Contents of Volume 122, Number 1

STUDIA MATHEMATICA

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Subscription information (1997): Vols. 122–126 (15 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223

Some weighted inequalities for general one-sided maximal operators

by

F. J. MARTÍN-REYES and A. DE LA TORRE (Málaga)

Abstract. We characterize the pairs of weights on $\mathbb{R}$ for which the operators

$$M^+_{h,k}f(x) = \sup_{c>x} h(x,c)\int_x^c f(s)k(x,s,c)\,ds$$

are of weak type (p,q), or of restricted weak type (p,q), $1 \le p < q < \infty$, between the Lebesgue spaces with the coresponding weights. The functions h and k are positive, h is defined on $\{(x,c) : x < c\}$, while k is defined on $\{(x,s,c) : x < s < c\}$. If $h(x,c) = (c-x)^{-\beta}$, $k(x,s,c) = (c-s)^{\alpha-1}$, $0 \le \beta \le \alpha \le 1$, we obtain the operator

$$M^+_{\alpha,\beta}f = \sup_{c>x} \frac{1}{(c-x)^\beta}\int_x^c \frac{f(s)}{(c-s)^{1-\alpha}}\,ds.$$

For this operator, under the assumption $1/p - 1/q = \alpha - \beta$, we extend the weak type characterization to the case $p = q$ and prove that in the case of equal weights and $1 < p < \infty$, weak and strong type are equivalent. If we take $\alpha = \beta$ we characterize the strong type weights for the operator $M^+_{\alpha,\alpha}$ introduced by W. Jurkat and J. Troutman in the study of C_α differentiation of the integral.

1. Introduction. In 1979 W. Jurkat and J. Troutman [JT] studied the operator

$$M^+_\alpha f(x) = \sup_{x<c} \frac{\alpha}{(c-x)^\alpha}\int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}}\,ds, \qquad 0 < \alpha \le 1.$$

They proved that there is a limit case $p = 1/\alpha$ below which the operator is not of weak type. For $p = 1/\alpha$ the operator still fails to be of weak type, but it is of restricted weak type, and for $p > 1/\alpha$ it is bounded. From these estimates they obtained results about a C_α version of Lebesgue's differentiation theorem. The restricted weak type of some particular weight for the

1991 *Mathematics Subject Classification*: Primary 42B25.
Key words and phrases: one-sided maximal operators, Cesàro averages, weights.
Research supported by DGICYT Grant (pb94-1496) and Junta de Andalucía.

case $\alpha = 1/2$ has also been recently used by A. Carbery, E. Hernandez and F. Soria in the study of the Kakeya maximal operator for radial functions [CHS].

Since the operator is clearly one-sided, it seems natural to study the good weights and to compare them with the classes A_p^+ introduced by Sawyer [S] for the one-sided Hardy–Littlewood maximal function.

The paper is organized as follows: In this section we introduce a very general operator and show that almost any one-sided operator is a particular case of our operator. In Section 2 we study weak and restricted weak type inequalities. Section 3 is devoted to the study of strong type inequalities in the case of "equal" weights for the operator $M^+_{\alpha,\beta}$ defined below. The paper ends with some remarks about the relationship between our weights and the weights for the fractional one-sided maximal operator.

DEFINITION 1.1. Let f be a locally integrable function defined on $\mathbb{R}$, and let $h(x,y)$ and $k(x,y,z)$ be two positive measurable functions defined on $\{(x,y) : x < y\}$ and $\{(x,y,z) : x < y < z\}$ respectively. If the function $s \to k(x,s,y)$ is locally integrable on (x,y) for any $x < y$, we define the maximal operator

$$M^+_{h,k}f(x) = \sup_{c>x} h(x,c) \int_x^c |f(s)| k(x,s,c)\, ds.$$

In the case $h(x,c) = (c-x)^{-\beta}$, $k(x,s,c) = (c-s)^{\alpha-1}$, $0 \le \beta \le \alpha \le 1$, we will call the operator $M^+_{\alpha,\beta}$ instead of $M^+_{h,k}$.

Our aim is to study the good weights for these operators.

EXAMPLES. (1) The starting point and the motivation for this paper is the case $\alpha = \beta \neq 1$. Then the operator $M^+_{\alpha,\beta}$ is the maximal operator associated to the Cesàro averages C_α. For Lebesgue measure it is known that it maps L^p into itself if $p > 1/\alpha$. In the limit case $p = 1/\alpha$ it maps $L_{p,1}$ into $L_{p,\infty}$ (cf. [JT]), but nothing was known about the weights for this operator.

(2) If $\alpha = \beta = 1$, the operator $M^+_{\alpha,\beta}$ is the one-sided Hardy–Littlewood maximal operator, denoted usually by M^+. The pairs of weights for which this operator is of weak or strong type are well known. See [S] and [MOT].

(3) If $\alpha = 1$ and $0 < \beta < 1$ then $M^+_{\alpha,\beta}$ is the fractional one-sided maximal operator. The pairs of weights (u,v) for which this operator is bounded from $L^p(v\,dx)$ to $L^q(u\,dx)$ were characterized by Andersen and Sawyer in the case $v^q = u^p$ and with the restriction $1/p - 1/q = 1 - \beta$ (see [AS]). The weak type with the same conditions can be seen in [MPT]. The strong type for $1 < p \le q$ was solved by the authors in [MT].

