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Numerical curves and their applications to algebraic curves

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Abstract. Hermite interpolation by bivariate algebraic polynomials and its applications to some problems of the theory of algebraic curves, such as the existence of algebraic curves with given singularities, is considered. The scheme $\mathcal{N} = \{n_1, \ldots, n_s; n\}$, i.e., the sequence of multiplicities of nodes associated with the degree of interpolating polynomials, is considered. We continue the investigation of canonical decomposition of schemes and define so called maximal schemes. Some numerical results concerning the factorization of schemes are established. This leads to determination of irreducibility or to finding the (exact) number of components of algebraic curves as well as to the characterization of all singular points of a wide family of algebraic curves. Also, the Hilbert function of schemes is discussed. At the end, the problem of regularity of schemes depending on the number of interpolation conditions is considered.

1. Introduction. We define a *scheme* $\mathcal{N} = \{n_1, \ldots, n_s; n\}$ as a collection of nonnegative integers, where $n_1, \ldots, n_s$ are the *members*, n is the *degree* and s is the *length* of $\mathcal{N}$. We denote by $\mathcal{S}$ the set of all schemes. We agree that

$$\{n_1, \ldots, n_s; n\} = \{n_1, \ldots, n_s, 0, \ldots, 0; n\}$$

with an arbitrary (finite) number of zeros. So dealing with a finite number of schemes from $\mathcal{S}$, we may assume that they have the same length or, if necessary, that the length of a given scheme is great enough.

We need some notation. For $\mathcal{N} = \{n_1, \ldots, n_s; n\}, \mathcal{M} = \{m_1, \ldots, m_s; m\} \in \mathcal{S}$ and $\lambda \in \mathbb{Z}_+$ we define

$$\mathcal{N} + \mathcal{M} := \{n_1 + m_1, \ldots, n_s + m_s; n + m\}, \qquad \lambda\mathcal{N} := \{\lambda n_1, \ldots, \lambda n_s; \lambda n\},$$
$$\mathcal{N}' = \{n_1, \ldots, n_s\}, \qquad \operatorname{supp}\mathcal{N} := \{\nu : n_\nu \neq 0,\ 1 \le \nu \le s\}.$$

The inequality $\mathcal{N} \le \mathcal{M}$ means that $n \le m$ and $n_\nu \le m_\nu,\ \nu = 1, \ldots, s$.

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We call $\mathcal{N} \in \mathcal{S}$ an *interpolation scheme* if the following equality holds:

$$\sum_{\nu=1}^{s} \overline{n}_\nu = \overline{n+1}, \tag{1.1}$$

where $\overline{m} = 0 + \ldots + m$.

We denote the by $\mathcal{IS}$ set of all interpolation schemes.

Interpolation schemes with $s = 1$, i.e., the schemes $\{n+1; n\}$, are called *Taylor schemes*. Let $\pi_n(\mathbb{R}^2)$ be the space of bivariate polynomials of total degree $\le n$.

DEFINITION 1.1. For an interpolation scheme $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathcal{IS}$ and a node set $\mathcal{Z} = \{z_\nu = (x_\nu, y_\nu)\}_{\nu=1}^{s} \subset \mathbb{R}^2$ the (*regular*) *Hermite interpolation problem* $(\mathcal{N}, \mathcal{Z})$ is to find a unique polynomial $P \in \pi_n(\mathbb{R}^2)$ satisfying

$$\left.\frac{\partial^{i+j} P}{\partial x^i \partial y^j}\right|_{z_\nu} = \lambda_{i,j,\nu}, \quad i + j < n_\nu, \ \nu = 1, \ldots, s, \tag{1.2}$$

for a given collection of values $\Lambda = \{\lambda_{i,j,\nu} : i + j < n_\nu, \ \nu = 1, \ldots, s\}$.

In what follows, we briefly express equalities of the form (1.2) by writing

$$D^{\mathcal{N}'} P|_{\mathcal{Z}} = \Lambda.$$

We assume that there is no interpolation condition at nodes z_ν with $n_\nu = 0$.

Remark 1.2. The following statements are equivalent for any $\mathcal{N} \in \mathcal{IS}$:

(i) $(\mathcal{N}, \mathcal{Z})$ is *singular*, i.e. not regular.
(ii) There exists a polynomial P such that

$$P \in \pi_n(\mathbb{R}^2), \quad P \ne 0, \quad D^{\mathcal{N}'} P|_{\mathcal{Z}} = 0. \tag{1.3}$$

Since the determinant of the system (1.2) of linear equations (with respect to the unknown coefficients of P) is a polynomial in variables $x_1, y_1, \ldots, x_s, y_s$, the regularity of the problem $(\mathcal{N}, \mathcal{Z})$ for some $\mathcal{Z}$ implies that it is regular for almost all $\mathcal{Z} \in R^{2s}$ (with respect to the Lebesgue measure in $\mathbb{R}^{2s}$).

Remark 1.3. The following statements are equivalent for any $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathcal{S}$:

(i) $n_\nu \le n$ for $\nu = 1, \ldots, s$.
(ii) There exists a node set $\mathcal{Z} \subset \mathbb{R}^{2s}$ and a polynomial P such that (1.3) holds.

This, in particular, implies that the Taylor schemes are the only interpolation schemes which are regular for every node set (see [LL84] and [JS91] for more general results).

