k \ge 2$ and $c_1 = 1$.

(ii) *There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=0,1,\ldots} \sum_{m=0}^{k} \binom{k}{m} |x^* A^m (1+A)^{-k} x| \le M_2.$$

If, in addition, A is densely defined then A is of C^0-scalar type on $[0,\infty)$ if and only if

(iii) *$-A$ generates a C_0-semigroup $(U(t))_{t\ge 0}$ such that $U(t)X \subseteq D(A)$ for every $t > 0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,*

$$\sup_{k=1,2,\ldots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t) x| \, dt \le M_3.$$

Proof. (i) Let A be of C^0-scalar type on $[0,\infty)$ with the corresponding algebra homomorphism Φ. Given $x \in X$, $x^* \in X^*$ and $f \in C_0[0,\infty)$ we put $l[x,x^*]f = x^*\Phi(f)x$. Then $l[x,x^*]$ is a continuous linear functional on $C_0[0,\infty)$ with $\|l[x,x^*]\| \le \|\Phi\| \cdot \|x\| \cdot \|x^*\|$. So we may apply the Riesz representation theorem to find a complex-valued function $\phi[x,x^*]$ of bounded variation on $[0,\infty)$ with $\mathrm{Var}(\phi[x,x^*]) \le M\|x\| \cdot \|x^*\|$, and so that $l[x,x^*](f) = \int_0^1 f(u)\, d\phi[x,x^*](u)$ for all $f \in C_0[0,\infty)$. We then have

$$t^{k-1} x^* A^k (t+A)^{-2k} x = t^{k-1} \int_0^\infty u^k (t+u)^{-2k} \, d\phi[x,x^*](u).$$

Hence, by changing the order of integration and integrating by parts we get

$$\begin{aligned} c_k \int_0^\infty t^{k-1} |x^* A^k (t+A)^{-2k} x| \, dt &\le c_k \int_0^\infty \int_0^\infty t^{k-1} u^k (t+u)^{-2k} \, d\,\mathrm{Var}_{\phi[x,x^*]}(u)\, dt \\ &= c_k \int_0^\infty \int_0^\infty s^{k-1} (s+1)^{-2k} \, ds \, d\,\mathrm{Var}_{\phi[x,x^*]}(u) \\ &= \frac{k-1}{k} \mathrm{Var}(\phi[x,x^*]) \le \|\Phi\| \cdot \|x\| \cdot \|x^*\|. \end{aligned}$$

Conversely, assume that (i) holds. For the proof that A is of C^0-scalar type on $[0,\infty)$, we need Widder's inversion theorem for the Stieltjes transform. Therefore, given $x \in X$ and $x^* \in X^*$ we calculate $S_k[x^*Rx]$ where $R(t) = (t+A)^{-1}$. For every $k = 1,2,\ldots$ we have

$$\begin{aligned} &\frac{d^{2k-1}}{dt^{2k-1}} (t^k (t+A)^{-1}) \\ &= \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{d^l}{dt^l}(t^k) \frac{d^{2k-1-l}}{dt^{2k-1-l}} ((t+A)^{-1}) \\ &= \sum_{l=0}^{k} \binom{2k-1}{l} \frac{k!(2k-1-l)!}{(k-l)!} t^{k-l} (-1)^{2k-1-l} (t+A)^{-(2k-l)} \\ &= (2k-1)!(-1)^{k-1} \sum_{l=0}^{k} \binom{k}{l} (-t)^{k-l} (t+A)^l (t+A)^{-2k} \\ &= (2k-1)!(-1)^{k-1} A^k (t+A)^{-2k}, \end{aligned}$$

whence

$$S_k[x^*Rx](t) = c_k t^{k-1} x^* A^k (t+A)^{-2k} x.$$

Hence by Widder's theorem x^*Rx is the Stieltjes transform of a unique normalized function $\phi[x,x^*]$ of bounded variation on $[0,\infty)$ with $\mathrm{Var}(\phi[x,x^*]) \le M\|x\| \cdot \|x^*\| + |A[x,x^*]|$, where $A[x,x^*] = \lim_{t\to 0^+} tx^*R(t)x$. For every $t > 0$ the operator $tR(t)$ is bounded and