(4) If $\alpha = 1$, $\beta = 0$ then $M^+_{\alpha,\beta}f(x) = \int_x^\infty f(s)\,ds$, which is the dual of the Hardy operator $\int_0^x f(s)\,ds$.

(5) If $h(x,c) = x^\eta$, $k = 1$, we obtain the modified Hardy operator $x^\eta \int_x^\infty f(s)\,ds$. Weighted weak type inequalities were studied in [AM] and weighted strong type inequalities follow from the results for the dual of the Hardy operator.

(6) If $h = 1$ and $k(x,s,c) = (s-x)^{\alpha-1}$, then the operator is the Weyl fractional integral studied in [LT] and [KG].

Of course, one could also consider the operators $M^-_{h,k}$ and $M^-_{\alpha,\beta}$ defined reversing the orientation.

Throughout the paper, the weights u and v will be positive, measurable functions and C will denote a constant that may change from one line to another. If p is any number between 1 and ∞, then p' will denote its conjugate exponent. For any measurable, positive function g and any measurable set E, $g(E)$ will stand for the integral of g over E and $M_g f$ will be the maximal operator $M_g f(x) = \sup_{x\in I}(1/g(I)) \int_I |f|g$, where the supremum is taken over all the intervals I containing x. We will use the fact that, since we are in dimension one, this operator is of weak type $(1,1)$ with respect to the measure $g\,dx$.

2. Weak and restricted weak type inequalities. In this section we characterize the pairs of weights for which the above operators are of weak or restricted weak type. The following theorem characterizes, under mild conditions on h and k, weak type weights in the case $p < q$.

THEOREM 2.1. *For $1 \le p, q < \infty$, we consider the following two conditions:*

(1) *There exists a constant C such that for any $f \in L^p(v)$,*

$$\sup_{\lambda>0} \lambda^q u(\{x : M^+_{h,k}f(x) > \lambda\}) \le C\Big(\int |f|^p v\Big)^{q/p}.$$

(2) *The pair (u,v) belongs to $A^+_{p,q,h,k}$, i.e., there exists a constant C such that for any three numbers $a < b < c$,*

$$(A^+_{p,q,h,k}) \qquad h(a,c)\Big(\int_a^b u\Big)^{1/q}\Big(\int_b^c v^{1-p'}(s)k^{p'}(a,s,c)\,ds\Big)^{1/p'} \le C \quad \text{if } p > 1,$$

and

$$h(a,c)\Big(\int_a^b u\Big)^{1/q} \le C \operatorname*{ess\,inf}_{s\in(b,c)} v(s)k^{-1}(a,s,c) \quad \text{if } p = 1.$$

If both h and k are increasing in the first variable then (1)⇒(2). If h and k decrease in their last variable and $p < q$, then (2)⇒(1).

Proof. Assume (1) holds. If $p > 1$, $a < b < c$ and we consider the function $f(s) = v^{1-p'}(s)k^{p'-1}(a,s,c)\chi_{(b,c)}(s)$, then for any $x \in (a,b)$ we have

$$M^+_{h,k}f(x) \ge h(x,c)\int_b^c v^{1-p'}(s)k^{p'-1}(a,s,c)k(x,s,c)\,ds$$
$$\ge h(a,c)\int_b^c v^{1-p'}(s)k^{p'}(a,s,c)\,ds = h(a,c)\int f^p v \equiv \lambda.$$

This means that $(a,b) \subset \{x : M^+_{h,k}f(x) \ge \lambda\}$. If $\lambda < \infty$ we apply (1) and obtain (2). If $\lambda = \infty$, then the function $m(s) = v^{-1}(s)k(a,s,c)$ is not in $L^{p'}(\chi_{(b,c)}v)$. This means that there exists $g \ge 0$, $g \in L^p(\chi_{(b,c)}v)$, such that $\int_b^c g(s)k(a,s,c)\,ds = \infty$. It follows that for any $x \in (a,b)$,

$$M^+_{h,x}g(x) \ge h(x,c)\int_x^c g(s)k(x,s,c)\,ds \ge h(x,c)\int_b^c g(s)k(a,s,c)\,ds = \infty,$$

contrary to (1). If $p = 1$, we take $a < b < c-t < c-r < c$ and consider the function $f = \chi_{(c-t,c-r)}$. Now if $x \in (a,b)$ then

$$M^+_{h,k}f(x) \ge h(x,c)\int_{c-t}^{c-r} k(x,s,c)\,ds \ge h(a,c)\int_{c-t}^{c-r} k(a,s,c)\,ds = \lambda.$$

Therefore $h(a,c)\int_{c-t}^{c-r} k(a,s,c)\,ds\,(\int_a^b u)^{1/q} \le C\int_{c-t}^{c-r} v$, and (2) follows from Lebesgue's differentiation theorem.

