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Initial value problem for the time dependent Schrödinger equation on the Heisenberg group

by

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Abstract. Let L be the full laplacian on the Heisenberg group $\mathbb{H}^n$ of arbitrary dimension n. Then for $f \in L^2(\mathbb{H}^n)$ such that $(I-L)^{s/2} f \in L^2(\mathbb{H}^n)$, $s > 3/4$, for a $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}tL} f(x)|^2\, dx \le C_\phi \|f\|^2_{W^s}.$$

On the other hand, the above maximal estimate fails for $s < 1/4$. If Δ is the sublaplacian on the Heisenberg group $\mathbb{H}^n$, then for every $s < 1$ there exists a sequence $f_n \in L^2(\mathbb{H}^n)$ and $C_n > 0$ such that $(I-L)^{s/2} f_n \in L^2(\mathbb{H}^n)$ and for a $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}t\Delta} f_n(x)|^2\, dx \ge C_n \|f_n\|^2_{W^s}, \quad \lim_{n\to\infty} C_n = +\infty.$$

Introduction. In his lectures *Some analytic problems related to statistical mechanics* [C] Lennart Carleson observed the following. Let H be a hamiltonian of a quantum system and let V_t be the time dependent Schrödinger group which describes the time evolution of the system $V_t f = e^{\sqrt{-1}tH} f$. Then for a general state $f \in \mathcal{H}$ although $\lim_{t\to 0} \|V_t f - f\|_{\mathcal{H}} = 0$, a better convergence like a.e. may not hold. Indeed, Carleson showed that if $\mathcal{H} = L^2(\mathbb{R})$ and the hamiltonian H is equal to d^2/dx^2, then there exists $f \in W^{1/8}$ for which $V_t f$ does not converge to f a.e. as $t \to 0$. On the other hand, he proved that if f belongs to the Sobolev space $W^{1/4+\varepsilon}$, $\varepsilon > 0$, then $\lim_{t\to 0} V_t f(x) = f(x)$ a.e.

The last theorem attracted a lot of attention. In 1983 Michael Cowling [Cw] put the Carleson theorem in a general framwork.

Let X be a measure space and H a self-adjoint, densely defined operator on $L^2(X)$. We introduce a scale of Sobolev spaces W^s, $s \in \mathbb{R}$, by

$$f \in W^s \quad \text{iff} \quad f \in L^2(X) \text{ and } |H|^{s/2} f \in L^2(X).$$

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THEOREM (M. Cowling, 1983). *Suppose $f \in W^s$ for some $s > 1$. Then*

$$(*) \qquad \lim_{t\to 0} e^{\sqrt{-1}tH} f(x) = f(x) \quad a.e.$$

In this generality one does not expect the theorem to be true for less regular functions. Indeed, as we shall show in this paper, for H being the sublaplacian on the Heisenberg group, Cowling's result is sharp. However, there are a number of results which allow for smaller s's in the case of some specific hamiltonians H.

The case of $X = \mathbb{R}$ and $H = d^2/dx^2$ is completely settled. B. E. J. Dahlberg and C. E. Kenig [DK] showed that given $s < 1/4$, there exists $f \in W^s$ such that $(*)$ fails. On the other hand, C. E. Kenig and A. Ruiz [KR] showed that if $\phi \in C_c(\mathbb{R})$ then

$$\int_{\mathbb{R}} |\phi(x)| \sup_{0<t<1} |e^{\sqrt{-1}tH} f(x)|^2\, dx \le C_\phi \|f\|^2_{W^{1/4}},$$

which implies $(*)$.

But already if $X = \mathbb{R}^d$ and $H = \sum_{k=1}^d \partial^2_{x_k}$ the known results are not as sharp. The best are for $d = 2$. In this case first Per Sjölin [S1] proved $(*)$ for $s = 1/2$, then J. Bourgain [B] found a very small but positive δ such that $(*)$ holds for $s > 1/2 - \delta$. For higher dimension $d \ge 3$ Per Sjölin [S1] proved $(*)$ for $s \ge 1/2 + \varepsilon$.

There seems to be no better estimate from below on s which is necessary for $(*)$ to hold than the one in the case $d = 1$.

There are a number of papers dealing with this circle of ideas: [S2], [S3], [SS], [V]. All of them deal with hamiltonians which are differential or pseudodifferential constant coefficient operators on $\mathbb{R}^d$, the estimates being heavily dependent on analysis of the Fourier transform of $e^{\sqrt{-1}tH} f$. Even though the original lectures of L. Carleson exhibit a physical example where the hamiltonian is d^2/dx^2 on the circle, other fundamental hamiltonians like harmonic oscillator or the laplacian or sublaplacian on the Heisenberg group are not treated. The point is that these operators are *not* translation invariant and the Fourier transform technique is not available. For these cases, however, the Laguerre transform is at hand. First it was applied to questions in harmonic analysis by A. Hulanicki and F. Ricci [HR], and proved to be a very efficient tool (see e.g. D. Müller [M]).

Our results are the following:

Let L be the full laplacian on the Heisenberg group $\mathbb{H}^n$ of arbitrary dimension n. Then for $f \in L^2(\mathbb{H}^n)$ such that $(I-L)^{s/2} f \in L^2(\mathbb{H}^n)$, $s > 3/4$, for any $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}tL} f(x)|^2\, dx \le C_\phi \|f\|^2_{W^s}.$$

On the other hand, the above maximal estimate fails for $s < 1/4$.

If Δ is the sublaplacian on the Heisenberg group $\mathbb{H}^n$, then for every $s < 1$ there exists a sequence $f_n \in L^2(\mathbb{H}^n)$ and $C_n > 0$ such that $(I-L)^{s/2} f_n \in L^2(\mathbb{H}^n)$ and for $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}t\Delta} f_n(x)|^2\, dx \ge C_n \|f_n\|^2_{W^s}, \qquad \lim_{n\to\infty} C_n = +\infty.$$

This seems to be the first example of a hypoelliptic symmetric second order differential operator for which Cowling's estimate is sharp.

Even though our results on the Heisenberg group cannot be transferred by a unitary representation, the methods can. Here we show only a similar result for the twisted laplacian on $\mathbb{C}^n$; the harmonic oscillator and other related operators like $\partial^2/\partial x^2 + x^{2n}\partial^2/\partial y^2$ for $n \ge 2$ will be treated in another paper.

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0. Preliminaries. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and consequently $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Thus for the standard symplectic form on $\mathbb{R}^{2n}$ we write $S(\mathbf{z}, \mathbf{w}) = 2\Im(\mathbf{z}\cdot\overline{\mathbf{w}})$. For $m = 0, 1, 2, \ldots$ let

$$L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{k!}$$

be the Laguerre polynomial of degree m. A function $f \in L^2(\mathbb{C}^n)$ is called *polyradial* if it is invariant under the natural action of $\mathbb{T}^n$ on $\mathbb{C}^n$: for any $\theta = (\theta_1, \ldots, \theta_n)$, $|\theta_i| = 1$,

$$f(\theta_1 z_1, \ldots, \theta_n z_n) = f(z_1, \ldots, z_n).$$

A polyradial function on $\mathbb{C}$ is called simply *radial*.

For every real $a \ne 0$ the *Laguerre functions*

$$l_{m,a}(z) = e^{-|a||z|^2} L_m(2|a|\cdot|z|^2), \qquad m = 0, 1, \ldots,$$

form an orthogonal basis of the subspace $L^2_r(\mathbb{C})$ of radial functions in $L^2(\mathbb{C})$ (cf. [E2]).

Consequently,

$$q_{\mathbf{m},a}(\mathbf{z}) = l_{m_1,a}(z_1)\ldots l_{m_n,a}(z_n), \qquad \mathbf{m} = (m_1, \ldots, m_n),\ \mathbf{z} = (z_1, \ldots, z_n),$$

is an orthonormal basis of the space $L^2_r(\mathbb{C}^n)$ of polyradial functions in $L^2(\mathbb{C}^n)$.