In view of Remark 1.2 the regularity and singularity of $(\mathcal{N}, \mathcal{Z})$ can be defined in the following more general way which enables us to remove the restriction that the scheme $\mathcal{N}$ is an interpolation scheme:

We call the problem $(\mathcal{N}, \mathcal{Z})$ *singular* if there exists a polynomial P satisfying (1.3); otherwise it is called *regular*.

Geometrically, the singularity of $(\mathcal{N}, \mathcal{Z})$ means that there exists an algebraic curve of degree $\le n$ passing through $\mathcal{Z}$ with multiplicity $\mathcal{N}'$ (i.e. passing through z_ν with multiplicity n_ν, $\nu = 1, \ldots, s$).

In what follows we will consider the singularity and regularity in this wider sense. Note that for interpolation schemes the two definitions of regularity and singularity coincide.

DEFINITION 1.4. We say that a scheme $\mathcal{N}$ is

(i) *regular* if $(\mathcal{N}, \mathcal{Z})$ is regular for some node set $\mathcal{Z}$,
(ii) *singular* if $(\mathcal{N}, \mathcal{Z})$ is singular for any node set $\mathcal{Z}$.

The problem of the full description of regular and singular schemes remains open.

Let us define the following "*less conditions*" class of schemes:

$$\mathrm{LC} := \Big\{\mathcal{M} = \{m_1, \ldots, m_s; m\} \in \mathcal{S} : \sum_{\nu=1}^{s} \overline{m}_\nu < \overline{m+1}\Big\}.$$

DEFINITION 1.5. A scheme $\mathcal{N}$ is called a *numerical curve* if there is a set $M \subset \mathrm{LC}$ such that

$$\mathcal{N} = \sum_{\mathcal{M} \in M} \mathcal{M}. \tag{1.4}$$

We denote the set of numerical curves by NC.

It is not hard to see that numerical curves are singular schemes. Indeed, for $\mathcal{N} = \mathcal{M} \in \mathrm{LC}$ and for every node set $\mathcal{Z}$, finding a polynomial $P = P_{\mathcal{M}}$ such that (1.3) holds reduces to solving a system of $\sum_{\nu=1}^{s} \overline{m}_\nu$ homogeneous linear equations in $\overline{m+1}$ unknowns. If $\mathcal{N}$ is a numerical curve with (1.4), then for every node set $\mathcal{Z}$ the polynomial

$$P := \prod_{\mathcal{M} \in M} P_{\mathcal{M}}$$

satisfies (1.3), which means that $\mathcal{N}$ is singular.

CONJECTURE 1.6 ([GHS90], [P92]). *Each singular scheme is a numerical curve.*

We have proved in [GHS92b] that this conjecture is true under the restriction that there are at most 9 knots with multiplicities ≥ 2.

DEFINITION 1.7. Let $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathcal{S}$.

(i) If $n_1 + n_2 \geq n+1$, $n_1 \leq n$ and $n_2 \leq n$, then the *reduction* of $\mathcal{N}$ with respect to the first two members is the scheme

$$\mathcal{N}^\times = \mathcal{N}^\times_{1,2} = \{n - n_2, n - n_1, n_3, \ldots, n_s; 2n - n_1 - n_2\}.$$

(ii) If

$$n_i + n_j \leq n, \quad 1 \leq i < j \leq 3, \tag{1.5}$$

then the *quadratic transformation* of $\mathcal{N}$ with respect to the first three members is the scheme (cf. [W50], Chapter 3, Theorem 7.2)

$$\mathcal{N}^* = \mathcal{N}^*_{1,2,3} = \{n - n_2 - n_3, n - n_1 - n_3, n - n_1 - n_2, n_4 \ldots, n_s; \\ 2n - n_1 - n_2 - n_3\},$$

It is not hard to check that

$$\mathcal{N}^\times = \mathcal{N}^\times_{1,2} = \{n_1 - r, n_2 - r, n_3, \ldots, n_s; n - r\},$$
$$\mathcal{N}^* = \mathcal{N}^*_{1,2,3} = \{n_1 - t, n_2 - t, n_3 - t, n_4, \ldots, n_s; n - t\},$$

with $r = n_1 + n_2 - n$ and $t = n_1 + n_2 + n_3 - n$.

We define a reduction or quadratic transformation with respect to other members in a similar way. We write

$$\operatorname{supp} Q = \{i, j, k\}$$

if Q is a quadratic transformation with respect to the members in the places $i,\ j,\ k$.

We say that a quadratic transformation Q is *positive* (for $\mathcal{N}$) if $\deg Q(\mathcal{N}) < \deg \mathcal{N}$, i.e. $t > 0$. Note that the positivity of Q (for $\mathcal{N}$) implies that

$$\operatorname{supp} Q \subset \operatorname{supp} \mathcal{N}. \tag{1.6}$$

We have the following

THEOREM 1.8 (see [GHS92]). *Let $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathcal{S}$.*

(i) *If $n_1 + n_2 = n + r \geq n+1$, $n_1 \leq n$ and $n_2 \leq n$, then the schemes $\mathcal{N}$ and $\mathcal{N}^\times$ are simultaneously singular or not.*

(ii) *If $n_i + n_j \leq n$ for $1 \leq i < j \leq 3$, then the schemes $\mathcal{N}$ and $\mathcal{N}^*$ are simultaneously singular or not.*

This theorem reduces the investigation of an arbitrary scheme to the following two cases:

1) $n_{\nu_0} \geq n+1$ for some ν_0,
2) $n_i + n_j + n_k \leq n$ for $1 \leq i < j < k \leq s$.