Assume now that (2) holds and h and k are decreasing in their last variable. For $f \in L^p(v)$, $f \ge 0$ and $x < c$, we define a sequence of points by taking $x_0 = c$ and $\int_x^{x_{i+1}} f^p v = \int_{x_{i+1}}^{x_i} f^p v$ for $i \ge 0$. If $p > 1$, we may write

$$h(x,c)\int_x^c f(s)k(x,s,c)\,ds$$
$$= h(x,c)\sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i} f(s)k(x,s,c)\,ds$$
$$\le h(x,c)\sum_{i=0}^{\infty}\Big(\int_{x_{i+1}}^{x_i} f^p v\Big)^{1/p}\Big(\int_{x_{i+1}}^{x_i} v^{1-p'}(s)k^{p'}(x,s,c)\,ds\Big)^{1/p'}$$
$$\le h(x,c)\sum_{i=0}^{\infty}\Big(\int_{x_{i+1}}^{x_i} f^p v\Big)^{1/p}\Big(\int_{x_{i+1}}^{x_i} v^{1-p'}(s)k^{p'}(x,s,x_i)\,ds\Big)^{1/p'}$$
$$\le Ch(x,c)\sum_{i=0}^{\infty}\frac{(\int_x^{x_{i+1}} f^p v)^{1/p}}{(\int_x^{x_{i+1}} u)^{1/q}h(x,x_i)}$$
$$\le C(M_u(f^p vu^{-1}))^{1/q}(x)\sum_{i=0}^{\infty}\Big(\int_x^{x_{i+1}} f^p v\Big)^{1/p-1/q}.$$

But it follows from the definition of the points x_i that $\int_x^{x_i} f^p v = 2^{-i}\int_x^c f^p v$, and therefore we have proved that for any $x < c$,

$$h(x,c)\int_x^c f(s)k(x,s,c)\,ds \le C(M_u(f^p vu^{-1}))^{1/q}(x)\Big(\int_{\mathbb{R}} f^p v\Big)^{1/p-1/q}.$$

This implies that $M^+_{h,k}f(x) \le C(M_u(f^p vu^{-1}))^{1/q}(x)(\int_{\mathbb{R}} f^p v)^{1/p-1/q}$ and then

$$u\{x : M^+_{h,k}f(x) > \lambda\}$$
$$\le u\Big\{x : M_u(f^p vu^{-1})(x) > C\lambda^q\Big(\int f^p v\Big)^{1-q/p}\Big\} \le \frac{C}{\lambda^q}\Big(\int f^p v\Big)^{q/p},$$

which is (1). If $p = 1$, the proof follows the same lines but we dominate $\int_{x_{i+1}}^{x_i} f(s)k(x,s,c)\,ds$ by

$$\int_{x_{i+1}}^{x_i} f(s)k(x,s,x_i)v^{-1}(s)v(s)\,ds \le C\int_{x_{i+1}}^{x_i} f(s)v(s)h^{-1}(x,x_i)\,ds\Big(\int_x^{x_{i+1}} u\Big)^{-1/q},$$

and then proceed as in the case $p > 1$. ■

Under the same conditions on h, k and still with $p < q$ we obtain a characterization for restricted weak type.

THEOREM 2.2. *For $1 \le p, q < \infty$, we consider the following two conditions:*

(1) *There exists a constant C such that for any measurable set E,*

$$\sup_{\lambda>0}\lambda^q u(\{x : M^+_{h,k}\chi_E(x) > \lambda\}) \le C(v(E))^{q/p}.$$

(2) *There exists a constant C such that for any three numbers $a < b < c$ and any measurable set E contained in (b,c),*

$$h(a,c)\int_E k(a,s,c)\,ds \le C\frac{(v(E))^{1/p}}{(u(a,b))^{1/q}}.$$

If both h and k are increasing in the first variable then (1)⇒(2). If h and k decrease in their last variable and $p < q$, then (2)⇒(1).

Proof. Assume that (1) holds with h and k increasing in their first variable. Fix $a < b < c$. By a limiting argument it is enough to consider $E \subset (b,d)$ with $d < c$. If we define $\lambda = h(a,c)\int_E k(a,s,c)\,ds$, then $\lambda < \infty$ and for any $x \in (a,b)$ we have

$$M^+_{h,k}\chi_E(x) \ge h(x,c)\int_x^c \chi_E(s)k(x,s,c)\,ds \ge h(a,c)\int_b^c \chi_E(s)k(a,s,c)\,ds.$$

This means that $(a,b)\subset\{x : M^+_{h,k}\chi_E(x)\ge\lambda\}$, and therefore

$$\Big(h(a,c)\int_E k(a,s,c)\,ds\Big)^q u(a,b)\le C(v(E))^{q/p},$$

which is (2).

Assume now that $p<q$, that h and k decrease in their last variable and that (2) holds. Given then a measurable set E and $x<c$, we define a sequence of points as in the preceding theorem, i.e. $x_0=c$, $\int_{x_{i+1}}^{x_i}\chi_E v=\int_x^{x_{i+1}}\chi_E v$. If we consider the sets $E_k=E\cap(x_{i+1},x_i)$, and apply (2) in the case $a=x$, $b=x_{i+1}$ and $c=x_i$, we have

$$\begin{aligned}
h(x,c)\int_x^c\chi_E(s)k(x,s,c)\,ds &= h(x,c)\sum_{i=0}^\infty\int_{x_{i+1}}^{x_i}\chi_E(s)k(x,s,c)\,ds\\
&\le h(x,c)\sum_{i=0}^\infty\int_{E_i}k(x,s,x_i)\,ds\\
&\le Ch(x,c)\sum_{i=0}^\infty\frac{1}{h(x,x_i)}\cdot\frac{(v(E_i))^{1/p}}{(u(x,x_{i+1}))^{1/q}}\\
&\le C\sum_{i=0}^\infty\Big(\frac{\int_{x_{i+1}}^{x_i}\chi_E v}{\int_x^{x_{i+1}}u}\Big)^{1/q}\Big(\int_{x_{i+1}}^{x_i}\chi_E v\Big)^{1/p-1/q}\\
&\le C(M_u(\chi_E vu^{-1}))^{1/q}(x)\Big(\int_E v\Big)^{1/p-1/q},
\end{aligned}$$

which implies

$$M^+_{h,k}\chi_E(x)\le C(M_u(\chi_E vu^{-1}))^{1/q}(x)\Big(\int_E v\Big)^{1/p-1/q},$$

and (1) follows from the weak type $(1,1)$ of the operator M_u with respect to the measure $u\,dx$. ∎

The proof of these theorems uses the fact that $1/p-1/q>0$. In order to deal with the case $p=q$ we restrict ourselves to the operator $M^+_{\alpha,\alpha}$ and exploit the relationship that in this case exists between the functions h and k.