$$|x^* tR(t) x| = \left| \int_0^\infty \frac{t}{t+u} \, d\phi[x,x^*](u) \right| \le \mathrm{Var}(\phi[x,x^*])$$

for every $x \in X$ and $x^* \in X^*$. Hence $\overline{A} = \sup_{0<t<\infty} \|tR(t)\| < \infty$ by the uniform boundedness principle. Consequently,

$$\mathrm{Var}(\phi[x,x^*]) \le \overline{M} \|x\| \cdot \|x^*\|,$$

where $\overline{M} = M + \overline{A}$. Now let $\Phi(f)x$ be the linear functional on X^* which assigns to every $x^* \in X^*$ the complex number $\int_0^\infty f(t)\, d\phi[x,x^*](t)$. Then $\Phi : C_0[0,\infty) \to \mathbf{L}(X, X^{**})$ is linear and we have the estimate

$$|\langle \Phi(f)x, x^* \rangle| \le \|f\| \, \mathrm{Var}(\phi[x,x^*]) \le \overline{M} \|f\| \cdot \|x\| \cdot \|x^*\|.$$

So we infer $\|\Phi\| \le \overline{M}$. Moreover, $\Phi(\varrho_s) = (s+A)^{-1} \in \mathbf{L}(X)$ because

$$\langle \Phi(\varrho_s)x, x^* \rangle = \int_0^\infty \frac{1}{s+t} \, d\phi[x,x^*](t) = x^*(s+A)^{-1}x.$$

Since $\{\varrho_s : 0 < s < \infty\}$ is a total subset of $C_0[0,\infty)$ we infer that $\Phi(f) \in \mathbf{L}(X)$ for every $f \in C_0[0,\infty)$, that is, $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$.

We now claim that Φ is an algebra homomorphism. By the resolvent equality, for all $0 < r, s < \infty$ with $r \neq s$ we have

$$\begin{aligned}\Phi(\varrho_r\varrho_s) &= (r-s)^{-1}\Phi(\varrho_s - \varrho_r)\\ &= (r-s)^{-1}((s+A)^{-1} - (r+A)^{-1}) = (r+A)^{-1}(s+A)^{-1},\end{aligned}$$

and

$$\begin{aligned}\Phi(\varrho_s^2) &= \lim_{h\to 0}\Phi(h^{-1}(\varrho_s - \varrho_{s+h}))\\ &= \lim_{h\to 0} h^{-1}((s+A)^{-1} - (s+h+A)^{-1}) = (s+A)^{-2}.\end{aligned}$$

So we may conclude $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C_0[0,\infty)$ because the set $\{\varrho_s : 0 < s < \infty\}$ is total in $C_0[0,\infty)$.

(ii) Let again Φ be the continuous algebra homomorphism corresponding to A. Given $x \in X$, $x^* \in X^*$ and $f \in C[0,1]$ put $l[x,x^*]f = x^*\Phi(f)x$. Then $l[x,x^*]$ is a continuous linear functional on $C[0,1]$ with $\|l[x,x^*]\| \le \|\Phi\|\cdot\|x\|\cdot\|x^*\|$. So we may apply the Riesz representation theorem to get a complex-valued function $\phi[x,x^*]$ of bounded variation on $[0,1]$ which represents $l[x,x^*]$, and with $\mathrm{Var}(\phi[x,x^*]) = \|l[x,x^*]\|$. Consequently,

$$\begin{aligned}\sum_{m=0}^{k}\binom{k}{m}|x^*A^m(1+A)^{-k}x| &= \sum_{m=0}^{k}\binom{k}{m}\Big|\int_0^\infty t^m(1+t)^{-k}\,d\phi[x,x^*](t)\Big|\\ &\le \int_0^\infty \sum_{m=0}^{k}\binom{k}{m} t^m(1+t)^{-k}\,d\,\mathrm{Var}_{\phi[x,x^*]}(t)\\ &= \mathrm{Var}(\phi[x,x^*]) \le \|\Phi\|\cdot\|x\|\cdot\|x^*\|.\end{aligned}$$

Thus, if A is of C^0-scalar type on $[0,\infty)$, then (ii) follows.