In the first case the scheme is obviously regular (Remark 1.3).

DEFINITION 1.9. A scheme $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathcal{S}$ satisfying the above condition 2) is called *basic*.

The set of basic schemes is denoted by BS.

CONJECTURE 1.10 [GHS90]. *A basic scheme is singular if and only if it belongs to* LC.

We have proved that Conjecture 1.10 is true in the case

$$\sum_{\nu: n_\nu > 1} n_\nu \leq 3n, \tag{1.7}$$

and that the above two conjectures are equivalent (see [GHS92b], and also [GHS95] for the wider concept of regularity).

Let us define the *intersection product* of schemes as follows:

$$\langle \mathcal{N}, \mathcal{M} \rangle := \sum_{\nu=1}^{s} n_\nu m_\nu - nm,$$

and set

$$\langle \mathcal{N} \rangle := \langle \mathcal{N}, \mathcal{N} \rangle = \sum_{\nu=1}^{s} n_\nu^2 - n^2, \qquad [\mathcal{N}] := \sum_{\nu=1}^{s} n_\nu - 3n.$$

The schemes $\mathcal{N}$ and $\mathcal{M}$ are called *orthogonal* ($\mathcal{N} \perp \mathcal{M}$) if $\langle \mathcal{N}, \mathcal{M} \rangle = 0$. We say that a set $\mathcal{A}$ of schemes is *orthogonal* if $\langle \mathcal{N}, \mathcal{M} \rangle = 0$ for any distinct $\mathcal{N}, \mathcal{M} \in \mathcal{A}$.

Denote by $\overline{\mathcal{N}}$ the difference of the number of conditions of $\mathcal{N}$ and the dimension of the space of algebraic curves of degree n, i.e. $\dim \pi_n(\mathbb{R}^2) - 1$:

$$\overline{\mathcal{N}} := \sum_{\nu=1}^{s} \overline{n}_\nu - \overline{n+1} + 1.$$

Thus, $\overline{\mathcal{N}} \leq 0$ means that $\mathcal{N} \in$ LC, and $\overline{\mathcal{N}} = 1$ if $\mathcal{N}$ is an interpolation scheme. Let us mention the following identities:

$$\begin{aligned} \langle \mathcal{N} + \mathcal{M} \rangle &= \langle \mathcal{N} \rangle + \langle \mathcal{M} \rangle + 2\langle \mathcal{N}, \mathcal{M} \rangle, \\ \overline{\mathcal{N} + \mathcal{M}} &= \overline{\mathcal{N}} + \overline{\mathcal{M}} + \langle \mathcal{N}, \mathcal{M} \rangle. \end{aligned} \tag{1.8}$$

Note also that

$$\overline{\mathcal{N}} = (\langle \mathcal{N} \rangle + [\mathcal{N}])/2. \tag{1.9}$$

It is not hard to check the following properties of quadratic transformation and reduction (see [GHS92b], [GHS95]):

LEMMA 1.11. *Let $\mathcal{N}, \mathcal{M} \in \mathcal{S}$.*

(i) $(\mathcal{N}^*)^* = \mathcal{N}$.
(ii) $\langle \mathcal{N}, \mathcal{M} \rangle = \langle \mathcal{N}^*, \mathcal{M}^* \rangle$.
(iii) $[\mathcal{N}] = [\mathcal{N}^*]$, $\langle \mathcal{N} \rangle = \langle \mathcal{N}^* \rangle$, $\overline{\mathcal{N}} = \overline{\mathcal{N}^*}$.
(iv) $[\mathcal{N}] = [\mathcal{N}^\times] - r$, $\langle \mathcal{N} \rangle = \langle \mathcal{N}^\times \rangle + r^2$, $\overline{\mathcal{N}} = \overline{\mathcal{N}^\times} + \overline{r-1}$.

DEFINITION 1.12. (i) The schemes $\mathcal{N}, \mathcal{M}$ are called *quadratically equivalent* ($\mathcal{N} \sim \mathcal{M}$) if one of them can be obtained from the other by means of quadratic transformations.

(ii) We say that $\mathcal{N}$ *reduces* to $\mathcal{M}$ ($\mathcal{N} \rightarrow \mathcal{M}$) if $\mathcal{M}$ can be obtained from $\mathcal{N}$ by means of reduction and (or) quadratic transformations.

Theorem 1.8 implies

Remark 1.13. If a scheme $\mathcal{N}$ reduces to a scheme $\mathcal{M}$, then they are simultaneously regular or not.

Let us denote by $E_{p,q}$ the scheme $\{e_1, \ldots, e_s; e\}$ with

$$e_p = e_q = e = 1, \quad e_\nu = 0 \quad \text{for } \nu = 1, \ldots, s, \ \nu \neq p, q.$$

Now we give the definition of prime numerical curves which play an essential role in our investigation.

A scheme which is quadratically equivalent to some $E_{p,q}$ is called a *prime numerical curve*. We denote the class of prime numerical curves by PNC:

$$\mathrm{PNC} := \{\mathcal{N} \in \mathcal{S} : \mathcal{N} \sim E_{p,q} \text{ for some } p, q\}.$$

Note that in view of Lemma 1.11, for $A \in \mathrm{PNC}$, we have

$$[A] = -1, \quad \langle A \rangle = 1, \quad \overline{A} = 0. \tag{1.10}$$

DEFINITION 1.14. (i) A scheme which is quadratically equivalent to some basic scheme is called an *e-basic scheme*.