THEOREM 2.3. *Let $1<p<\infty$ and $0<\alpha\le1$. The following are equivalent*:

(1) *There exists a constant C such that for any $f\in L^p(v)$ and any positive λ*,

$$u\{x : M^+_{\alpha,\alpha}f(x)>\lambda\}\le C\lambda^{-p}\int|f|^p v.$$

(2) *There exists a constant C such that for any three numbers $a<b<c$*,

$$(A^+_{p,p,\alpha,\alpha})\qquad\Big(\int_a^b u\Big)^{1/p}\Big(\int_b^c\frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\Big)^{1/p'}\le C(c-a)^\alpha.$$

Before proving the theorem we observe that $A^+_{p,p,\alpha,\alpha}$ is nothing but a particular case of $A^+_{p,q,h,k}$, and that, in this case, both h and k satisfy the monotonicity conditions of Theorems 2.1 and 2.2.

Proof. That (1) implies (2) is the first part of Theorem 2.1, for this particular case. For the converse we will prove that (2) implies that there exists $C>0$ such that for every $a<b$ and any $f\ge0$ the following inequality holds:

$$\int_a^b\frac{f(s)}{(b-s)^{1-\alpha}}\,ds\le C(M_u(f^pvu^{-1}))^{1/p}(a)(b-a)^\alpha, \tag{2.4}$$

i.e. $M^+_{\alpha,\alpha}f(a)\le C(M_u(f^pvu^{-1}))^{1/p}(a)$. From this inequality one obtains (1) as in Theorem 2.1. To prove (2.4) we define a sequence $x_0=b>x_1>x_2>\ldots>a$ by the identity

$$\int_a^{x_{i+1}}u=\int_{x_{i+1}}^{x_i}u=\frac12\int_a^{x_i}u.$$

On each interval (x_{i+1},x_i) we have

$$\begin{aligned}
&\int_{x_{i+1}}^{x_i}\frac{f(s)}{(b-s)^{1-\alpha}}\,ds\\
&=\int_{x_{i+1}}^{x_i}\frac{f(s)}{(x_i-s)^{1-\alpha}}\Big(\frac{x_i-s}{b-s}\Big)^{1-\alpha}ds\\
&\le\Big(\frac{x_i-x_{i+2}}{b-x_{i+2}}\Big)^{1-\alpha}\int_{x_{i+1}}^{x_i}\frac{f(s)}{(x_i-s)^{1-\alpha}}v^{1/p}(s)v^{-1/p}(s)\,ds\\
&\le\Big(\frac{x_i-x_{i+2}}{b-x_{i+2}}\Big)^{1-\alpha}\Big(\int_{x_{i+1}}^{x_i}f^pv\Big)^{1/p}\Big(\int_{x_{i+1}}^{x_i}\frac{v^{-p'/p}(s)}{(x_i-s)^{(1-\alpha)p'}}\,ds\Big)^{1/p'}\\
&\le C\Big(\frac{x_i-x_{i+2}}{b-x_{i+2}}\Big)^{1-\alpha}(x_i-x_{i+2})^\alpha\Big(\int_{x_{i+1}}^{x_i}f^pv\Big)^{1/p}\Big(\int_{x_{i+2}}^{x_{i+1}}u\Big)^{-1/p}\\
&\le C\frac{x_i-x_{i+2}}{(b-x_{i+2})^{1-\alpha}}\Big(\int_a^{x_i}f^pv\Big)^{1/p}\Big(\int_a^{x_i}u\Big)^{-1/p}\\
&\le C\frac{x_i-x_{i+2}}{(b-x_{i+2})^{1-\alpha}}(M_u(f^pvu^{-1}))^{1/p}(a).
\end{aligned}$$

(For passing from line 2 to 3 of the above we have used the fact that the function $s \to ((x_i - s)/(c-s))^{1-\alpha}$ is decreasing.) Adding in i from $i=0$ to ∞ we get

$$\begin{aligned}\int_a^b \frac{f(s)}{(b-s)^{1-\alpha}}\,ds &\le C(M_u(f^p v u^{-1}))^{1/p}(a) \sum_i \frac{x_i - x_{i+2}}{(b-x_{i+2})^{1-\alpha}} \\ &\le C(M_u(f^p v u^{-1}))^{1/p}(a) \sum_i \int_{x_{i+2}}^{x_i} \frac{1}{(b-s)^{1-\alpha}}\,ds \\ &\le C(M_u(f^p v u^{-1}))^{1/p}(a) \int_a^b \frac{1}{(b-s)^{1-\alpha}}\,ds \\ &= C(M_u(f^p v u^{-1}))^{1/p}(a)(b-a)^\alpha. \ \blacksquare\end{aligned}$$

Restricted weak type can also be characterized in this case.