Assume now that (ii) is satisfied. We show first that this assumption implies that $(1+A)^{-1}$ is of C-scalar type on $[0,1]$. To this end put $\mu_n = (1+A)^{-n}$. Then by induction it follows that $\Delta^k\mu_n = (-1)^kA^k(1+A)^{-(k+n)}$ for $k, n = 0, 1, \ldots$, whence

$$\Lambda_k[\mu]_m = \binom{k}{m}A^{k-m}(1+A)^{-k} \quad \text{for } k, m = 0, 1, \ldots \text{ with } k \ge m.$$

Hence (ii) implies, for every $x \in X$ and $x^* \in X^*$,

$$\sup_{k=0,1,\ldots}\sum_{m=0}^{k}|\Lambda_k[x^*\mu x]_m| \le M\|x\|\cdot\|x^*\|.$$

So we may conclude from Widder's Theorem 2(iii) that there exists a unique complex-valued normalized function $\phi[x,x^*]$ of bounded variation on $[0,1]$ with $\mathrm{Var}(\phi[x,x^*]) \le M\|x\|\cdot\|x^*\|$ and so that

$$x^*\mu_n x = \int_0^1 t^n\,d\phi[x,x^*](t).$$

Since

$$\Big|\int_0^1 f(t)\,d\phi[x,x^*](t)\Big| \le \|f\|_\infty \mathrm{Var}(\phi[x,x^*]) \le M\|f\|_\infty\|x\|\cdot\|x^*\|,$$

and since $\phi[x,x^*]$ depends linearly on x and x^*, we can define a continuous linear operator $\Psi : C[0,1] \to \mathbf{L}(X, X^{**})$ by

$$\langle\Psi(f)x, x^*\rangle = \int_0^\infty f(t)\,d\phi[x,x^*](t).$$

We then have, for every $n = 0, 1, \ldots$,

$$\langle\Psi(\tau_n)x, x^*\rangle = \int_0^1 t^n\,d\phi[x,x^*](t) = x^*(1+A)^{-n}x,$$

whence $\Psi(\tau_n) = (1+A)^{-n} \in \mathbf{L}(X)$. Since the set $\{\tau_n : n = 0, 1, \ldots\}$ is total in $C[0,1]$ we conclude that Ψ is an operator from $C[0,1]$ into $\mathbf{L}(X)$. Moreover, for every $l, n = 0, 1, \ldots$,

$$\Psi(\tau_l\tau_n) = \Psi(\tau_{l+n}) = (1+A)^{-(l+n)} = (1+A)^{-l}(1+A)^{-n} = \Psi(\tau_l)\Psi(\tau_n).$$

This equation shows that the bounded operator Ψ is, in addition, an algebra homomorphism, because the set $\{\tau_n : n = 0, 1, \ldots\}$ is total in $C[0,1]$. Note that $\Psi(\tau_0) = (1+A)^{-0} = \mathrm{Id}$ and $\Psi(\tau_1) = (1+A)^{-1}$, so that $(1+A)^{-1}$ is of C^0-scalar type on $[0,1]$.

To show that A is of C^0-scalar type on $[0,\infty)$ put $\Gamma f(t) = f(1/t - 1)$ for $0 < t \le 1$ and $\Gamma f(0) = 0$. Then $\Gamma : C_0[0,\infty) \to C[0,1]$ is a continuous algebra homomorphism. Hence $\Phi = \Psi \circ \Gamma : C_0[0,\infty) \to \mathbf{L}(X)$ is also a continuous algebra homomorphism, and for every $0 < s < \infty$ we have

$$\begin{aligned}\Phi(\varrho_s) &= \Psi(\Gamma(\varrho_s)) = \Psi(\tau_1(1+(s-1)\tau_1)^{-1})\\ &= (1+A)^{-1}(1+(s-1)(1+A)^{-1})^{-1} = (s+A)^{-1}.\end{aligned}$$

It follows, by Lemma 5, that A is of C^0-scalar type on $[0,\infty)$.