We denote by $d\mathbf{z}$ the Lebesgue measure on $\mathbb{C}^n$ and for $a \neq 0$ we define the *twisted convolution*

$$f \times_a g(\mathbf{z}) = \int f(\mathbf{z}-\mathbf{w})g(\mathbf{w})e^{iaS(\mathbf{z},\mathbf{w})}\,d\mathbf{w}, \quad f,g \in C_c^\infty(\mathbb{C}^n).$$

We have the following orthogonality relation for the Laguerre functions (cf. [M]):

(0.1) $$|a|^n q_{\mathbf{k},a} \times_a q_{\mathbf{m},a}(\mathbf{z}) = \delta_{\mathbf{k},\mathbf{m}} q_{\mathbf{m},a}(\mathbf{z}).$$

Fix a real $a \neq 0$ and let

$$Q_{\mathbf{m},a} f(\mathbf{z}) = |a|^n q_{\mathbf{m},a} \times_a f(\mathbf{z}).$$

Since the functions $q_{\mathbf{m},a}$ decay exponentially at infinity, each $Q_{\mathbf{m},a}$ is bounded on $L^2(\mathbb{C}^n)$. It is not difficult to verify that for a fixed $a \neq 0$,

(0.2) *The operators $Q_{\mathbf{m},a}$ are mutually orthogonal projectors and $\sum_{\mathbf{m}} Q_{\mathbf{m},a} = \mathrm{Id}$.*

Indeed, by (0.1) we have

(0.3) $$Q_{\mathbf{k},a} Q_{\mathbf{m},a} = \delta_{\mathbf{k},\mathbf{m}} Q_{\mathbf{m},a},$$

whence, in particular, $Q_{\mathbf{m},a}^2 = Q_{\mathbf{m},a}$. Moreover,

(0.4) $Q_{\mathbf{m},a}$ is a symmetric operator.

Indeed, we note that the kernel $\mathcal{K}$ of $Q_{\mathbf{m},a}$ is of the form

(0.5) $$\mathcal{K}(\mathbf{z}',\mathbf{z}'') = |a|^n q_{\mathbf{m},a}(\mathbf{z}'-\mathbf{z}'')e^{iaS(\mathbf{z}',\mathbf{z}'')}$$

and is symmetric. To complete the proof of (0.2) we assume that for $g \in L^2(\mathbb{C}^n)$ we have $Q_{\mathbf{m},a} g = 0$ for each $\mathbf{m}$. Since $q_{\mathbf{m},a}$ form a basis of $L_r^2(\mathbb{C}^n)$, we have $h_t \times_a g = 0$ for all polyradial h_t. If h_t is an approximate identity, passing to the limit we obtain $g = 0$.

We introduce a separate notation for the operators $Q_{\mathbf{m},a}$ in the case $\mathbf{m} = m \in \mathbb{N}$, i.e. $n = 1$. We then write

$$Q_{\mathbf{m},a} = P_{m,a} f = |a| l_{m,a} \times_a f.$$

The *Heisenberg group* $\mathbb{H}^n$ is defined as $\mathbb{C}^n \times \mathbb{R}$, with the group product

$$(\mathbf{z},s)(\mathbf{w},t) = (\mathbf{z}+\mathbf{w}, s+t+2\Im(\mathbf{z}\cdot\overline{\mathbf{w}})), \quad \text{where } \mathbf{z} = (z_1,\ldots,z_n),\ z_j = x_j + iy_j.$$

Then the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ is the Haar measure on $\mathbb{H}^n$.

Let K belong to $L^1(\mathbb{H}^n)$ or be a distribution with compact support. Let $F(\mathbf{z},t) = g(\mathbf{z})e^{iat}$, $g \in C_c^\infty(\mathbb{C}^n)$. Then for K and F on $\mathbb{H}^n$ their convolution is equal to

$$K * F(\mathbf{z},t) = \int e^{iat} K^a \times_a g(\mathbf{z})\,da,$$

where

(0.6) $$K^a(\mathbf{z}) = \int e^{-iat} K(\mathbf{z},t)\,dt.$$

Polyradial functions form a commutative subalgebra $\mathcal{A}$ of the group algebra $L^1(\mathbb{H}^n)$ (see [HR]). The Gelfand space of $\mathcal{A}$ is identified with $\mathbb{R}^* \times \mathbb{N}^n \cup (\{0\} \times \mathbb{R}_+^n)$. The Gelfand transform $\mathcal{G}$ of the elements of $\mathcal{A}$ is described in the following way. Let

$$\chi_{a,\mathbf{m}}(z,t) = q_{\mathbf{m},a}(z)e^{iat}.$$

We have

(0.7) $$\mathcal{G}f(a,\mathbf{m}) = \int f(\mathbf{z},t)\chi_{a,\mathbf{m}}(\mathbf{z},t)\,d\mathbf{z}\,dt \quad \text{for } (a,\mathbf{m}) \in \mathbb{R}^* \times \mathbb{N}^n,$$

(0.8) $$\mathcal{G}f(0,\varrho) = \int f(\mathbf{z},t)e^{2i\Re(\mathbf{z}\cdot\overline{\mathbf{w}})}\,d\mathbf{z}\,dt \quad \text{for } \varrho \in \mathbb{R}_+^n,\ \mathbf{w} \in \mathbb{C}^n,\ |w_j| = \varrho_j.$$

Hence, by (0.7) and (0.1), for $f \in \mathcal{A}$ we have

(0.9) $$f * \chi_{\mathbf{m},a} = \mathcal{G}f(\mathbf{m},a)\chi_{\mathbf{m},a}.$$

The theory of representations of the Heisenberg group implies the Fourier inversion formula and the Plancherel formula for $f \in \mathcal{A}$:

(0.10) $$f(\mathbf{z},t) = C\int \sum_{\mathbf{m}} \mathcal{G}f(a,\mathbf{m})\overline{\chi}_{a,\mathbf{m}}(\mathbf{z},t)|a|^n\,da = C \int_{\mathbb{R}^* \times \mathbb{N}^n} \mathcal{G}f(a,\mathbf{m})\overline{\chi}_{a,\mathbf{m}}(\mathbf{z},t)\,d\Lambda(a,\mathbf{m}),$$

(0.11) $$\|f\|_2^2 = C\int \sum_{\mathbf{m}} |\mathcal{G}f(a,\mathbf{m})|^2|a|^n\,da = C \int_{\mathbb{R}^* \times \mathbb{N}^n} |\mathcal{G}f(a,\mathbf{m})|^2\,d\Lambda(a,\mathbf{m}),$$

where the Plancherel measure $d\Lambda(a,\mathbf{m}) = |a|^n\delta_{\mathbf{m}} \times da$ is supported on $\mathbb{R}^* \times \mathbb{N}^n$. Consequently, for every subset $\mathbf{A}$ of $\mathbb{R}^* \times \mathbb{N}^n$ the operator

$$P_{\mathbf{A}} f = \mathcal{G}^{-1}(\mathbf{1}_{\mathbf{A}}\mathcal{G}f)$$

is an orthogonal projection in $L_r^2(\mathbb{H}^n)$ given by convolution with a radial kernel. By a simple use of a radial approximate identity it is immediate to verify that convolution with such a kernel is an orthogonal projection in the whole of $L^2(\mathbb{H}^n)$.

Of course our $\mathbf{A}$ is given by a family $A_{\mathbf{m}}$ of subsets of $\mathbb{R}^*$. Consequently, for $f \in L^2(\mathbb{H}^n)$,

(0.12) $$P_{\mathbf{A}} f(\mathbf{z},u) = \sum_{\mathbf{m}} \int_{A_{\mathbf{m}}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\,da.$$

See [M] and [HR] for the proofs.

Let

$$X_i = \partial_{x_i} + 2y_i\partial_t, \quad Y_i = \partial_{y_i} - 2x_i\partial_t \quad \text{for } 1 \le i \le n,$$

and

$$L=\sum_{i=1}^{n}X_i^2+Y_i^2+T^2,\qquad \Delta=\sum_{i=1}^{n}X_i^2+Y_i^2$$

be the *elliptic* laplacian and the sublaplacian on $\mathbb{H}^n$. The closures of L and Δ on $C_c^\infty(\mathbb{H}^n)$ are selfadjoint operators (see [NS]). Therefore iL and $i\Delta$ generate groups $\{V_t\}_{t\in\mathbb{R}}$ and $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on $L^2(\mathbb{H}^n)$. Since L commutes with the action of $\mathbb{T}^n$ on $\mathbb{H}^n$, by (0.7), as is well known and can be computed directly,

$$L\chi_{\mathbf{m},a}=\lambda_{|\mathbf{m}|}(a)\chi_{\mathbf{m},a}, \tag{0.13}$$

where

$$\lambda_s(a)=|a|(2s+n)+a^2,\quad |\mathbf{m}|=m_1+\ldots+m_n,\quad s\ge 0.$$

Hence, since L commutes with left translations, for every $t\in\mathbb{R}$ and $f\in\mathcal{S}$, for fixed $a\in\mathbb{R}^*$, $N\in\mathbb{N}$, $\phi\in C_c^\infty(R^*)$, for the function g defined by

$$g(\mathbf{z},u)=\sum_{|\mathbf{m}|\le N}\int_{\mathbb{R}^*}\phi(a)e^{iua}Q_{\mathbf{m},a}f^a(\mathbf{z})\,da$$

we have

$$V_tg(\mathbf{z},u)=\sum_{|\mathbf{m}|\le N}\int_{\mathbb{R}^*}\phi(a)e^{iua}e^{it\lambda_{|\mathbf{m}|}(a)}Q_{\mathbf{m},a}f^a(\mathbf{z})\,da.$$

Observe that g is a Schwartz function and the set of such g is dense in $L^2(\mathbb{H}^n)$. Consequently, we can assume that a function $f\in\mathcal{S}$ has the additional property that the set $\{(a,\mathbf{m}):Q_{\mathbf{m},a}f^a$ do not vanish identically$\}$ is a compact subset of $\mathbb{R}^*\times\mathbb{N}$.