THEOREM 2.5. *Let $1 \le p < \infty$ and $0 < \alpha \le 1$. The following are equivalent*:

(1) *Restricted weak type, i.e. there exists C such that for any measurable set E and any $\lambda > 0$,*

$$\int_{\{x : M^+_{\alpha,\alpha}\chi_E(x) > \lambda\}} u \le \frac{C}{\lambda^p} \int \chi_E v.$$

(2) *There exists C such that for any $a < b < c$ and any measurable set $E \subset (b,c)$,*

$$\frac{\int_E (c-s)^{\alpha-1}\,ds}{(c-a)^\alpha} \le C \frac{(v(E))^{1/p}}{(\int_a^b u)^{1/p}}.$$

Proof. We only need to prove that (2) implies (1). For any given interval (a,b) we define a sequence x_i as in the proof of (2.4). It follows that if E is any measurable set and $E_i = E \cap (x_{i+1}, x_i)$, then

$$\begin{aligned}\int_{x_{i+1}}^{x_i} \frac{\chi_E(s)}{(b-s)^{1-\alpha}}\,ds &\le \left(\frac{x_i - x_{i+2}}{b - x_{i+2}}\right)^{1-\alpha} \int_{E_i} \frac{1}{(x_i - s)^{1-\alpha}}\,ds \\ &\le C \frac{x_i - x_{i+2}}{(b-x_{i+2})^{1-\alpha}} \cdot \frac{(\int_{E_i} v)^{1/p}}{(\int_{x_{i+2}}^{x_{i+1}} u)^{1/p}} \\ &\le C \frac{x_i - x_{i+2}}{(b-x_{i+2})^{1-\alpha}} \left(\frac{\int_a^{x_i} v\chi_{E_i}}{\int_a^{x_i} u}\right)^{1/p} \\ &\le C \frac{x_i - x_{i+2}}{(b-x_{i+2})^{1-\alpha}} (M_u(v\chi_E u^{-1}))^{1/q}(a).\end{aligned}$$

As in the preceding theorem this implies

$$M^+_{\alpha,\alpha}\chi_E(a) \le C(M_u(\chi_E v u^{-1}))^{1/p}(a),$$

and restricted weak type follows. ■

3. The case of equal weights. By equal weights we mean $v^q = u^p$. In this section we will prove that in this case, and under the restrictions $1/p - 1/q = \alpha - \beta$, and $p > 1$, weak and strong type for the operator $M^+_{\alpha,\beta}$ are equivalent. The natural approach is to prove a "$p - \varepsilon$" property. The problem is that this property in the theory of A^+_p weights relies on some kind of reverse Hölder inequality for the function $v^{1-p'}$, but the role of this function is now played by $v^{1-p'}(s)/(c-s)^{(1-\alpha)p'}$, which is not a fixed function but a one-parameter family depending on the variable point c. Still the "$p - \varepsilon$" property can be proved using the following lemma:

LEMMA 3.1. *Let $1 < p < \infty$ and $0 \le \alpha - \beta \le 1/p$. Assume $A_{p,q,\alpha,\beta}$ holds, i.e. there exists C_1 such that for any $a < b < c$,*

$$\left(\int_a^b u\right)^{1/q} \left(\int_b^c \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\,ds\right)^{1/p'} \le C_1 (c-a)^\beta. \tag{3.2}$$

Then there exists a constant C_2 depending only on C_1 and such that for any $x < y < z < c$,

$$\left(\int_x^y u\right)^{1/q} \left(\int_y^z \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\,ds\right)^{1/p'} \le C_2 \left(\int_x^z \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\,ds\right)^{1+\beta-\alpha} \tag{3.3}$$

Proof. We consider $x < y < z < c$ and define points $x_0 = z$, $\int_x^{x_{i+1}} u = \int_{x_{i+1}}^{x_i} u$ if $i \ge 0$. Let N be such that $x_N \le y < x_{N-1}$. It follows from the definition that $\int_{x_{N+1}}^{x_N} u = \frac{1}{2}\int_{x_N}^{x_{N-1}} u = \frac{1}{4}\int_x^{x_{N-1}} u > \frac{1}{4}\int_x^y u$, while for $i < N-1$ we have $\int_{x_{i+1}}^{x_i} u = \int_x^{x_{i+1}} u > \int_x^y u$. Keeping this in mind we have

$$\begin{aligned}\int_y^z \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\,ds &\le \sum_{i=0}^{N-1} \int_{x_{i+1}}^{x_i} \frac{v^{1-p'}(s)}{(x_i - s)^{(1-\alpha)p'}} \left(\frac{x_i - s}{c - s}\right)^{(1-\alpha)p'} ds \\ &\le \sum_{i=0}^{N-1} \left(\frac{x_i - x_{i+2}}{c - x_{i+2}}\right)^{(1-\alpha)p'} \int_{x_{i+1}}^{x_i} \frac{v^{1-p'}(s)}{(x_i - s)^{(1-\alpha)p'}}\,ds\end{aligned}$$

$$\le C_1 \sum_{i=0}^{N-1} \left(\frac{x_i - x_{i+2}}{c - x_{i+2}}\right)^{(1-\alpha)p'} (x_i - x_{i+2})^{\beta p'} \frac{1}{(\int_{x_{i+2}}^{x_{i+1}} u)^{p'/q}}$$

$$\le 4^{p'/q} C_1 \Big(\int_x^y u\Big)^{-p'/q} \sum_{i=0}^{N-1} \left(\frac{x_i - x_{i+2}}{(c - x_{i+2})^{(1-\alpha)/(1+\beta-\alpha)}}\right)^{(1+\beta-\alpha)p'}$$

$$\le C_2 \Big(\int_x^y u\Big)^{-p'/q} \left(\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+2}} \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\, ds\right)^{(1+\beta-\alpha)p'}$$

$$\le C_2 \Big(\int_x^y u\Big)^{-p'/q} \left(\int_x^z \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\, ds\right)^{(1+\beta-\alpha)p'} . \ \blacksquare$$