(iii) Let A be of C^0-scalar type on $[0,\infty)$ with the corresponding algebra homomorphism Φ and assume A to be densely defined. Since

$$\|\lambda(\lambda+A)^{-1}\| = \|\Phi(\lambda\varrho_\lambda)\| \le \|\Phi\|\cdot\|\lambda\varrho_\lambda\|_\infty \le \|\Phi\|$$

for all $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| \le 3\pi/4$, Theorem 5.3 of [12] tells us that $-A$ generates an analytic semigroup. In particular, $D(A^n) \subseteq U(t)X$ for all $t > 0$ and

$$U^{(n)}(t) = (-1)^nA^nU(t).$$

We denote by $\varepsilon_t \in C_0[0,\infty)$, $t>0$, the function $\varepsilon_t(u)=e^{-ut}$. Put $V(t)=\Phi(\varepsilon_t)$ for $t>0$ and let $V(0)=\mathrm{Id}$. We next show that $U=V$.

Given $x\in X$ and $x^*\in X^*$ let $\phi[x,x^*]$ be the complex-valued normalized function of bounded variation on $[0,\infty)$ which represents the linear functional $x^*\Phi x$ on $C_0[0,\infty)$. Then for $s>0$ we have

$$\begin{aligned}\int_0^\infty e^{-st}x^*V(t)x\,dt &= \int_0^\infty e^{-st}x^*\Phi(\varepsilon_t)x\,dt\\ &= \int_0^\infty e^{-st}\int_0^\infty e^{-tu}\,d\phi[x,x^*](u)\,dt\\ &= \int_0^\infty\int_0^\infty e^{-t(s+u)}\,dt\,d\phi[x,x^*](u)\\ &= \int_0^\infty \frac{1}{s+u}\,d\phi[x,x^*](u)\\ &= x^*\Phi(\varrho_s)x = x^*(s+A)^{-1}x.\end{aligned}$$

Hence the function $s\mapsto(s+A)^{-1}$ is the Laplace transform of V. Since $-A$ generates the C_0-semigroup U we know, by [12], p. 17, that $s\mapsto(s+A)^{-1}$ is also the Laplace transform of U. By the injectivity of the Laplace transform, $V(t)=U(t)$ follows for all $t>0$, since U and V are strongly continuous in $(0,\infty)$. In particular, $U^{(n)}(t)=(-1)^n\Phi(\tau^n\varepsilon_t)$ for all $t>0$.

Now, by Widder's Theorem 2(ii) the inequality

$$\sup_{k=1,2,\ldots}\frac{1}{(k-1)!}\int_0^\infty t^{k-1}|x^*A^kU(t)x|\,dt\le\|\Phi\|\cdot\|x\|\cdot\|x^*\|$$

follows, because $L_k[U](t)=\frac{1}{k!}A^kU(k/t)(k/t)^{k+1}$. We have thus proved that (iii) holds if A is a densely defined operator of C_0-scalar type on $[0,\infty)$.

Conversely, assume that A is densely defined and (iii) holds. Since $T(t)X$ is contained in $D(A)$ for every $t>0$ it follows, by [1], Proposition 1.1.10, that U is infinitely often differentiable in $t>0$, $U(t)X\subseteq D(A^n)$, and $U^{(n)}(t)=(-1)^nA^nU(t)$. Hence condition (iii) is equivalent to

$$\sup_{k=1,2,\ldots}\int_0^\infty|x^*L_k[U](t)x|\,dt\le M\|x\|\cdot\|x^*\|.$$

So Widder's theorem implies that for every $x\in X$ and $x^*\in X^*$ there exists a unique complex-valued normalized function $\phi[x,x^*]$ on $[0,\infty)$ with $\mathrm{Var}(\phi[x,x^*])\le M\|x\|\cdot\|x^*\|+|B[x,x^*]|$, where $B[x,x^*]=\lim_{t\to\infty}x^*U(t)x$, and so that

$$x^*U(t)x=\int_0^\infty e^{-tu}\,d\phi[x,x^*](u)\quad\text{ for all } 0<t<\infty.$$