The same argument allows us to write the formula for the unitary group generated by $i\Delta$:

$$U_tf(\mathbf{z},u)=\sum_{\mathbf{m}}\int_{\mathbb{R}^*}e^{iua}e^{it|a|(2|\mathbf{m}|+n)}Q_{\mathbf{m},a}f^a(\mathbf{z})\,da.$$

See [M], Lemma 2.2, for the proof.

Let $s\ge 0$. We define a scale of Sobolev spaces putting

$$\|f\|_{W^s}=\|(I-L)^{s/2}f\|_{L^2}.$$

Since $Q_{\mathbf{m},a}$ are mutually orthogonal projectors, by (0.13), (0.2) and the Plancherel theorem applied to the variable u,

$$\|f\|_{W^s}^2=\sum_{\mathbf{m}}\int(1+\lambda_{|\mathbf{m}|}(a))^s\|Q_{\mathbf{m},a}f^a\|_{L^2(\mathbb{C}^n)}^2\,da.$$

1. Lemmas.

First we recall the following classical facts.

LEMMA 1 (Schur). *For a measure space (Ω,μ) let T be an operator on $L^2(\mu)$ given by*

$$Tf(x)=\int K(x,y)f(y)\,d\mu(y).$$

If

$$\sup_y\int|K(x,y)|\,d\mu(x)\le C,\qquad \sup_x\int|K(x,y)|\,d\mu(y)\le C,$$

then $\|T\|\le C$.

Let $0<\alpha<1$. The *fractional derivative* of order α is defined by

$$\partial^\alpha f(s)=\int_{\mathbb{R}}(f(s-t)-f(s))|t|^{-(1+\alpha)}\,dt.$$

LEMMA 2 (Sobolev). *Let $\gamma>0$ be a Schwartz function and $1/2<\alpha<1$. Then*

$$\sup_{-1\le t\le 1}|f(t)|^2\le C_\alpha\Big(\int_{\mathbb{R}}|\partial^\alpha f(t)|^2\gamma(t)\,dt+\int_{\mathbb{R}}|f(t)|^2\gamma(t)\,dt\Big).$$

LEMMA 3 (van der Corput). (i) *Suppose that $F,\phi\in C^1[a,b]$, $\partial\phi$ is monotonic and $|\partial\phi|\ge\lambda$. Then*

$$\Big|\int_a^b e^{i\phi(s)}F(s)\,ds\Big|\le\frac{C}{\lambda}\Big(|F(a)|+\int_a^b|F'(s)|\,ds\Big).$$

(ii) *Let $F\in C^1[a,b]$, $\phi\in C^2[a,b]$, $|\partial^2\phi|\ge\lambda$. Then*

$$\Big|\int_a^b e^{i\phi(s)}F(s)\,ds\Big|\le\frac{C}{\lambda^{1/2}}\Big(|F(a)|+\int_a^b|F'(s)|\,ds\Big).$$

The constant C does not depend on a,b,ϕ,λ and F.

The following is a classical formula (cf. [E2]).

LEMMA 4.

$$L_m(\lambda x)=\sum_{k=0}^{m}\binom{m}{k}L_k(x)\lambda^k(1-\lambda)^{m-k}. \tag{1.1}$$

In the next two lemmas we present estimates for the mth Laguerre function in the region $|z|\le(m+1)^{1/2}$.

LEMMA 5. *Let $1\le|z|\le(m+1)^{1/2}$. Then*

$$|l_{m,1}(z)|\le C(m+1)^{-1/4}|z|^{-1/2}.$$

Proof. Let $0 < \varepsilon \leq \varphi \leq \pi/2 - \varepsilon(m+1)^{-1/2}$. Then by a theorem of Szegő [Sz], for $x = (4m+2)\cos^2\varphi$, we have

$$e^{-x/2}L_m(x) = \{(-1)^m(\pi\sin\varphi)^{-1/2}(\sin((m+1/2)(\sin 2\varphi - 2\varphi) + 3\pi/4)\} \times (x(m+1))^{-1/4} + (x(m+1))^{-1/2}O(1)).$$

LEMMA 6. *Let $|z| \leq 1$. Then*

$$l_{m,1}(z) = J_0(2^{1/2}|z|(m+1/2)^{1/2}) + O((m+1)^{-3/4}),$$

where J_0 is the zero Bessel function.

Proof. Follows from an asymptotic formula for Laguerre polynomials (cf. [Sz]):

$$e^{-x/2}L_m(x) = J_0((2x(m+1/2))^{1/2}) + O((m+1)^{-3/4}).$$

LEMMA 7. *There is a constant C such that for $A \geq 1$ we have*

$$\int |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \leq CA(m+1)^{-1/2}.$$

Proof. By Lemma 5, we obtain

$$\int_{1\leq|z|\leq(m+1)^{1/2}} |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \leq C\int \frac{1}{|z|(m+1)^{1/2}} e^{-|z|^2/A^2}\,dz \leq CA(m+1)^{-1/2}.$$

Also

$$\int_{|z|\geq(m+1)^{1/2}} |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \leq e^{-m/A^2}\int |l_{m,1}(z)|^2\,dz \leq CA(m+1)^{-1/2}.$$

On the other hand, from Lemma 6, using the estimate for the Bessel function (see [Sz])

$$|J_0(x)| \leq C(1+|x|)^{-1/2}$$

we obtain

$$|l_{m,1}(z)| \leq C(1+|z|^{1/2}(m+1)^{1/4})^{-1}.$$

Hence

$$\int_{|z|\leq 1} |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \leq C(m+1)^{-1/2}.$$

For a function ϕ, let M_ϕ denote the operator of multiplication by ϕ. Let $\phi \in C_c^\infty(\mathbb{C})$ with $\operatorname{supp}(\phi) \subset B(1)$, and

$$T_{m,a}f(z) = M_\phi P_{m,a}f(z) = \phi(z)|a|l_{m,a} \times_a f(z).$$

LEMMA 8. *For $4 \leq |a| \leq m+1$ we have*

$$\|T_{m,a}\|^2_{L^2\to L^2} \leq C|a|^{1/2}(m+1)^{-1/2}.$$

Proof. Since $P_{m,a}$ is an orthogonal projector, $T_{m,a}T^*_{m,a} = M_\phi P_{m,a} M_\phi$. Hence, by (0.5), the kernel K of $T_{m,a}T^*_{m,a}$ is given by the formula

$$K(z_1,z_2) = \phi(z_1)|a|l_{m,a}(z_1-z_2)e^{-iaS(z_1,z_2)}\phi(z_2). \tag{1.2}$$

We write

$$1 = e^{-(z_1-z_2)^2}e^{(z_1-z_2)^2} = e^{-(z_1-z_2)^2}\sum_\alpha c_\alpha z_1^{\alpha_1} z_2^{\alpha_2}.$$

Thus

$$K(z_1,z_2) = \sum_\alpha c_\alpha z_1^{\alpha_1}\phi(z_1)e^{-(z_1-z_2)^2}|a|l_{m,a}(z_1-z_2)e^{-iaS(z_1,z_2)}\phi(z_2)z_2^{\alpha_2}.$$

Consequently, the operator $T_{m,a}T^*_{m,a}$ is the sum of operators

$$c_\alpha M_\phi M_{z_1^{\alpha_1}} T_{K_1} M_{z_2^{\alpha_2}} M_\phi,$$

where

$$T_{K_1} = f \times_a K_1 \quad\text{and}\quad K_1(z) = e^{-|z|^2}|a|l_{m,a}(z).$$