We have used the fact that $(1+\beta-\alpha)p' \ge 1$, which is nothing but another way of writing $\alpha - \beta \le 1/p$. Observe that if $\beta \ne 0$, we can take $z = c$ in (3.3) and obtain (3.2), so actually they are equivalent. We also point out that $A^+_{p,q,\alpha,\beta}$ is nothing but $A^+_{p,q,h,k}$ when $h(x,c) = (c-x)^{-\beta}$ and $k(x,s,c) = (c-s)^{\alpha-1}$. Therefore $A_{p,q,\alpha,\beta}$ characterizes weak type if $p < q$ or $\alpha = \beta$, $p = q$.

In order to characterize strong type in the case of equal weights, we need to recall a few facts about weights for the operator

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{\int_x^{x+h} g} \int_x^{x+h} fg,$$

where g is any locally integrable function. The main result needed is the following.

THEOREM 3.4. *The operator M_g^+ is of weak type (p,p), $1 < p < \infty$, with respect to the weights (u,u) if, and only if, the following condition is satisfied: There exists C such that for any $x < y < z$,*

$$(A_p^+(g)) \qquad \Big(\int_x^y u\Big)^{1/p} \Big(\int_y^z u^{1-p'} g^{p'}\Big)^{1/p'} \le C \int_x^z g.$$

Furthermore, if (u,u) satisfies $A_p^+(g)$, then there exists $\varepsilon > 0$, depending only on the constant C of the definition of $A_p^+(g)$ and not on the particular function g, such that (u,u) satisfies $A^+_{p-\varepsilon}(g)$.

For the proof see [MOT]. The same result holds also for functions and weights supported on a half-line $(-\infty, c)$ (cf. [A]).

THEOREM 3.5. *Let $0 < \beta \le \alpha \le 1$ and $1 < p < \infty$. If $1/p - 1/q = \alpha - \beta$ and $v^q = u^p$, then the following are equivalent:*

(1) *There exists a constant C such that for all $f \in L^p(v)$,*

$$\Big(\int (M^+_{\alpha,\beta} f)^q u\Big)^{1/q} \le C\Big(\int |f|^p v\Big)^{q/p}.$$

(2) *The pair (u,v) satisfies $A^+_{p,q,\alpha,\beta}$.*

Proof. (1)⇒(2) follows from Theorems 2.1 and 2.3. Conversely, we know from Lemma 3.1 that $A^+_{p,q,\alpha,\beta}$ implies (and is actually equivalent to)

$$\Big(\int_x^y u\Big)^{1/q} \left(\int_y^z \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\, ds\right)^{1/p'} \le C\left(\int_x^z \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\, ds\right)^{1+\beta-\alpha},$$

for all x, y, z, c, with $x < y < z < c$, where the constant C does not depend on x, y, z, c. If $v^q = u^p$, this is equivalent to

$$\Big(\int_x^y u\Big)^{1/q} \left(\int_y^z \frac{u^{(p/q)(1-p')}(s)}{(c-s)^{(1-\alpha)p'}}\, ds\right)^{1/p'} \le C\left(\int_x^z \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\, ds\right)^{1+\beta-\alpha}.$$

Defining $r = (p'+q)/p'$ and $r' = (p'+q)/q$, and keeping in mind that $1 + \beta - \alpha = 1/p' + 1/q$, the last inequality can be written as

$$(3.6) \qquad \Big(\int_x^y u\Big)^{1/r} \left(\int_y^z \frac{u^{1-r'}(s)}{(c-s)^{(1-\alpha)r'/(1+\beta-\alpha)}}\, ds\right)^{1/r'} \le C\left(\int_x^z \frac{1}{(c-s)^{(1-\alpha)/(1+\beta-\alpha)}}\, ds\right).$$

This is $A_r^+(g_c)$, where g_c is the function $(c-s)^{(\alpha-1)/(1+\beta-\alpha)}\chi_{(-\infty,c)}(s)$. Because of the "$p - \varepsilon$" property, the same inequality holds with r replaced by a smaller exponent r_1 that does not depend on c and with a constant that depends only on C and r_1 but not on x, y, z, c. On the other hand, it follows from Hölder's inequality that (3.6) holds with r replaced by $r_2 > r$ with the same constant. If we now follow the above chain of equivalences backwards we obtain $A^+_{p_i,q_i,\alpha,\beta}$ with $p_i = r_i/(r_i(\alpha-\beta)+1+\beta-\alpha)$, $q_i = r_i/(1+\beta-\alpha)$, where still $1/p_i - 1/q_i = \alpha - \beta$ $(i = 1, 2)$. It follows from Theorems 2.1 and 2.3 that the operator $M^+_{\alpha,\beta}$ is of weak type (p_1, q_1) and (p_2, q_2) with respect to the measures $u\,dx$, $v\,dx$. The interpolation theorem of Marcinkiewicz [SW] asserts that then $M^+_{\alpha,\beta}$ is of strong type (p_t, q_t) with

$$\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_2}, \qquad \frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_2}.$$