We deduce from this equation that $\sup_{t>0}|x^*U(t)x|\le\mathrm{Var}(\phi[x,x^*])$. Thus the uniform boundedness principle implies $\overline{B}=\sup_{t>0}\|U(t)\|<\infty$ and we infer that $\mathrm{Var}(\phi[x,x^*])\le\overline{M}\|x\|\cdot\|x^*\|$, where $\overline{M}=M+\overline{B}$. It follows that $\Phi: C_0[0,\infty)\to\mathbf{L}(X,X^{**})$ defined by $\langle\Phi(f)x,x^*\rangle=\int_0^\infty f(t)\,d\phi[x,x^*](t)$ is a continuous linear operator with $\Phi(\varepsilon_t)=U(t)\in\mathbf{L}(X)$. Since the set $\{\varepsilon_t: 0<t<\infty\}$ is total in $C_0[0,\infty)$ we infer $\Phi f\in\mathbf{L}(X)$ for all $f\in C_0[0,\infty)$.

From $\Phi(\varepsilon_s)=U(s)$, and from the fact that U is generated by $-A$, we deduce

$$\begin{aligned}x^*(s+A)^{-1}x&=\int_0^\infty e^{-st}x^*U(t)x\,dt=\int_0^\infty e^{-st}\int_0^\infty e^{-tu}\,d\phi[x,x^*](u)\,dt\\&=\int_0^\infty\frac{1}{s+u}\,d\phi[x,x^*](u)=x^*\Phi(\varrho_s)x.\end{aligned}$$

Hence $\Phi(\varrho_s)=(s+A)^{-1}$. That Φ is an algebra homomorphism follows as in the proof of (i). Consequently, A is of C^0-scalar type on $[0,\infty)$. ■

Remark 7. (i) If A is a bounded operator with spectrum contained in some interval $[m,M]$ then A is of C^0-scalar type on $[m,M]$ if and only $\sup\|p(A)\|<\infty$, where the supremum is taken over all polynomials p on $[m,M]$ with $\sup_{m\le t\le M}|p(t)|\le 1$. This follows immediately from the observation that the polynomials are dense in $C[m,M]$.

Following an idea of Schäfer [16] we can give a second characterization of bounded operators being of C^0-scalar type on $[m,M]$, which uses Widder's characterization of moment sequences. It is clear that A is of C^0-scalar type on $[m,M]$ if and only if $B=(M-m)^{-1}(m-A)$ is of C^0-scalar type on $[0,1]$. Now consider the operator-valued sequence $\mu_n=B^n$, $n=0,1,2,\ldots$ Then $\Lambda_k[\mu]_m=\binom{k}{m}B^m(1-B)^{k-m}$ for $m,k=0,1,2,\ldots$ with $m\le k$. Now applying Widder's characterization of moment sequences it can be seen as in the proof of Theorem 6(ii) that B is of C^0-scalar type on $[0,1]$ if and only if there exists $M>0$ such that

$$\sup_{k=0,1,\ldots}\sum_{m=0}^k\binom{k}{m}|x^*B^m(1-B)^{k-m}x|\le M\|x\|\cdot\|x^*\|\quad\text{ for all } x\in X,\ x^*\in X^*.$$

(ii) deLaubenfels [2] defined an operator A to be C^0-scalar on $[0,\infty)$ if $-A$ generates a uniformly bounded semigroup and $(1+A)^{-1}$ is of C^0-scalar type on $[0,1]$. From the proof of Theorem 6(iii) we see that a densely defined operator A with $(-\infty,0)\subseteq\varrho(A)$ is of C_0-scalar type on $[0,\infty)$ in our sense if and only if A is C^0-scalar on $[0,\infty)$ in the sense of deLaubenfels.