Since c_α converges to zero faster than exponentially, it suffices to estimate the norm of T_{K_1}. Dilating we see that the norm of T_{K_1} is the same as the norm of the 1-twisted convolution operator by

$$\mathcal{K}(z) = e^{-|z|^2|a|^{-1}} l_{m,1}(z).$$

The radial function $\mathcal{K}(z)$ has a decomposition

$$\mathcal{K}(z) = \sum_{k=0}^\infty c_{k,m,a} l_{k,1}(z),$$

where

$$c_{k,m,a} = \int e^{-|z|^2|a|^{-1}} l_{k,1}(z) l_{m,1}(z)\,dz. \tag{1.3}$$

So

$$\mathcal{K}(z) \times_1 f(z) = \sum_{k=0}^\infty c_{k,m,a} P_{m,1} f(z).$$

Since $P_{m,1}$, $m = 0,1,\ldots$, are mutually orthogonal projectors, the norm of the operator $f \mapsto \mathcal{K} \times_1 f$ is equal to $\sup_k |c_{k,m,a}|$. By the Schwarz inequality, we obtain

$$|c_{k,m,a}| \leq \|e^{-|z|^2/(2|a|)} l_{k,1}(z)\|_{L^2} \|e^{-|z|^2/(2|a|)} l_{m,1}(z)\|_{L^2}.$$

Now, by Lemma 7, if $10k \geq m$, then

$$|c_{k,m,a}| \leq C\left(\frac{|a|}{m+1}\right)^{1/4}\left(\frac{|a|}{k+1}\right)^{1/4} \leq C\left(\frac{|a|}{m+1}\right)^{1/2}.$$

It remains to estimate the coefficients $c_{k,m,a}$ for $10k \le m$. Observe that by the definition of $l_{m,1}(z)$, for $\lambda = (1 + (2|a|)^{-1})^{-1}$, (1.3) turns into

$$c_{k,m,a} = C \int_0^\infty e^{-\lambda^{-1}x} L_m(x) L_k(x)\, dx.$$

Then

$$c_{k,m,a} = C\lambda \int_0^\infty e^{-x} L_m(x\lambda) L_k(x\lambda)\, dx,$$

whence, in virtue of (1.1), because the L_k form an orthonormal basis with weight e^{-x}, we obtain

$$\begin{aligned} c_{k,m,a} &= C\lambda \sum_{s_1=0}^{m} \sum_{s_2=0}^{k} \binom{m}{s_1} \binom{k}{s_2} \lambda^{s_1+s_2} \\ &\quad \times (1-\lambda)^{m+k-(s_1+s_2)} \int_0^\infty e^{-x} L_{s_1}(x) L_{s_2}(x)\, dx \\ &= C\lambda \sum_{s=0}^{k} \binom{m}{s} \binom{k}{s} \lambda^{2s} (1-\lambda)^{m+k-2s}. \end{aligned}$$

Now, if $|a| \ge 4$ then $2/3 \le \lambda \le 1$ so for $10k \le m$ we have

$$\begin{aligned} |c_{k,m,a}| &\le \sum_{s=0}^{k} 2^m 2^k (1-\lambda)^{m+k-2k} \le k 2^m 2^k 3^{-m-k} \\ &\le k \left(\frac{2}{3}\right)^{m+k} \le 2^{-\varepsilon m} \le 2^{-\varepsilon m} |a| \end{aligned}$$

for some positive constant ε.

LEMMA 9. *For $|a| \le 4$ we have*

$$\|T_{m,a}\|^2 \le C \left(\frac{|a|}{m+1} \right)^{1/2}. \tag{1.4}$$

Proof. In order to estimate the norm of $T_{m,a} T^*_{m,a}$ we use (1.2) and the asymptotic formula for the Laguerre functions given in Lemma 6.

Let $|a| \le 4$. By the Taylor series expansion for $e^{iaS(z_1,z_2)}$ we have

$$\begin{aligned} K(z_1, z_2) &= \sum_{k,l} z_1^k \phi(z_1) |a| l_{m,a}(z_1 - z_2) \phi(z_2) z_2^l a_{k,l} |a|^k \\ &= \sum_{k,l} a_{k,l} |a|^k K_{k,l}(z_1, z_2). \end{aligned}$$

Since the $a_{k,l}$'s decay faster than exponentially, and the norms of the operators $M_\phi M_{z^k}$ grow at most exponentially, it suffices to estimate the norm of the operator K given by the kernel

$$A(z_1, z_2) = \psi(z_1) |a| l_{m,a}(z_1 - z_2) \psi(z_2),$$

where $\psi \in C_c^\infty$ with $\psi(z) = 1$ on $\operatorname{supp} \phi$. Now using Lemma 6 we obtain

$$\begin{aligned} \psi(z_1)|a| l_{m,a}(z_1 - z_2)\psi(z_2) &= C\psi(z_1)|a| J_0(2|a|^{1/2}|z_1 - z_2|(2m+1)^{1/2})\psi(z_2) \\ &\quad + \psi(z_1)\psi(z_2) O(|a|(m+1)^{-3/4}). \end{aligned}$$

Observe that the error term in the last formula gives an operator of norm of order $|a|(m+1)^{-3/4}$, so it is negligible.

Hence, for a function $\widetilde{\phi} \in S(\mathbb{C})$ with $\widetilde{\phi} = 1$ on $\operatorname{supp} \psi$ we write

$$\begin{aligned} &\psi(z_1)|a| J_0(|a|^{1/2}|z_1 - z_2|(2m+1)^{1/2})\psi(z_2) \\ &\qquad = \widetilde{\phi}(z_1 - z_2)\psi(z_1)|a| J_0(|a|^{1/2}|z_1 - z_2|(2m+1)^{1/2})\psi(z_2). \end{aligned}$$

Thus we may drop $\psi(z_1), \psi(z_2)$ and we estimate the norm of the convolution operator by the function

$$R = \widetilde{\phi}(z)|a| J_0(|a|^{1/2}|z|(2m+1)^{1/2}).$$

By definition, J_0 is the Fourier transform of the normalized Lebesgue measure supported on the unit circle. Hence

$$\widehat{R} = \widehat{\widetilde{\phi}} * |a|\mu,$$

where μ is the normalized Lebesgue measure supported by the circle of radius $|a(2m+1)|^{1/2}$. We write (using a smooth resolution of identity $1 = \sum_{j \in \mathbb{Z}^2} k(z-j)$, $\operatorname{supp} k \subset B(2)$)

$$\widehat{\widetilde{\phi}} = \sum_j \alpha_j \phi_j,$$

where $\sum_j |\alpha_j| < \infty$, $\|\phi_j\|_{L^\infty} \le 1$ and the support of ϕ_j is contained in the disc of radius two. A trivial geometric argument shows that for $|(2m+1)a| \ge 1$, $\|\phi_j * \mu\|_{L^\infty} \le C|(2m+1)a|^{-1/2}$. These imply that the L^∞ norm of $\widehat{R}$ is bounded by $C|a|^{1/2}|m+1|^{-1/2}$. If $|(2m+1)a| \le 1$ then $\|\phi_j * \mu\|_{L^\infty} \le C$ and consequently $\|\widehat{R}\|_{L^\infty} \le C|a| \le C|m+1|^{-1/2}|a|^{1/2}$. This proves the lemma.

Let

$$K_{M,N}(u,\tau) = \sum_{s=M}^{3M} \int_{N \le |a| \le 2N} e^{iua + i\tau\lambda_s(a)}\, da,$$

where $\lambda_s(a) = (2s+n)|a| + a^2$.

LEMMA 10. *Let $|u| \le 1/8$ and $|\tau| \le 1$. Then*

$$|K_{M,N}(u,\tau)| \le C\min\{MN, (M+N)^{3/2}N^{-1}|u|^{-3/2}, M(M+N)^{1/2}|u|^{-1/2}\} + \min\{M|u|^{-1}, MN\},$$

Proof. The estimate $|K_{M,N}(u,\tau)| \le CMN$ is trivial. It suffices to consider only the integral over the set $N \le a \le 2N$. We split the argument into cases:

(a) $2|u| \le \tau(M+N)$,
(b) $8\tau(M+N+n) \le |u|$,
(c) $|u| \le 8\tau(M+N+n)$ and $\tau(M+N) \le 2|u|$.