(ii) A scheme which can be reduced to some basic scheme is called an *r-basic scheme*.

We denote by BS^* and $\mathrm{BS}^{\times *}$ the classes of e-basic and r-basic schemes respectively. Of course we have $\mathrm{BS} \subset \mathrm{BS}^* \subset \mathrm{BS}^{\times *}$. Note that Theorem 1.8 and Remark 1.3 imply

Remark 1.15. If $\mathcal{N}$ is singular then $\mathcal{N} \in \mathrm{BS}^{\times *}$.

The results reported below in this section are proved in [GHS95]. The following theorem shows, in particular, that the quadratic transformation can be applied to any triplet of members of a prime numerical curve different from $E_{p,q}$ and of an e-basic scheme (setting $A = E_{p,q}$).

THEOREM 1.16. *If A, B are distinct prime numerical curves and $\mathcal{M}$ is an e-basic scheme, then*

(i) $\langle A, B \rangle \leq 0$,
(ii) $\langle A, \mathcal{M} \rangle \leq 0$.

Part (i) of the following theorem gives a characterization of e-basic schemes, and (ii) readily follows from (i).

THEOREM 1.17. (i) *A scheme $\mathcal{N}$ is e-basic if and only if $\langle A, \mathcal{N} \rangle \leq 0$ for all $A \in \mathrm{PNC}$.*

(ii) *The sum of e-basic schemes is e-basic.*

The next theorem establishes the canonical decomposition of r-basic schemes:

THEOREM 1.18. *Let $\mathcal{N} \in \mathrm{BS}^{\times *}$. Then there exists a finite set $\Omega_{\mathcal{N}}$ of prime numerical curves, an e-basic scheme $\mathcal{N}^{\downarrow}$ and natural numbers $\mu_A = \mu_{A,\mathcal{N}}$ such that*

$$\mathcal{N} = \sum_{A \in \Omega_{\mathcal{N}}} \mu_A A + \mathcal{N}^{\downarrow}, \quad \{\mathcal{N}^{\downarrow}\} \cup \Omega_{\mathcal{N}} \text{ is orthogonal.} \tag{1.11}$$

Moreover, $\mathcal{N} \rightarrow \mathcal{N}^{\downarrow}$ and the decomposition (1.11) is unique.

In view of (1.8), from (1.11) we have

$$\overline{\mathcal{N}} = \sum_{A \in \Omega_{\mathcal{N}}} \overline{\mu_A - 1} + \overline{\mathcal{N}^{\downarrow}}. \tag{1.12}$$

Note that a prime numerical curve A belongs to $\Omega_{\mathcal{N}}$ in the canonical decomposition (1.11) if and only if $\langle A, \mathcal{N} \rangle > 0$; moreover, we then have $\mu_A = \langle A, \mathcal{N} \rangle$.

We also have

$$\mathcal{N}^{\downarrow} \in \mathrm{LC} \quad \text{and} \quad \langle \mathcal{N}^{\downarrow} \rangle \leq 0 \quad \text{if } \mathcal{N} \in \mathrm{NC}. \tag{1.13}$$

2. Some numerical results. We start with a result on factorization of r-basic schemes.

THEOREM 2.1. *Suppose a scheme $\mathcal{N} \in \mathrm{BS}^*$ with $\overline{\mathcal{N}} = 0$ has a representation*

$$\mathcal{N} = \mathcal{N}_1 + \ldots + \mathcal{N}_q, \quad \mathcal{N}_i \in \mathrm{NC}, \ q \geq 2. \tag{2.1}$$

Then

$$\mathcal{N} \sim \lambda C_0, \tag{2.2}$$

where C_0 is a scheme of degree 3 having nine nonzero members equal to 1 each, and λ is the greatest common divisor of the members and the degree of $\mathcal{N}$. Moreover, we have

$$\mathcal{N}_i \sim \lambda_i C_0, \quad \sum_{i=1}^{q} \lambda_i = \lambda. \tag{2.3}$$

For the proof we need the following lemma (cf. [GHS95]).

LEMMA 2.2. *Suppose that $\mathcal{N} = \{n_1, \ldots, n_s; n\}$ is a basic scheme with $\sum_{\nu=1}^{s} n_\nu \leq 3n$. Then $\langle \mathcal{N} \rangle \leq 0$, equality being possible if and only if $\mathcal{N} = \lambda C_0$ or $\mathcal{N} = \lambda\{1; 1\}$ for some $\lambda \in \mathbb{Z}_+$.*

Proof. Suppose the members of $\mathcal{N}$ are in decreasing order. Then

$$\begin{aligned} n^2 &\ge n(n_1+n_2+n_3)-(n-n_1)(n_1-n_2)-(n-n_2)(n_2-n_3)\\ &= n_1^2+n_2^2+n_3(3n-n_1-n_2)\\ &\ge n_1^2+n_2^2+n_3(n_3+\ldots+n_s)\ge n_1^2+\ldots+n_s^2. \end{aligned}$$

It is easy to see that $\langle\mathcal{N}\rangle=0$ only if either $n_1=\ldots=n_s=n/3$ and $s=9$, or $n_1=n$ and $n_2=\ldots=n_s=0$.

In particular, in view of (1.9), (1.10) and Lemma 1.11, Lemma 2.2 implies

COROLLARY 2.3. (i) *Suppose* $\mathcal{N}\in\mathrm{BS}^*$, $\overline{\mathcal{N}}=0$ *and* $\langle\mathcal{N}\rangle=0$. *Then* $\mathcal{N}\sim\lambda C_0$.