We note that not every operator of C_0-scalar type on $[0,\infty)$ generates a C_0-semigroup. Consider the operator A on $C_0[0,\infty)$ with $D(A) = \{v \in C_0[0,\infty) : \lim_{t\to 0+} v(t)/t \text{ exists}\}$ and $Av(t) = v(t)/t$ for $t > 0$ and $Av(0) = \lim_{t\to 0+} v(t)/t$. Since $(z + A)^{-1}v(t) = (t/(tz+1))v(t)$ we see that every $z \in \mathbb{C} \setminus [0,\infty)$ is contained in the resolvent set of A. Indeed, $\sigma(A) = [0,\infty)$. Moreover, A is of C_0-scalar type on $[0,\infty)$, because the operator $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ defined by $\Phi(f)v(t) = f(1/t)v(t)$ if $t > 0$, and $\Phi(f)v(0) = 0$, is a continuous algebra homomorphism with $\Phi(\varrho_s)v(t) = (t/(ts+1))v(t)$. But A does not generate a C_0-semigroup, because A is not densely defined.

(iii) The proof of Theorem 6 shows that for densely defined operators the condition (iii) in this theorem is equivalent to

(iv) $-A$ generates an analytic semigroup U, and there exists $M_4 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$,

$$\sup_{k=1,2,\ldots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t) x| \, dt \le M_4.$$

2. Characterization of scalar-type operators. In this section we prove the main theorem. The key for the proof is the relationship between vector measures and operators on spaces of continuous functions. For a deeper discussion of this relationship we refer the reader to the monograph of Diestel and Uhl [8]. Here we only recall some basic facts concerning the representation of operators on spaces of continuous functions.

If Σ is a σ-algebra and $F : \Sigma \to X$ has the property

$$F\Big(\bigcup_{k=1}^\infty B_k\Big) = \sum_{k=1}^\infty F(B_k)$$

whenever (B_k) is a sequence of pairwise disjoint members of Σ, then F is called a *countably additive vector measure* on Σ. If Σ is the Borel σ-algebra on a locally compact Hausdorff space then we call F *weakly regular* if $x^* \circ F$ is a regular complex Borel measure for every $x^* \in X^*$.

The following theorem would be a combination of results presented in [8] if we considered operators on $C(K)$, where K is a compact Hausdorff space. But for our investigation of scalar-type operators we have to study operators on $C_0[0,\infty)$.

THEOREM 8. *If $T : C_0[0,\infty) \to X$ is a bounded operator, then T is weakly compact if and only if T can be represented by a unique weakly regular countably additive vector measure F on $[0,\infty)$, that is, $Tf = \int_0^\infty f(t)\,F(dt)$ for $f \in C_0[0,\infty)$. In particular, if X has no subspace isomorphic to c_0, then every bounded operator from $C_0[0,\infty)$ into X can be represented by a unique weakly regular countably additive vector measure F.*

Proof. In this proof we identify the dual space of $C_0[0,\infty)$ with the space of regular Borel measures on $[0,\infty)$ endowed with the variation norm. This is possible by the Riesz representation theorem. Thus every Borel function g on $[0,\infty)$ can be identified with a member in the second dual of $C_0[0,\infty)$ by putting $\langle g,\nu\rangle = \int_0^\infty g(t)\,\nu(dt)$ for all regular Borel measures ν. In particular, this identification is possible for characteristic functions χ_E of Borel sets E.

If $T : C_0[0,\infty) \to X$ is weakly compact then the second dual T^{**} of T takes all its values in X. Given a Borel set B let $F(B) = T^{**}(\chi_B)$. Then for every $x^* \in X^*$ we have

$$x^*F(B) = \langle \chi_B, T^*x^*\rangle = T^*x^*(B).$$

Here we identify $T^*x^* \in C_0[0,\infty)^*$ with a regular Borel measure on $[0,\infty)$. Hence F is weakly regular. Moreover, it follows that F is weakly countably additive, and by the Orlicz–Pettis theorem [8], Corollary I.4.4, we infer that F is countably additive. Since for all $x^* \in X^*$,

$$x^*Tf = T^*x^*(f) = \int_0^\infty f(t)\, x^*F(dt),$$

we see that T is represented by F.

Conversely, if there exists a countably additive vector measure F on $[0,\infty)$ which represents T, then define $T_n f = \int_0^n f(t)\,F(dt)$ for every natural number n. The operators T_n from $C_0[0,\infty)$ into X are weakly compact, by the Bartle–Dunford–Schwartz Theorem [8], Theorem VI.2.5, and they converge in norm to T. Since the operator ideal of weakly compact operators is closed with respect to the operator norm, the weak compactness of T is established.