Choosing t so that $1/r = t/r_1 + (1-t)/r_2$, we get $p_t = p$, $q_t = q$. $\blacksquare$

4. Final remarks. In this section we want to point out some facts about $A^+_{p,q,\alpha,\beta}$ weights. First of all we observe that although the weak type characterization for $p < q$ holds without any relation between p, q on the one hand and α, β on the other, actually if $\alpha - \beta < 1/p - 1/q$ then the class $A^+_{p,q,\alpha,\beta}$ is empty. To see this one takes $a = c - 2h$, $b = c - h$ and then $A^+_{p,q,\alpha,\beta}$ implies

$$\left(\frac{1}{h}\int_{c-2h}^{c-h} u\right)^{1/q}\left(\frac{1}{h}\int_{c-h}^{c} v^{1-p'}\right)^{1/p'} \le Ch^{\beta-\alpha+1/p-1/q},$$

which is impossible unless $uv^{-1} = 0$ a.e. Furthermore, since the function $1/(c-s)^{(1-\alpha)p'}$ must be locally integrable near c, the exponent $(1-\alpha)p'$ must be less than 1, which means that $p > 1/\alpha$. Therefore there is always a critical index $p = 1/\alpha$ for which weak type fails, but still restricted weak type may hold. It is easy to see that if $\alpha = \beta = 1/p$ then Lebesgue measure satisfies the condition for restricted weak type (p,p), which is one of the main results in [JT].

We end our work by studying the relationship between the good weights for $M^+_{\alpha,\alpha}$ and those for M^+. We will fix $0 < \alpha < 1$ and simplify our notation writing $A^+_{p,\alpha}$ instead of $A^+_{p,p,\alpha,\alpha}$. The good weights for M^+ wil be denoted, as usual, by A^+_p.

The first observation is that if $(u,u) \in A^+_{p,\alpha}$ then

$$\left(\int_a^b u\right)^{1/p}\left(\int_b^c u^{1-p'}\right)^{1/p'}$$
$$\le \left(\int_a^b u\right)^{1/p}\left(\int_b^c \frac{u^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\right)^{1/p'} ds\,(c-a)^{1-\alpha} \le C(c-a).$$

Therefore $A^+_{p,\alpha} \subset A^+_p$. That this inclusion is proper can be seen by considering weights of the form $u(x) = |x|^\gamma$. It is easy to see that then (u,u) is in A^+_r if, and only if, $-1 < \gamma < r - 1$. Now if $r > \alpha p$ and we choose γ so that $\alpha p - 1 < \gamma < r - 1$, the pair (u,u) satisfies A^+_r while

$$\left(\int_{-b}^{0} u\right)^{1/p}\left(\int_0^b \frac{u^{1-p'}(s)\,ds}{(b-s)^{(1-\alpha)p'}}\right)^{1/p'}$$

is of the order of b raised to the exponent

$$\frac{\gamma+1}{p} + \frac{1+\gamma(1-p')-(1-\alpha)p'}{p'},$$

and the condition $\gamma > \alpha p - 1$ implies that this exponent is strictly greater than α. The conclusion is that $A^+_{p,\alpha}$ is not contained in A^+_r whenever $r > \alpha p$. This means that the natural candidate for the inclusion in the other direction is $A^+_{\alpha p}$. To see that $A^+_{\alpha p} \subset A^+_{p,\alpha}$ we will use the characterization of restricted weak type. If (u,v) is such that the operator M^+ is of restricted weak type (r,r), then Theorem 2.5 asserts that there exists C such that for any $x < y < z$ and $E \subset (y,z)$,

$$|E| \le C(z-x)\left(\frac{v(E)}{u(x,y)}\right)^{1/r}.$$

It follows that

$$\int_E \frac{ds}{(z-s)^{1-\alpha}} \le |E|^\alpha \le C(z-x)^\alpha\left(\frac{v(E)}{u(x,y)}\right)^{\alpha/r}.$$

This means that M^+_α is of restricted weak type (s,s) with $s = r/\alpha$. Therefore if $(u,u) \in A^+_s$ with $s < \alpha p$ then M^+ is of restricted weak type (s,s) and M^+_α is of restricted weak type $(s/\alpha, s/\alpha)$. Since $s/\alpha < p$, interpolation gives weak type (p,p). We have thus proved that $s < \alpha p$ implies $A^+_s \subset A^+_{\alpha,p}$. Finally, if $(u,u) \in A^+_{\alpha p}$, $\alpha p > 1$, there exists $s < \alpha p$ so that $(u,u) \in A^+_s$, which implies $A^+_{\alpha p} \subset A^+_{p,\alpha}$. The natural question of whether or not $A^+_{p,\alpha}$ is equal to $A^+_{\alpha p}$ is open. We have a positive answer for restricted weak type in the limit case, even for different weights. More precisely: let us write RA^+_p and $RA^+_{\alpha,p}$ for the classes of weights (u,v) for which restricted weak type (p,p) holds for the operators M^+ and M^+_α respectively. We have seen above that $RA^+_{\alpha s} \subset RA^+_{\alpha,s}$. We claim that in the limit case $\alpha s = 1$ they agree. Observe first that RA^+_1 is just A^+_1, because in this case weak type and restricted weak type are the same. It is then enough to check that $RA^+_{\alpha,1/\alpha}$ is A^+_1. But $RA^+_{\alpha,1/\alpha}$ means that there exists C such that for any $a < b < c$ and $E \subset (b,c)$,

$$\int_E \frac{ds}{(c-s)^{1-\alpha}} \le C(c-a)^\alpha\left(\frac{v(E)}{u(a,b)}\right)^\alpha.$$