(ii) $\mathrm{PNC}\cap\mathrm{BS}=\emptyset$.

Proof of Theorem 2.1. Since the sum of numerical curves is a numerical curve, we can suppose that there are only two summands in (2.1). Also, in order to prove (2.3) we can suppose that one of the two summands coincides with $\mathcal{N}_i$, for any fixed i, $1\le i\le q$.

Thus, we assume $q=2$ in (2.1). Consider the canonical decompositions

$$\mathcal{N}_i=\sum_{A\in\Omega_i}\mu_A A+\mathcal{N}_i^{\downarrow},\qquad i=1,2,$$

where (see (1.13))

$$\mathcal{N}_i^{\downarrow}\in\mathrm{BS}^*\cap\mathrm{LC},\quad \langle\mathcal{N}_i^{\downarrow}\rangle\le 0,\quad i=1,2. \tag{2.4}$$

Thus, we get the representation

$$\mathcal{N}=\sum_{A\in\Omega_3}\mu_A A+\mathcal{M}, \tag{2.5}$$

with $\Omega_3:=\Omega_1\cup\Omega_2$ and $\mathcal{M}:=\mathcal{N}_1^{\downarrow}+\mathcal{N}_2^{\downarrow}$. According to Theorem 1.17 we have $\mathcal{M}\in\mathrm{BS}^*$. Using identities (1.8) and Cauchy's inequality we obtain

$$\overline{\mathcal{M}}=\overline{\mathcal{N}_1^{\downarrow}}+\overline{\mathcal{N}_2^{\downarrow}}+2\langle\mathcal{N}_1^{\downarrow},\mathcal{N}_2^{\downarrow}\rangle\le 0. \tag{2.6}$$

Equality here holds if and only if

$$\overline{\mathcal{N}_1^{\downarrow}}=\overline{\mathcal{N}_2^{\downarrow}}=\langle\mathcal{N}_1^{\downarrow}\rangle=\langle\mathcal{N}_2^{\downarrow}\rangle=0,\quad \mathcal{N}_1=\gamma\mathcal{N}_2. \tag{2.7}$$

Note also that in this case, according to Corollary 2.3, we have

$$\mathcal{N}_i^{\downarrow}\sim\gamma_i C_0,\qquad i=1,2. \tag{2.8}$$

Since $\mathcal{N}\in\mathrm{BS}^*$, the canonical decomposition of $\mathcal{N}$ is trivial: $\mathcal{N}=\mathcal{N}$. Hence the decomposition (2.5) cannot be orthogonal if $\Omega_3\ne\emptyset$. On the other hand, we can get an orthogonal and hence the canonical decomposition of $\mathcal{N}$ by changing the representation (2.5) as follows:

1) If there is $B\in\Omega_3$ such that $\langle B,\mathcal{M}\rangle<0$, then

$$\mathcal{N}=\sum_{A\in\Omega_3,\,A\ne B}\mu_A A+(\mu_B-1)B+\mathcal{M}_1,$$

with $\mathcal{M}_1:=\mathcal{M}+B$.

2) If there are $B,D\in\Omega_3$ such that $\langle B,D\rangle<0$, then

$$\mathcal{N}=\sum_{A\in\Omega_3,\,A\ne B,D}\mu_A A+(\mu_B-1)B+(\mu_D-1)D+\mathcal{M}_2,$$

with $\mathcal{M}_2:=\mathcal{M}+B+D$.

In both cases, using Theorem 1.17 and identities (1.8) we get

$$\mathcal{M}_i\in\mathrm{BS}^*,\quad \langle\mathcal{M}_i\rangle\le 0,\quad \overline{\mathcal{M}_i}<0,\quad i=1,2.$$

Thus, after a finite number of steps we will obtain the canonical decomposition $\mathcal{N}=\mathcal{N}$ with $\overline{\mathcal{N}}<0$. This contradiction proves that the set Ω_3 of prime numerical curves in (2.5) is empty and $\mathcal{N}=\mathcal{M}=\mathcal{N}_1+\mathcal{N}_2=\mathcal{N}_1^{\downarrow}+\mathcal{N}_2^{\downarrow}$. Since $\overline{\mathcal{N}}=0$, we have equality in (2.6) and therefore (2.7) and (2.8) hold. Theorem 2.1 is proved.

From now on we will also use the canonical decomposition of an r-basic scheme $\mathcal{N}$ with a slightly changed last summand:

$$\mathcal{N}=\sum_{A\in\Omega_{\mathcal{N}}}\mu_A A+\mu^{\Downarrow}\mathcal{N}^{\Downarrow}, \tag{2.9}$$

where $\mu^{\Downarrow}=\lambda$ and $\mathcal{N}^{\Downarrow}=\frac{1}{\lambda}\mathcal{N}^{\downarrow}$ if $\mathcal{N}^{\downarrow}\sim\lambda C_0$, and $\mu^{\Downarrow}=1$ and $\mathcal{N}^{\Downarrow}=\mathcal{N}^{\downarrow}$ otherwise.

The following theorem shows that the canonical decomposition of a numerical curve is in a certain sense the unique representation as a sum of numerical curves, provided that $\overline{\mathcal{N}^{\downarrow}}=0$.