For (a) and (b) we write $\phi_s(a) = ua + ((2s+n)a + a^2)\tau$ and see that in either case $|\partial_a\phi_s| \ge \frac{1}{2}|u|$ so by the van der Corput lemma, Lemma 3(i),

$$\Big|\int_N^{2N} e^{i\phi_s(a)}\,da\Big| \le \frac{C}{|u|}.$$

Hence

$$|K_{M,N}(u,\tau)| \le \sum_{s=M}^{3M}\Big|\int_N^{2N} e^{i\phi_s(a)}\,da\Big| \le \frac{CM}{|u|}.$$

In case (c) we have

$$K_{M,N}(u,\tau) = \int_N^{2N} e^{iua+i\tau a^2+i(4M+n)a\tau}\frac{\sin\left((M+\frac{1}{2})a\tau\right)}{\sin\left(\frac{1}{2}a\tau\right)}\,da$$
$$= (2i)^{-1}\int_N^{2N} e^{iua+i\tau a^2+(5M+1/2+n)a\tau}\frac{da}{\sin\left(\frac{1}{2}a\tau\right)} - (2i)^{-1}\int_N^{2N} e^{iua+i\tau a^2+(3M-1/2+n)a\tau}\frac{da}{\sin\left(\frac{1}{2}a\tau\right)}.$$

Let $\phi_+(a) = ua+\tau a^2+(5M+1/2+n)a\tau$, $\phi_-(a) = ua+\tau a^2+(3M-1/2+n)a\tau$. Then $\partial_a^2\phi_+(a) = \partial_a^2\phi_-(a) = 2\tau$. For $a \in [N,2N]$ and τ satisfying (c) the function $a \to 1/\sin\left(\frac{1}{2}a\tau\right)$ is monotonic. From the van der Corput lemma, Lemma 3(ii), we obtain

$$\int_N^{2N} e^{i\phi_+(a)}\frac{da}{\sin\left(\frac{1}{2}a\tau\right)} \le \frac{C}{\tau^{1/2}}\left(\frac{1}{\sin\left(\frac{1}{2}N\tau\right)} + \int_N^{2N}\left|\partial_a\left(\frac{1}{\sin\left(\frac{1}{2}a\tau\right)}\right)\right|da\right)$$
$$\le \frac{C}{\tau^{1/2}}\frac{1}{\sin\left(\frac{1}{2}N\tau\right)} \le C(M+N)^{3/2}N^{-1}|u|^{-3/2}.$$

An analogous estimate holds for $\phi_-(a)$. Also we have, by Lemma 3(ii),

$$\sum_{s=M}^{3M}\Big|\int_N^{2N} e^{iua+i\tau a^2+i\tau(2s+1)a}\,da\Big| \le \frac{CM}{\tau^{1/2}} \le CM(M+N)^{1/2}|u|^{-1/2}$$

and all these prove the lemma.

LEMMA 11. *Let $\phi \ge 0$, $\phi \in C_c^\infty(\mathbb{R})$. Define a maximal function*

$$V : L^2(\mathbb{R}, l^2) \ni f \mapsto Vf \in L^2(\mathbb{R})$$

by

$$Vf(u) = \sup_{0\le t\le 1}\phi(u)\Big|\sum_{s=M}^{3M}\int_N^{2N} e^{i(ua+t\lambda_s(a))}f_s(a)\,da\Big|.$$

Then

$$\|V\|_{L^2(\mathbb{R},l^2)\to L^2(\mathbb{R})} \le C((M+N)(M/N)^{1/2} + M\log N)^{1/2}.$$

Proof. It suffices to prove the estimate for functions ϕ supported in the interval $[-1/8, 1/8]$. We use the Kolmogorov–Plessner–Silvestroff method. Let $\tau(u)$ be a continuous function. We are going to estimate the norm of the linear operator $T : L^2(\mathbb{R}, l^2) \to L^2(\mathbb{R})$,

$$Tf(u) = \phi(u)\sum_{s=M}^{3M}\int_N^{2N} e^{i(ua+\tau(u)\lambda_s(a))}f_s(a)\,da.$$

The operator TT^* acts on L^2 and has the kernel

$$TT^*(u_1,u_2) = \phi(u_1)\phi(u_2)\sum_{s=M}^{3M}\int_N^{2N} e^{i(u_1-u_2)a+i(\tau(u_1)-\tau(u_2))\lambda_s(a)}\,da$$
$$= \phi(u_1)\phi(u_2)K_{M,N}(u_1-u_2, \tau(u_1)-\tau(u_2)).$$

By the Schur Lemma,

$$\|TT^*\| \le \sup_{u_1}\phi(u_1)\int|K_{M,N}(u_1-u_2,\tau(u_1)-\tau(u_2))|\phi(u_2)\,du_2$$
$$\le C\int(\min\{MN, (M+N)^{3/2}N^{-1}|u|^{-3/2}, M(M+N)^{1/2}|u|^{-1/2}\} + \min\{M|u|^{-1}, MN\})\phi(u)\,du$$
$$\le C(M+N)(M/N)^{1/2} + M\log N.$$

Hence

$$\|T\| \le C((M+N)(M/N)^{1/2} + M\log N)^{1/2}.$$

2. Main theorem. For a fixed $\phi \in C_c^\infty(\mathbb{H}^n)$ we define the local maximal function of the group V_t:

$$Mf(\mathbf{z},u) = \phi(\mathbf{z},u)\sup_{0\le t\le 1}|V_tf(\mathbf{z},u)|.$$

We have

THEOREM 1. *Let $s > 3/4$ and $f \in W^s$. Then*

$$\|Mf\|_{L^2} \le C\|f\|_{W^s}.$$

Proof. Let $f \in L^2(\mathbb{H}^n)$. To estimate $\|Mf\|_{L^2(\mathbb{H}^n)}$ we introduce a family of projections $P_{\mathbf{A}_\alpha}$ given by a partition $\{\mathbf{A}_\alpha\}$ of $\mathbb{R}^* \times \mathbb{N}^n$. Then we write

$$\|Mf\|_{L^2(\mathbb{H}^n)} \le \sum_\alpha \|MP_{\mathbf{A}_\alpha} f\|_{L^2(\mathbb{H}^n)}$$

and we estimate each $\|MP_{\mathbf{A}_\alpha} f\|_{L^2(\mathbb{H}^n)}$ separately.

We write $s \approx 2^k$ iff $2^k \le s < 2^{k+1}$. Let

$$\mathbf{A}_{k,l} = \{(a,\mathbf{m}) \in \mathbb{R}^* \times \mathbb{N}^n : |\mathbf{m}| \approx 2^k,\ |a| \approx 2^l\}, \qquad k \le 3l,$$
$$\mathbf{A}_0 = \{(a,\mathbf{m}) \in \mathbb{R}^* \times \mathbb{N}^n : |a| < 1\},$$
$$\mathbf{A}^c = \text{the complement of } \bigcup_{k,l} \mathbf{A}_{k,l} \cup \mathbf{A}_0.$$

We observe that

$$\text{(2.1)} \qquad \mathbf{A}^c = \bigcup_{\mathbf{m}} A_{\mathbf{m}} \times \{\mathbf{m}\} \quad \text{with } A_{\mathbf{m}} \subset \{a \in \mathbb{R}^* : 1 \le |a| \le 10|\mathbf{m}|^{1/3}\}.$$

We write

$$P_{k,l}f(\mathbf{z},u) = P_{\mathbf{A}_{k,l}}f(\mathbf{z},u) = \sum_{\{\mathbf{m}:|\mathbf{m}|\approx 2^k\}} \int_{\{|a|\approx 2^l\}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\,da,$$
$$P_0 f(\mathbf{z},u) = P_{\mathbf{A}_0} f(\mathbf{z},u) = \sum_{\mathbf{m}} \int_{\{|a|\le 1\}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\,da,$$
$$P_1 f(\mathbf{z},u) = P_{\mathbf{A}^c} f(\mathbf{z},u) = \sum_{\mathbf{m}} \int_{A_{\mathbf{m}}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\,da,$$

and we note that

$$P_0 + P_1 + \sum_{k,l} P_{k,l} = \mathrm{Id}.$$

The maximal function of the theorem splits into the maximal functions

$$S_{k,l}f(\mathbf{z},u) = \sup_{0\le t\le 1} |\psi(\mathbf{z})\phi(u) P_{k,l} V_t f(\mathbf{z},u)|,$$
$$\psi \in C_c^\infty(\mathbb{C}^n),\ \phi \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}\phi \subset B(1),$$
$$S_0 f(\mathbf{z},u) = \sup_{0\le t\le 1} |\psi(\mathbf{z}) P_0 V_t f(\mathbf{z},u)|, \quad \psi \in C_c^\infty(\mathbb{C}^n),$$
$$S_1 f(\mathbf{z},u) = \sup_{0\le t\le 1} |\psi(\mathbf{z}) P_1 V_t f(\mathbf{z},u)|, \quad \psi \in C_c^\infty(\mathbb{C}^n).$$