Choosing $E = (b,c)$ we get

$$\frac{1}{c-a}\int_a^b u \le C\frac{1}{c-b}\int_b^c v.$$

Taking the limit when $c \downarrow b$ we have $(1/(b-a))\int_a^b u \le Cv(b)$ for a.e. b, which is A^+_1.

References

[A] K. F. Andersen, *Weighted inequalities for maximal functions associated with general measures*, Trans. Amer. Math. Soc. 326 (1991), 907–920.

[AM] K. F. Andersen and B. Muckenhoupt, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. 72 (1982), 9–26.

[AS] K. F. Andersen and E. T. Sawyer, *Weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators*, Trans. Amer. Math. Soc. 308 (1988), 547–557.

[CHS] A. Carbery, E. Hernandez and F. Soria, *Estimates for the Kakeya maximal operator and radial functions in* $\mathbb{R}^n$, in: Harmonic Analysis (Sendai, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, 41–50.

[JT] W. Jurkat and J. Troutman, *Maximal inequalities related to generalized a.e. continuity*, Trans. Amer. Math. Soc. 252 (1979), 49–64.

[KG] V. Kokilashvili and M. Gabidzashvili, *Two weight weak type inequalities for fractional type integrals*, Math. Inst. Czech. Acad. Sci. Prague 45 (1989).

[LT] M. Lorente and A. de la Torre, *Weighted inequalities for some one-sided operators*, Proc. Amer. Math. Soc. 124 (1996), 839–848.

[MOT] F. J. Martín-Reyes, P. Ortega Salvador and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. 319 (1990), 517–534.

[MPT] F. J. Martín-Reyes, L. Pick and A. de la Torre, A_∞^+ *condition*, Canad. J. Math. 45 (1993), 1231–1244.

[MT] F. J. Martín-Reyes and A. de la Torre, *Two weight norm inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. 117 (1993), 483–489.

[S] E. T. Sawyer, *Weighted inequalities for the one sided Hardy–Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53–61.

[SW] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Pres 1971.

Análisis Matemático
Facultad de Ciencias
Universidad de Málaga
29071 Málaga, Spain
E-mail: martin@anamat.cie.uma.es
torre@anamat.cie.uma.es

Received June 6, 1995
Revised version June 15, 1996 (3480)

STUDIA MATHEMATICA 122 (1) (1997)

Initial value problem for the time dependent Schrödinger equation on the Heisenberg group

by

JACEK ZIENKIEWICZ (Wrocław)

Abstract. Let L be the full laplacian on the Heisenberg group $\mathbb{H}^n$ of arbitrary dimension n. Then for $f \in L^2(\mathbb{H}^n)$ such that $(I-L)^{s/2} f \in L^2(\mathbb{H}^n)$, $s > 3/4$, for a $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}tL} f(x)|^2\, dx \le C_\phi \|f\|^2_{W^s}.$$

On the other hand, the above maximal estimate fails for $s < 1/4$. If Δ is the sublaplacian on the Heisenberg group $\mathbb{H}^n$, then for every $s < 1$ there exists a sequence $f_n \in L^2(\mathbb{H}^n)$ and $C_n > 0$ such that $(I-L)^{s/2} f_n \in L^2(\mathbb{H}^n)$ and for a $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}t\Delta} f_n(x)|^2\, dx \ge C_n \|f_n\|^2_{W^s}, \qquad \lim_{n\to\infty} C_n = +\infty.$$

Introduction. In his lectures *Some analytic problems related to statistical mechanics* [C] Lennart Carleson observed the following. Let H be a hamiltonian of a quantum system and let V_t be the time dependent Schrödinger group which describes the time evolution of the system $V_t f = e^{\sqrt{-1}tH} f$. Then for a general state $f \in \mathcal{H}$ although $\lim_{t\to 0} \|V_t f - f\|_{\mathcal{H}} = 0$, a better convergence like a.e. may not hold. Indeed, Carleson showed that if $\mathcal{H} = L^2(\mathbb{R})$ and the hamiltonian H is equal to d^2/dx^2, then there exists $f \in W^{1/8}$ for which $V_t f$ does not converge to f a.e. as $t \to 0$. On the other hand, he proved that if f belongs to the Sobolev space $W^{1/4+\varepsilon}$, $\varepsilon > 0$, then $\lim_{t\to 0} V_t f(x) = f(x)$ a.e.

The last theorem attracted a lot of attention. In 1983 Michael Cowling [Cw] put the Carleson theorem in a general framwork.

Let X be a measure space and H a self-adjoint, densely defined operator on $L^2(X)$. We introduce a scale of Sobolev spaces W^s, $s \in \mathbb{R}$, by

$$f \in W^s \quad \text{iff} \quad f \in L^2(X) \text{ and } |H|^{s/2} f \in L^2(X).$$

1991 *Mathematics Subject Classification*: Primary 22E30.