THEOREM 2.4. *Suppose a scheme* $\mathcal{N}\in\mathrm{BS}^{\times *}$ *with* $\overline{\mathcal{N}^{\downarrow}}=0$ *has a representation*

$$\mathcal{N}=\mathcal{N}_1+\ldots+\mathcal{N}_q,\qquad \mathcal{N}_i\in\mathrm{NC},\ q\ge 2.$$

Then $\mathcal{N}$ *is divisible by each* $\mathcal{N}_i$, *i.e.*

$$\mathcal{N}_i=\sum_{A\in\Omega_{\mathcal{N}}}\mu_{i,A}A+\mu_i^{\Downarrow}\mathcal{N}^{\Downarrow},$$

with nonnegative coefficients satisfying

$$\sum_{i=1}^{q}\mu_{i,A}=\mu_A,\qquad \sum_{i=1}^{q}\mu_i^{\Downarrow}=\mu^{\Downarrow}.$$

Proof. As in the proof of Theorem 2.1 we suppose that $q = 2$ and consider the canonical decompositions

$$\mathcal{N}_i = \sum_{A\in\Omega_i} \mu_A A + \mathcal{N}_i^{\downarrow}, \quad i = 1,2,$$

where again

$$\mathcal{N}_i^{\downarrow} \in \mathrm{BS}^* \cap \mathrm{LC}, \quad \langle \mathcal{N}_i^{\downarrow}\rangle \le 0, \quad i = 1,2.$$

Then we get the representation

$$\mathcal{N} = \sum_{A\in\Omega_3} \mu_A A + (\mathcal{N}_1^{\downarrow} + \mathcal{N}_2^{\downarrow}) \tag{2.10}$$

with $\Omega_3 := \Omega_1 \cup \Omega_2$. Changing (2.10) as in the proof of Theorem 2.1 we find that for some set $D \subset \Omega_3$ the representation

$$\mathcal{N} = \sum_{A\in\Omega_3\setminus D} \mu_A A + \mathcal{N}^{\downarrow}$$

is canonical, where $\mathcal{N}^{\downarrow} = \mathcal{N}_1^{\downarrow} + \mathcal{N}_2^{\downarrow} + \sum_{A\in D}\mu_A A$. Now, in view of the condition $\overline{\mathcal{N}^{\downarrow}} = 0$, Theorem 2.1 implies that $D = \emptyset$ and $\mathcal{N}_i^{\downarrow} = 0$ or $\mathcal{N}_i^{\downarrow} \sim \lambda_i C_0$, for $i = 1,2$. This completes the proof.

The following theorem announced in [H94] establishes the interesting fact that for every $\mathcal{N} \in \mathrm{BS}^*$ there is only one (up to the order of members) basic scheme, denoted by $\mathcal{N}^b = \{n_1^b,\ldots,n_s^b;n^b\}$, such that $\mathcal{N} \sim \mathcal{N}^b$.

THEOREM 2.5. *If basic schemes $\mathcal{N}$ and $\mathcal{M}$ are quadratically equivalent, then they have the same degrees and the same members, maybe in different order.*

First we need two lemmas.

LEMMA 2.6. *Suppose $A = \{\alpha_1,\ldots,\alpha_s;\alpha\} \in \mathrm{PNC}$ $(\alpha > 1)$.*

(i) *There exist quadratic transformations Q_i, $i = 1,\ldots,l$, and $p,q \in \operatorname{supp} A$ such that*

$$Q_l\ldots Q_1 A = E_{p,q}, \quad Q_i \text{ is positive}, \ i = 1,\ldots,l. \tag{2.11}$$

(ii) *If $\alpha_p = \alpha_q = 1$ for some p,q then there exist quadratic transformations Q_i, $i = 1,\ldots,l$, with*

$$\operatorname{supp} Q_i \cap \{p,q\} = \emptyset, \quad i = 1,\ldots,l, \tag{2.12}$$

such that (2.11) *holds.*

(iii) *If $\alpha_p = 1$, then for some $q \in \operatorname{supp} A$, $q \ne p$, there exist quadratic transformations Q_i, $i = 1,\ldots,l$, with*

$$\operatorname{supp} Q_i \cap \{p\} = \emptyset, \quad i = 1,\ldots,l, \tag{2.13}$$

such that (2.11) *holds.*

Proof. The statement (i) follows from Theorem 1.16(i) and the fact that the sum of the three maximal members of every prime numerical curve is greater than its degree (see Corollary 2.3).

To prove (ii) and (iii) we use induction on $\deg A$. The case $\deg A = 1$ is trivial. Suppose $\alpha > 1$ and the lemma holds if $\deg A < \alpha$.

Note first that each prime numerical curve with degree greater than one has at least five nonzero members. This follows from the fact that the unique scheme from which $\{1,1;1\}$ can be obtained by means of quadratic transformations is $\{1,1,1,1,1;2\}$, and every prime numerical curve can be transformed, without enlarging its support, to $\{1,1;1\}$.

Therefore, we can choose the three maximal members of A such that the quadratic transformation Q_1 with this triplet satisfies (2.12) or (2.13) with $i = 1$. According to Corollary 2.3(ii), Q_1 is positive, i.e. $\deg(Q_1 A) < \alpha$, and the induction hypothesis completes the proof.

The following lemma completes Theorem 1.16(ii) from [GHS95].