We are going to estimate the norms

$$\|S_{k,l}\|_{W^{3/4+\varepsilon}\to L^2}, \quad \|S_0\|_{W^{3/4+\varepsilon}\to L^2} \quad \text{and} \quad \|S_1\|_{W^{3/4+\varepsilon}\to L^2}.$$

Then we sum up the estimates. With no loss of generality we may consider only the $\mathbf{m}$'s in $I_1 = \{\mathbf{m} : m_1 = \max(m_1,\dots,m_n)\}$. We have

$$\begin{aligned} S_{k,l}f(\mathbf{z},u) &= \sup_{0\le t\le 1}\Big|\psi(\mathbf{z})\phi(u) \sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|\approx 2^k\}} \int_{\{|a|\approx 2^l\}} e^{iua} e^{it\lambda_{|\mathbf{m}|}(a)} Q_{\mathbf{m},a} f^a(\mathbf{z})\,da\Big| \\ &= \sup_{0\le t\le 1}\Big|\psi(\mathbf{z})\phi(u) \sum_{s=2^k}^{2^{k+1}} \int_{\{|a|\approx 2^l\}} e^{iua+it\lambda_s(a)} \Big(\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a} f^a(\mathbf{z})\Big)\,da\Big|. \end{aligned}$$

Now we apply Lemma 11 with $M = 2^k$, $N = 2^l$, and

$$f_s(a) = \sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a} f^a(\mathbf{z}) \quad \text{for } M \le s < 2M.$$

We have

$$\int_{-1}^{1} |S_{k,l}f(\mathbf{z},u)|^2\,du \le C((M+N)(M/N)^{1/2} + M\log N) \times \sum_{s=2^k}^{2^{k+1}} \int_{\{|a|\approx 2^l\}} \Big|\Big(\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a} f^a(\mathbf{z})\Big)\Big|^2\,da\,\Psi(z_1),$$

where $\Psi \in C_c^\infty(\mathbb{C})$ depends only on the first coordinate of $\mathbf{z} = (z_1,\dots z_n)$ and $\psi(\mathbf{z})^2 \le \Psi(z_1)$. Integrating with respect to $d\mathbf{z}$ we obtain

$$\begin{aligned} \iint |S_{k,l}f(\mathbf{z},u)|^2\,du\,d\mathbf{z} &\le C((M+N)(M/N)^{1/2} + M\log N) \\ &\quad \times \sum_{s=2^k}^{2^{k+1}} \int_{\{|a|\approx 2^l\}} \int \Big|\Big(\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a} f^a(\mathbf{z})\Big)\Big|^2 \Psi(z_1)\,d\mathbf{z}\,da. \end{aligned}$$

Let $A = \{(\mathbf{m},\mathbf{r}) : m_2 = r_2,\dots,m_n = r_n,\ \mathbf{m},\mathbf{r} \in I_1\}$. We fix a and we note that $|\mathbf{m}| = |\mathbf{r}|$ and $(\mathbf{m},\mathbf{r}) \in A$ imply $\mathbf{m} = \mathbf{r}$. By the orthogonality relations for $P_{m,a}$ we have

$$\begin{aligned} &\int Q_{\mathbf{m},a} f(\mathbf{z}) \overline{Q_{\mathbf{r},a} f(\mathbf{z})}\,dz_2\dots dz_n \\ &\quad = \int P_{m_1,a}P_{m_2,a}\dots P_{m_n,a} f(\mathbf{z}) \overline{P_{r_1,a}P_{r_2,a}\dots P_{r_n,a} f(\mathbf{z})}\,dz_2\dots dz_n = 0 \end{aligned}$$

if $(m_2,\dots m_n) \ne (r_2,\dots,r_n)$. In the formula above $P_{m_i,a}$ acts on the variable z_i.

We write

$$\int\Big|\Big(\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a}f(\mathbf{z})\Big)\Big|^2\Psi(z_1)\,d\mathbf{z}$$
$$=\int\Big(\sum_{\{\mathbf{r}\in I_1:|\mathbf{r}|=s\}}\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}} Q_{\mathbf{m},a}f(\mathbf{z})\overline{Q_{\mathbf{r},a}f(\mathbf{z})}\Big)\,dz_2\ldots dz_n\,\Psi(z_1)\,dz_1$$
$$=\int\Big(\sum_{\{(\mathbf{m},\mathbf{r})\in A:|\mathbf{m}|=|\mathbf{r}|=s,\mathbf{m}\in I_1,\mathbf{r}\in I_1\}} Q_{\mathbf{m},a}f(\mathbf{z})\overline{Q_{\mathbf{r},a}f(\mathbf{z})}\Big)\,dz_2\ldots dz_n\,\Psi(z_1)\,dz_1$$
$$=\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\int|Q_{\mathbf{m},a}f(\mathbf{z})|^2\,dz_2\ldots dz_n\,\Psi(z_1)\,dz_1.$$

Now again we write $Q_{\mathbf{m},a}f=P_{m_1,a}(P_{m_2,a}\ldots P_{m_n,a})$ and we apply Lemma 8 to the operator $M_\Psi P_{m_1,a}$ to obtain

$$\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\int|Q_{\mathbf{m},a}f(\mathbf{z})|^2\,dz_2\ldots dz_n\,\Psi(z_1)\,dz_1$$
$$\le\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\int\min\bigg\{\Big(\frac{|a|}{m_1+1}\Big)^{1/2},1\bigg\}|Q_{\mathbf{m},a}f(\mathbf{z})|^2\,d\mathbf{z}.$$

Consequently,

$$\iint|S_{k,l}f(\mathbf{z},u)|^2\,du\,d\mathbf{z}\tag{2.2}$$
$$\le C((M+N)(M/N)^{1/2}+M\log N)$$
$$\times\sum_{s=2^k}^{2^{k+1}}\int_{\{|a|\approx 2^l\}}\int\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\min\bigg\{\Big(\frac{|a|}{m_1+1}\Big)^{1/2},1\bigg\}$$
$$\times|Q_{\mathbf{m},a}f^a(\mathbf{z})|^2\,d\mathbf{z}\,da.$$

Let $1\le k\le l$. Then for $N\le|a|\le 2N$ and $M\le|\mathbf{m}|\le 2M$ we can easily verify that

$$(M+N)(M/N)^{1/2}+M\log N\le C(MN)^{1/2}\log N\le C2^{-\varepsilon l}\lambda_{|\mathbf{m}|}(a)^{1/2+\varepsilon}$$

so by (2.2) and (2.1),

$$\|S_{k,l}f\|_{L^2}^2\le C2^{-\varepsilon l}\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|\approx 2^k\}}\int_{\{|a|\approx 2^l\}}\lambda_{|\mathbf{m}|}(a)^{1/2+\varepsilon}\|Q_{\mathbf{m},a}f^a\|^2\,da$$
$$\le C2^{-\varepsilon l}\|f\|^2_{W^{3/4+\varepsilon}}.$$

Let $\frac{1}{3}k\le l<k$. Then $N<M$ and for a and $\mathbf{m}$ as before we have

$$((M+N)(M/N)^{1/2}+M\log N)(N/M)^{1/2}\le CM\log N\le C2^{-\varepsilon k}\lambda_{|\mathbf{m}|}(a)^{3/4+3\varepsilon}.$$

So since $|a|/m_1\le 2nN/M$,

$$\|S_{k,l}f\|_{L^2}^2\le C2^{-\varepsilon k}\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|\approx 2^k\}}\int_{\{|a|\approx 2^l\}}\lambda_{|\mathbf{m}|}(a)^{3/4+2\varepsilon}\|Q_{\mathbf{m},a}f^a\|^2\,da$$
$$\le C2^{-\varepsilon l}\|f\|^2_{W^{3/4+2\varepsilon}}.$$

Consequently, we have

$$\sum_{k\ge 0}\sum_{l\ge(1/3)k}\|S_{k,l}\|_{W^{3/4+\varepsilon}\to L^2}<\infty.$$

Now we are going to estimate the norm of S_0. We use the Sobolev lemma. We have

$$|S_0f(\mathbf{z},u)|^2\le C\Big(\int_{\mathbb{R}}|\partial_t^{1/2+\varepsilon}V_tP_0f(\mathbf{z},u)|^2\gamma(t)\,dt$$
$$+\int_{\mathbb{R}}|V_tP_0f(\mathbf{z},u)|^2\gamma(t)\,dt\Big)\psi(\mathbf{z}).$$

In what follows we assume that $\widehat{\gamma}$ is supported in the interval $[-1,1]$. Integrating with respect to $d\mathbf{z}du$, by the Plancherel theorem applied to the Fourier transform in the central variable, we have

$$\int|S_0f(\mathbf{z},u)|^2\,d\mathbf{z}\,du$$
$$\le\int|\partial_t^{(\varepsilon+1)/2}P_0V_tf(\mathbf{z},u)|^2\psi(\mathbf{z})\gamma(t)\,d\mathbf{z}\,du\,dt$$
$$=C\iiint\Big|\sum_m I_{\{0\le|a|\le 1\}}(a)Q_{\mathbf{m},a}f^a(\mathbf{z})\partial_t^{(1+\varepsilon)/2}e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2\,da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z}$$
$$\le C\iiint\Big|\sum_{\mathbf{m}} I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a)\lambda_{\mathbf{m}}(a)^{(1+\varepsilon)/2}$$
$$\times Q_{\mathbf{m},a}f^a(\mathbf{z})e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2\,da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z}+\|f\|^2_{L^2}.$$

In the last inequality we have used the fact that for $|a|\le C|\mathbf{m}|^{-1}$, we have $\lambda_{|\mathbf{m}|}(a)\le C$.