Now assume that c_0 is not contained in X and let T be a bounded operator from $C_0[0,\infty)$ into X. To prove the second part of the theorem, by the first part it is sufficient to establish the weak compactness of T. We extend T continuously to an operator $T_l : C_l[0,\infty) \to X$ by $T_l(f) = T(f - f_\infty)$, where $f_\infty(t) = \lim_{s\to\infty} f(s)$ for all $0 \le t < \infty$. Define $\Gamma : C[0,1] \to C_l[0,\infty)$ by $\Gamma f(t) = f(1/(1+t))$ and let $S = T_l \circ \Gamma : C[0,1] \to X$. Since c_0 is not contained in X the operator S is weakly compact by [8], Theorem VI.2.15. Hence T_l is also weakly compact because Γ is continuously invertible. Now the weak compactness of T follows from the equality of the ranges of T and T_l. ■

If $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ is a bounded operator, let $\Phi[x]f = \Phi(f)x$ for $x \in X$ and $f \in C_0[0,\infty)$. We call the operators $\Phi[x] : C_0[0,\infty) \to X$ the *components* of Φ.

THEOREM 9. *Let A be a linear operator on X with $(-\infty,0) \subseteq \varrho(A)$. Then A is of scalar type on $[0,\infty)$ if and only if A is of C^0-scalar type on $[0,\infty)$ and the components $\Phi[x]$ of the corresponding algebra homomorphism Φ are weakly compact for every $x \in X$. In particular, if X has no subspace isomorphic to c_0 then A is of scalar type on $[0,\infty)$ if and only if A is of C^0-scalar type on $[0,\infty)$.*

Remark 10. For bounded A, the first half of this theorem is shown in [10], Theorem 4.6.24, and the second half first appeared in [9]. For possibly unbounded A the second half of Theorem 9 is an immediate consequence of [6], Theorem 3.3 and Corollary 3.5.

Proof of Theorem 9. If A is a scalar-type operator with spectral measure E on $[0,\infty)$ then it is well known that $\Phi f = \int_0^\infty f(t)\, E(dt)$ defines a continuous algebra homomorphism $\Phi : C_0[0,\infty) \to \mathbf{L}(X)$ with $\Phi(\varrho_s) = (s+A)^{-1}$. This is proven for example in [11]. Moreover, for every $x \in X$ the X-valued measure $E[x]$ with $E[x](B) = E(B)x$ is countably additive by assumption. Since $\Phi[x]f = \int_0^\infty f(t)\, E[x](dt)$ it follows, by Theorem 8, that $\Phi[x]$ is weakly compact.

Conversely, assume A is of C^0-scalar type on $[0,\infty)$ with the corresponding algebra homomorphism Φ. If Φ has weakly compact components or if X has no subspace isomorphic to c_0 then, by Theorem 8, for every $x \in X$ there exists a countably additive X-valued measure $E[x]$ on $[0,\infty)$ with $\Phi[x]f = \int_0^\infty f(t)\, E[x](dt)$. Given a Borel measurable set B in $[0,\infty)$ we now define $E(B) : X \to X$ by $E(B)x = E[x](B)$. Then $E(B)$ is linear and, by [8], Proposition I.1.11 and Theorem I.1.13,

$$\|E(B)\| = \sup_{\|x\|\le 1} \|E[x](B)\| \le \sup_{\|x\|\le 1} \|\Phi[x]\| \le \|\Phi\|. \tag{3}$$

Hence E is an operator-valued, strongly countably additive bounded measure defined on the Borel subsets of $[0,\infty)$ with

$$(s+A)^{-1}x = \int_0^\infty \frac{1}{s+t} E(dt)x.$$

Moreover, we have $\|t(t+A)^{-1}\| = \|\Phi(t\varrho_t)\| \le \|\Phi\|$.