LEMMA 2.7. *Suppose $\mathcal{N} = \{n_1,\ldots,n_s;n\} \in \mathrm{BS}$ and $A = \{\alpha_1,\ldots,\alpha_s;\alpha\} \in \mathrm{PNC}$. Then*

$$\langle \mathcal{N}, A\rangle \le \max_{1\le i<j\le s}(n_i + n_j - n).$$

More precisely, we have

$$\langle \mathcal{N}, A\rangle \le \langle \mathcal{N}, E_{p,q}\rangle = n_p + n_q - n \tag{2.14}$$

provided that (2.11) *holds; moreover, equality holds in* (2.14) *if and only if*

$$Q_i\mathcal{N} = \mathcal{N}, \quad i = 1,\ldots,l. \tag{2.15}$$

Proof. It is enough to show that for every $B \in \mathrm{PNC}$ and any quadratic transformation Q which is positive for B, we have

$$\langle \mathcal{N}, B\rangle \le \langle \mathcal{N}, QB\rangle \tag{2.16}$$

and equality holds iff $Q\mathcal{N} = \mathcal{N}$.

Suppose that $\operatorname{supp} Q = \{i,j,k\}$, $B = \{\beta_1,\ldots,\beta_s;\beta\}$ and

$$\tau := \beta_i + \beta_j + \beta_k - \beta > 0, \quad t := n_i + n_j + n_k - n \le 0.$$

Then

$$\begin{aligned}\langle \mathcal{N}, QB\rangle &= n_1(\beta_1 - \tau) + n_2(\beta_2 - \tau) + n_3(\beta_3 - \tau) - n(\beta - \tau) + \sum_{\nu=4}^{s} n_\nu \beta_\nu \\ &= \langle \mathcal{N}, B\rangle - \tau t \ge \langle \mathcal{N}, B\rangle.\end{aligned}$$

Equality holds iff $t = 0$, i.e. $Q\mathcal{N} = \mathcal{N}$.

Lemmas 2.6 and 2.7 imply

COROLLARY 2.8. *Suppose $\mathcal{N} \in \mathrm{BS}$, $\mathcal{N} \neq 0$, $A \in \mathrm{PNC}$ and $\langle \mathcal{N}, A\rangle = 0$. Then, taking into account only nonzero members, we have either $\mathcal{N} = \{n_1, n_2; n_1 + n_2\}$ and $A = E_{1,2}$, or $\mathcal{N} = \{n_1; n_1\}$ and $A = E_{1,q}$.*

COROLLARY 2.9. *Suppose $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathrm{BS}$ and $A \in \mathrm{PNC}$.*

(i) *If A has a member equal to one outside the support of $\mathcal{N}$, then*

$$\langle \mathcal{N}, A\rangle \le \max_{1\le i\le s} n_i - n.$$

(ii) *If A has two members equal to one outside the support of $\mathcal{N}$, then*

$$\langle \mathcal{N}, A\rangle \le -n.$$

Let us define:

- E — the scheme with zero members and with degree one;
- χ_p — the scheme with zero degree and zero members except for the pth member which is one;
- $E_p := \chi_p + E$.

Proof of Theorem 2.5. Let $\mathcal{N} = \{n_1, \ldots, n_s; n\}$, $\mathcal{M} = \{m_1, \ldots, m_s; m\}$ and

$$\mathcal{N} = T_1 \ldots T_k \mathcal{M}, \tag{2.17}$$

where T_i are quadratic transformations. We can suppose that

$$\operatorname{supp} T_i \subset \{1, \ldots, s\}, \qquad i = 1, \ldots, l. \tag{2.18}$$

Let us prove first that $n = m$. By Lemma 1.11(ii) and Corollary 2.9(ii) we have

$$-n = \langle E_{s+1,s+2}, \mathcal{N}\rangle = \langle T_k \ldots T_1 E_{s+1,s+2}, \mathcal{M}\rangle = \langle A, \mathcal{M}\rangle \le -m, \tag{2.19}$$

where

$$A = \{\alpha_1, \ldots, \alpha_{s+2}; \alpha\} := T_k \ldots T_1 E_{s+1,s+2} \in \mathrm{PNC} \tag{2.20}$$

with $\alpha_{s+1} = \alpha_{s+2} = 1$. Similarly we get $m \ge n$. Hence we have equality in (2.19) and Lemma 2.7 implies

$$Q_i \mathcal{M} = \mathcal{M}, \qquad i = 1, \ldots, l, \tag{2.21}$$

where the positive quadratic transformations Q_i are defined by Lemma 2.6(i) (with $s+2$ instead of s, $p = s+1$, $q = s+2$):

$$Q_l \ldots Q_1 A = E_{s+1,s+2}, \qquad \operatorname{supp} Q_i \subset \{1, \ldots, s\}. \tag{2.22}$$

From (2.17) and (2.20)–(2.22) we obtain

$$\mathcal{M} = Q_l \ldots Q_1 T_k \ldots T_1 \mathcal{N}, \qquad E_{s+1,s+2} = Q_l \ldots Q_1 T_k \ldots T_1 E_{s+1,s+2},$$

and hence $E = Q_l \ldots Q_1 T_k \ldots T_1 E$. Therefore for each $\gamma \in \mathbb{Z}_+$ we have

$$\mathcal{M}_\gamma = Q_l \ldots Q_1 T_k \ldots T_1 \mathcal{N}_\gamma, \tag{2.23}$$

where

$$\mathcal{N}_\gamma := \{n_1, \ldots, n_s; n + \gamma\} = \mathcal{N} + \gamma E,$$
$$\mathcal{M}_\gamma := \{m_1, \ldots, n_s; m + \gamma\} = \mathcal{M} + \gamma E.$$