Without loss of generality we may assume that in the above sum the multiindices $\mathbf{m}$ belong to I_1. Thus

$$\iiint\Big|\sum_{\mathbf{m}} I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a)\lambda_{|\mathbf{m}|}(a)^{(1+\varepsilon)/2}$$
$$\times Q_{\mathbf{m},a}f^a(\mathbf{z})e^{i\lambda_{\mathbf{m}}(a)t}\Big|^2\,da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z}$$
$$=\iint\sum_{\mathbf{m}\in I_1}\sum_{\mathbf{r}\in I_1}I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a)I_{\{C/|\mathbf{r}|\le|a|\le 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times Q_{\mathbf{m},a}f^a(\mathbf{z})\overline{Q_{\mathbf{r},a}f^a(\mathbf{z})}\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a)-\lambda_{|\mathbf{r}|}(a))\,da\,\psi(z_1)\,d\mathbf{z}.$$

By orthogonality of $P_{m,a}$ the last expression is equal to

$$\int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times\int Q_{\mathbf{m},a}f^a(\mathbf{z})\overline{Q_{\mathbf{r},a}f^a(\mathbf{z})}\psi(z_1)\,d\mathbf{z}\,\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a)-\lambda_{|\mathbf{r}|}(a))\,da$$
$$=\int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times\int Q_{\mathbf{m},a}f^a(\mathbf{z})\overline{Q_{\mathbf{r},a}f^a(\mathbf{z})}\psi(z_1)\,d\mathbf{z}\,\widehat{\gamma}(2m_1|a|-2r_1|a|)\,da$$
$$\le\int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times\left(\frac{|a|}{|\mathbf{m}|}\cdot\frac{|a|}{|\mathbf{r}|}\right)^{1/4}\|Q_{\mathbf{m},a}f^a\|\cdot\|Q_{\mathbf{r},a}f^a\|\widehat{\gamma}(2(m_1-r_1)|a|)\,da.$$

To verify the last inequality we use Lemma 9. The Schwarz inequality implies that the last expression is bounded by

$$S=\int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times((|a|/|\mathbf{m}|)^{1/2}\|Q_{\mathbf{m},a}f^a\|^2+(|a|/|\mathbf{r}|)^{1/2}\|Q_{\mathbf{r},a}f^a\|^2)$$
$$\times\widehat{\gamma}(2(m_1-r_1)|a|)\,da.$$

For fixed $\mathbf{r}$ we have

$$\sum_{\{\mathbf{m}:(\mathbf{m},\mathbf{r})\in A\}} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times(|a|/|\mathbf{m}|)^{1/2}\widehat{\gamma}((m_1-r_1)|a|)\le C\lambda_{|\mathbf{r}|}(a)^{1/2+\varepsilon}. \tag{2.3}$$

In order to verify (2.3) we observe that for $\mathbf{m}$, $\mathbf{r}$, and a we can write

$$c\lambda_{|\mathbf{m}|}(a)\le|\mathbf{m}|\cdot|a|\le C\lambda_{|\mathbf{m}|}(a),$$
$$c\lambda_{|\mathbf{r}|}(a)\le|\mathbf{r}|\cdot|a|\le C\lambda_{|\mathbf{r}|}(a),\qquad c|\mathbf{m}|\le|\mathbf{r}|\le C|\mathbf{m}|,$$

and also

$$\sharp\{\mathbf{m}:(\mathbf{m},\mathbf{r})\in A,\ |r_1-m_1|\cdot|a|\in\operatorname{supp}\widetilde{\phi}\}\le C/|a|.$$

Now (2.3) follows by an easy calculation.

By (2.3), S is dominated by

$$2\int\sum_{\mathbf{r}} I_{\{C\mathbf{m}^{-1}\le|a|\le1\}}(a)\lambda_{|\mathbf{r}|}(a)^{3/4+\varepsilon}\|Q_{\mathbf{r},a}f^a\|^2\,da\le\|f\|^2_{W_{3/4+\varepsilon}}.$$

In order to estimate S_1f we use again the Sobolev lemma. By the Plancherel

formula applied to the central variable, we have

$$\int|S_1f(\mathbf{z},u)|^2\,d\mathbf{z}\,du\le\iint|\partial_t^{(\varepsilon+1)/2}\phi(u)P_1V_tf(\mathbf{z},u)|^2\,d\mathbf{z}\,du\,\gamma(t)\,dt+\|f\|^2_{L^2}.$$

Again we consider only $\mathbf{m}$'s in I_1. Also since $\operatorname{supp}\widehat{\gamma}\subset\{u:|u|\le1\}$ and $|a|\ge1$ we have $\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a)-\lambda_{|\mathbf{r}|}(a))=0$ for $|\mathbf{m}|\ne|\mathbf{r}|$. Using (as before) orthogonality of $P_{m,a}$ we have

$$\iiint\Big|\sum_{\mathbf{m}}I_{A_{\mathbf{m}}}(a)\lambda_{|\mathbf{m}|}(a)^{(1+\varepsilon)/2}Q_{\mathbf{m},a}f^a(\mathbf{z})e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2\,da\,dt\,\psi(\mathbf{z})\,d\mathbf{z}$$
$$=\iint\sum_{\mathbf{m}}\sum_{\mathbf{r}}I_{A_{\mathbf{m}}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{(1+\varepsilon)/2}$$
$$\times Q_{\mathbf{m},a}f^a(\mathbf{z})\overline{Q_{\mathbf{r},a}f^a(\mathbf{z})}\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a)-\lambda_{|\mathbf{r}|}(a))\,da\,\Psi(z_1)\,d\mathbf{z}$$
$$\le\int\sum_{\mathbf{m}}I_{\{1\le|a|\le1+|\mathbf{m}|^{1/3}\}}(a)\lambda_{|\mathbf{m}|}(a)^{1+\varepsilon}\int|Q_{\mathbf{m},a}f^a(\mathbf{z})|^2\Psi(z_1)\,d\mathbf{z}\,da.$$

As before, we apply Lemmas 8 and 9 to the operator $M_\Psi P_{m_1,a}$ to obtain

$$\int|S_1f(\mathbf{z},u)|^2\,d\mathbf{z}\,du$$
$$\le\int\sum_{\mathbf{m}}I_{\{1\le|a|\le1+|\mathbf{m}|^{1/3}\}}(a)\lambda_{|\mathbf{m}|}(a)^{1+\varepsilon}(|a|/|\mathbf{m}|)^{1/2}\int|Q_{\mathbf{m},a}f^a(\mathbf{z})|^2\,d\mathbf{z}\,da$$
$$\le\sum_{\mathbf{m}}\int\lambda_{|\mathbf{m}|}(a)^{3/4+\varepsilon}\int|Q_{\mathbf{m},a}f^a(\mathbf{z})|^2\,d\mathbf{z}\,da\le\|f\|^2_{W^{3/4+\varepsilon}}.$$

Thus the proof of Theorem 1 is complete.

Example 1. It is interesting to observe that the analog of Theorem 1 fails for the sublaplacian Δ on $\mathbb{H}^n$.