Now proceed as in Kantorovitz' proof of Theorem 1.1 in [14] to show that E is a spectral measure for A with

$$D(A) = \Big\{x \in X : \lim_{n\to\infty} \int_0^n t\, E(dt)x \text{ exists}\Big\}$$

and $Ax = \lim_{n\to\infty} \int_0^n t\, E(dt)x$ for $x \in D(A)$. ■

The proof of the main theorem now follows from Theorem 6 combined with Theorem 9.

Proof of Theorem 1. Let A be of scalar type on $[0,\infty)$. Then A is of C^0-scalar type on $[0,\infty)$; that is, assertion (i) of Theorem 1 holds. Moreover, $-A$, by deLaubenfels [2], generates a C_0-semigroup, whence A has to be densely defined. Hence the assertions (ii)–(iv) of Theorem 1 follow from Theorems 6 and 9.

If, conversely, one of assertions (i)–(iv) of Theorem 1 holds then, by Theorem 6, A is of C^0-scalar type on $[0,\infty)$. Since X has no subspace isomorphic to c_0, the operator A is of scalar type on $[0,\infty)$, by Theorem 9. ■

We finally note that the condition $c_0 \not\subseteq X$ in Theorem 1 cannot be omitted; more precisely:

THEOREM 11. *If X contains an isomorphic copy of c_0, then there exists a bounded operator A on X which is of C^0-scalar type, but not of scalar type. In particular, this operator A satisfies each of conditions* (i)–(iv) *of Theorem* 1, *but it is not a scalar-type operator.*

The first part of this theorem is proved in [9] (see also [5]) and the second part is an immediate consequence of Theorem 6.

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Purely non-atomic weak L^p spaces

by

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Abstract. Let (Ω, Σ, μ) be a purely non-atomic measure space, and let $1 < p < \infty$. If $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic, as a Banach space, to $L^{p,\infty}(\Omega', \Sigma', \mu')$ for some purely atomic measure space (Ω', Σ', μ'), then there is a measurable partition $\Omega = \Omega_1 \cup \Omega_2$ such that $(\Omega_1, \Sigma \cap \Omega_1, \mu|_{\Sigma\cap\Omega_1})$ is countably generated and σ-finite, and that $\mu(\sigma) = 0$ or ∞ for every measurable $\sigma \subseteq \Omega_2$. In particular, $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to $\ell^{p,\infty}$.

1. Introduction. In [3], the author proved that the spaces $L^{p,\infty}[0,1]$ and $L^{p,\infty}[0,\infty)$ are both isomorphic to the atomic space $\ell^{p,\infty}$. Subsequently, it was observed that if (Ω, Σ, μ) is countably generated and σ-finite, then $L^{p,\infty}(\Omega, \Sigma, \mu)$ is isomorphic to either $\ell^{p,\infty}$ or ℓ^∞ [4, Theorem 7]. In the present paper, we show that the isomorphism of atomic and non-atomic weak L^p spaces does not hold beyond the countably generated, σ-finite situation.

Before giving the precise statement of the main theorem, let us agree on some terminology. Throughout this paper, every measure space under discussion is assumed to be *non-trivial* in the sense that it contains a measurable subset of finite non-zero measure. A measurable subset σ of a measure space (Ω, Σ, μ) is an *atom* if $\mu(\sigma) > 0$, and either $\mu(\sigma') = 0$ or $\mu(\sigma \setminus \sigma') = 0$ for each measurable subset σ' of σ. A *purely non-atomic* measure space is one which contains no atoms. We say that a collection S of measurable sets *generates* a measure space (Ω, Σ, μ) if Σ is the smallest σ-algebra containing S as well as the μ-null sets. A measure space (Ω, Σ, μ) is *purely atomic* if it is generated by the collection of all of its atoms; it is *countably generated* if there is a sequence (σ_n) in Σ which generates (Ω, Σ, μ). For any measure space (Ω, Σ, μ), and $1 < p < \infty$, the *weak* L^p *space* $L^{p,\infty}(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions f such that

$$\|f\| = \sup_{c>0} c(\mu\{|f| > c\})^{1/p} < \infty.$$

It is well known that $\|\cdot\|$ is equivalent to a norm under which $L^{p,\infty}(\Omega, \Sigma, \mu)$ is a Banach space. However, since we are only concerned with isomorphic

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