Let

$$f_k(\mathbf{z},u)=\int\phi_k(a)q_{0,a}(\mathbf{z})e^{iau}|a|^n\,da,$$

where $\phi_k(\mathbf{z})=\phi(2^{-k}|z|)$, $\phi\in C_c^\infty(1,2)$ is nonnegative, k is a positive integer and $q_{0,a}(\mathbf{z})=e^{-|a||\mathbf{z}|^2}\ge0$. Then

$$(I-\Delta)^{s/2}f_k(\mathbf{z},u)=\int\phi_k(a)q_{0,a}(\mathbf{z})e^{iau}(1+|a|)^{s/2}|a|^n\,da,$$
$$U_tf_k(\mathbf{z},u)=e^{it\Delta}f_k(\mathbf{z},u)=\int\phi_k(a)q_{0,a}(\mathbf{z})e^{iau}e^{iat}|a|^n\,da,$$

and if $(\mathbf{z},u)$ belongs to the unit ball B in the Heisenberg group, we have

$$Mf_k(\mathbf{z},u)=\sup_{0\le t\le1}|e^{it\Delta}f_k(\mathbf{z},u)|=\int\phi_k(a)q_{0,a}(\mathbf{z})|a|^n\,da$$

so

$$Mf_k(\mathbf{z},u)\ge e^{-2^{k+1}|\mathbf{z}|^2}\int\phi_k(a)|a|^n\,da\ge C2^k2^{nk}e^{-2^{k+1}|\mathbf{z}|^2}.$$

Now the L^2 norms of $Mf_k(\mathbf{z},u)$ and $(I-\Delta)^{s/2}f_k(\mathbf{z},u)$ can be estimated. We have

$$\|(I-\Delta)^{s/2}f_k\|^2 = \iint \phi_k(a)^2 q_{0,a}(\mathbf{z})^2(1+|a|)^s|a|^n\,d\mathbf{z}\,da = \int \phi_k(a)^2(1+|a|)^s|a|^n\,da \le 2^{nk+k+ks}$$

and

$$\|Mf_k\|^2 \ge C \int\limits_{|\mathbf{z}|\le 1}\int\limits_{|u|\le 1} 2^{2nk+2k}e^{-2^{k+2}|\mathbf{z}|^2}\,d\mathbf{z}\,du \ge C2^{nk+2k},$$

which disproves the estimate

$$\|Mf_k\|_{L^2(B)} \le C\|(I-\Delta)^{s/2}f_k\|_{L^2}$$

for $s<1$.

EXAMPLE 2. The theorem below gives the answer to a question posed in [MR].

Let $g\in C_c^\infty(\mathbb{C}^n)$ and $f(\mathbf{z},u)=g(\mathbf{z})\exp(-iu)$. We define a twisted laplacian $\widetilde{\Delta}$ on $C_c^\infty(\mathbb{C}^n)$ as the differential operator satisfying the condition $\Delta f(\mathbf{z},u)=\exp(-iu)\widetilde{\Delta}g(\mathbf{z})$. The formula makes sense, since the coefficients of Δ do not depend on t (cf. [MR]). For the theory of unitary groups generated by twisted laplacians see [MR].

The closure of $\widetilde{\Delta}$ on $C_c^\infty(\mathbb{C}^n)$ is a selfadjoint operator. The spectral decomposition of $\widetilde{\Delta}^{s/2}$ is given by the formula

$$\widetilde{\Delta}^{s/2}f(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^n}(2|\mathbf{m}|+n)^{s/2}Q_{\mathbf{m},1}f(z).$$

Consequently,

$$V_tf(\mathbf{z}) = \sum_{\mathbf{m}} e^{i(2|\mathbf{m}|+n)t}Q_{\mathbf{m},1}f(\mathbf{z}).$$

The Sobolev space of order s is defined as the domain of $\widetilde{\Delta}^{s/2}$, and

$$\|f\|_{W^s}^2 = \|\widetilde{\Delta}^{s/2}f\|^2 = \sum_{\mathbf{m}}(2|\mathbf{m}|+n)^{s/2}\|Q_{\mathbf{m},1}f\|^2.$$

Observe that for $s\ge 0$, W^s is contained in $L^2(\mathbb{C}^n)$ and the function $t\mapsto V_tf(z)$ is 2π-periodic.

For a fixed $\psi\in C_c^\infty(\mathbb{C}^n)$ we put

$$Mf(\mathbf{z}) = \psi(\mathbf{z})\sup_{0\le t\le 2\pi}|V_tf(\mathbf{z})|.$$

We are going to prove

THEOREM 2. *Let $s>1/2$ and $f\in W^s$. Then*

$$\|Mf\|_{L^2}\le C\|f\|_{W^s}.$$

Proof. Assume that the function f is real-valued. We use an estimate of the Sobolev type:

$$|\sup_{0\le t\le 2\pi} V_tf(\mathbf{z})|^2 \le C\int\limits_0^{2\pi}|\partial_t^{(1+\varepsilon)/2}V_tf(\mathbf{z})|^2\,dt + C\int\limits_0^{2\pi}|V_tf(\mathbf{z})|^2\,dt.$$

We multiply the inequality above by $|\psi|^2$ and integrate both sides with respect to $d\mathbf{z}$. Then an application of the Plancherel formula with respect to the t variable, as in the proof of Theorem 1, yields

$$\int\limits_{\mathbb{C}^n}|Mf(\mathbf{z})|^2\,d\mathbf{z} \le \sum_s(2s+n)\int\limits_{\mathbb{C}^n}\phi(\mathbf{z})^2\Big|\sum_{\{\mathbf{m}:|\mathbf{m}|=s\}}Q_{\mathbf{m},1}f(\mathbf{z})\Big|^2\,d\mathbf{z} + \int\limits_{\mathbb{C}^n}\int\limits_0^{2\pi}|V_tf(\mathbf{z})|^2\,d\mathbf{z}.$$

Since V_t is an isometry, the second summand is dominated by $\|f\|_{L^2}^2$.

Again with no loss of generality we may consider only the multiindices $\mathbf{m}$ in I_1. Majorizing $\psi(\mathbf{z})^2$ by $\Psi(z_1)$ (as in the proof of Theorem 1) we write

$$\begin{aligned}
&\sum_s(2s+n)^{1+\varepsilon}\int\limits_{\mathbb{C}^n}\psi(\mathbf{z})^2\Big|\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}Q_{\mathbf{m},1}f(\mathbf{z})\Big|^2\,d\mathbf{z}\\
&\le \sum_s(2s+n)^{1+\varepsilon}\Psi(z_1)\Big|\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}Q_{\mathbf{m},1}f(\mathbf{z})\Big|^2\,d\mathbf{z}\\
&\le \sum_s(2s+n)^{1+\varepsilon}\Psi(z_1)\sum_{\{(\mathbf{m},\mathbf{r})\in A:|\mathbf{m}|=|\mathbf{r}|=s\}}Q_{\mathbf{m},1}f(\mathbf{z})Q_{\mathbf{r},1}f(\mathbf{z})\,d\mathbf{z}\\
&= \sum_s(2s+n)^{1+\varepsilon}\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\int\Psi(z_1)|Q_{\mathbf{m},1}f(\mathbf{z})|^2\,d\mathbf{z}.
\end{aligned}$$

Now, by Lemma 4, we dominate the last sum by

$$\sum_s(2s+n)^{1/2+\varepsilon}\sum_{\{\mathbf{m}\in I_1:|\mathbf{m}|=s\}}\|Q_{\mathbf{m},1}f\|_{L^2}^2 \le \|f\|_{W^{1/2+\varepsilon}}^2.$$

The theorem follows.

EXAMPLE 3. We show that $s\ge 1/4$ is necessary. In fact, we show that the example given by Dahlberg and Kenig in [DK] works in our setting as well. For the sake of simplicity we state our estimate for $\mathbb{H}^1$.

Let $\mathcal{G}f(a,m)=1$ if $m=0$ and $N\le a\le N+N^{1/2}$, $\mathcal{G}f(a,m)=0$ otherwise. Then by the Plancherel formula we see that

$$\|f\|_{W^s}^2 = \int\limits_N^{N+N^{1/2}}(1+|a|+a^2)^s|a|\,da \le CN^{1+1/2+2s}.$$

We have

$$V_t f(z,u) = e^{-iu^2/(4t)-iu/(2t)} \int\limits_N^{N+N^{1/2}} e^{it(a+1/2+u/(2t))^2} e^{-|a||z|^2} |a|\, da.$$

On the other hand, choosing $t(z,u) = -u/(N+1/2)$ we see that for $|u| \le 1/10$ the real part of $e^{it(a+1/2-u/(2t))^2}$ is bounded from below by $1/2$. Also $t(z,u) \in [-1,1]$ and

$$|V_{t(z,u)} f(z,u)| \ge \frac{1}{2} \int\limits_N^{N+N^{1/2}} e^{-|a||z|^2} |a|\, da \ge CN^{3/2} e^{-2N|z|^2}.$$

Hence $\|Mf\|^2 \ge CN^2$. This completes the proof.

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