Thus, we have (2.17) with sufficiently large $n = m$ and it remains to prove that

$$n_\nu = m_\nu, \qquad \nu = 1, \ldots, s.$$

Let $\#\operatorname{supp}\mathcal{N} \ge \#\operatorname{supp}\mathcal{M}$. We will prove the above equalities by induction on $\#\operatorname{supp}\mathcal{N}$. The case $\#\operatorname{supp}\mathcal{N} = 0$ is already proved ($n = m$). Let n_{i_0} and m_{j_0} be the maximal members of $\mathcal{N}$ and $\mathcal{M}$ respectively. In view of the relation (2.23) we can assume that

$$4n_{i_0} \le n, \quad 4m_{j_0} \le m \quad (n = m). \tag{2.24}$$

According to Lemma 1.11(ii) and Corollary 2.9(i) we have

$$\begin{aligned} n_{i_0} - n = \langle E_{i_0,s+1}, \mathcal{N}\rangle &= \langle T_k \ldots T_1 E_{i_0,s+1}, \mathcal{M}\rangle = \langle A, \mathcal{M}\rangle \\ &\le m_{j_0} - m = m_{j_0} - n, \end{aligned} \tag{2.25}$$

where (see (2.18))

$$A = \{\alpha_1, \ldots, \alpha_{s+1}; \alpha\} := T_k \ldots T_1 E_{i_0,s+1} \in \mathrm{PNC}, \tag{2.26}$$

with $\alpha_{s+1} = 1$. Thus, $n_{i_0} \le m_{j_0}$. Similarly we get $m_{j_0} \le n_{i_0}$. Hence we have equality in (2.25) and Lemma 2.7 implies that

$$Q_i \mathcal{M} = \mathcal{M}, \qquad i = 1, \ldots, l, \tag{2.27}$$

where the positive quadratic transformations Q_i are defined by Lemma 2.6(ii) (with $s+1$ instead of s, $p = s+1$):

$$Q_l \ldots Q_1 A = E_{j_0,s+1}, \qquad \operatorname{supp} Q_i \subset \{1, \ldots, s\}.$$

This, combined with (2.17), (2.26) and (2.27), implies

$$\begin{aligned} \mathcal{M} &= Q_l \ldots Q_1 T_k \ldots T_1 \mathcal{N}, \\ E_{j_0,s+1} &= Q_l \ldots Q_1 T_k \ldots T_1 E_{i_0,s+1}, \end{aligned} \tag{2.28}$$

and since the supports of all transformations in (2.28) are in $\{1, \ldots, s\}$,

$$E_{j_0} = Q_l \ldots Q_1 T_k \ldots T_1 E_{i_0}. \tag{2.29}$$

From (2.28) and (2.29), taking into account that $n_{i_0} = m_{j_0}$, we get

$$\mathcal{M}_0 := \mathcal{M} - m_{j_0} E_{j_0} = Q_l \ldots Q_1 T_k \ldots T_1 (\mathcal{N} - n_{i_0} E_{i_0}) =: \mathcal{N}_0.$$

It follows from (2.24) that $\mathcal{N}_0$ and $\mathcal{M}_0$ are basic schemes and since

$$\#\operatorname{supp}\mathcal{M}_0 \le \#\operatorname{supp}\mathcal{N}_0 = \#\operatorname{supp}\mathcal{N} - 1,$$

we can use the induction hypothesis to complete the proof.

LEMMA 2.10. *Suppose* $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathrm{BS}^*$, $A = \{\alpha_1, \ldots, \alpha_s; \alpha\} \in$ PNC *and*

$$\langle A, \mathcal{N} \rangle = 0, \quad n_p = 0, \quad \alpha_p \neq 0, \tag{2.30}$$

for some p, $1 \le p \le s$. *Then* $\mathcal{N} \sim \lambda E_q$ *with some* q, *where* λ *is the greatest common divisor of the members and the degree of* $\mathcal{N}$ *and*

$$A = \left\{ \frac{n_1}{\lambda}, \ldots, \frac{n_{p-1}}{\lambda}, 1, \frac{n_{p+1}}{\lambda}, \ldots, \frac{n_s}{\lambda}; \frac{n}{\lambda} \right\} = \frac{1}{\lambda}\mathcal{N} + \chi_p. \tag{2.31}$$

Proof. Suppose $\mathcal{N}^b = T_k \ldots T_1\mathcal{N}$, where T_i are positive quadratic transformations with $p \notin \operatorname{supp} T_i$. The transformation T_i is applicable to each prime numerical curve which has a nonzero member outside $\operatorname{supp} T_i$, hence we can consider the scheme

$$B = \{\beta_1, \ldots, \beta_s; \beta\} := T_k \ldots T_1 A. \tag{2.32}$$

We have

$$\langle B, \mathcal{N}^b \rangle = 0, \quad \mathcal{N}^b \in \mathrm{BS},\ B \in \mathrm{PNC},\ \beta_p = \alpha_p \neq 0,\ n_p^b = 0.$$

According to Corollary 2.8 we have

$$\mathcal{N}^b = n^b E_q, \quad B = E_{p,q} = \frac{1}{n^b}\mathcal{N}^b + \chi_p.$$

This, combined with (2.32), completes the proof.

COROLLARY 2.11. *For every scheme* $\mathcal{N} = \{n_1, \ldots, n_s; n\}$ *with* $\mathcal{N} \sim \lambda E_q$ *there exist exactly* $s - \#\operatorname{supp}\mathcal{N}$ *prime numerical curves* A *satisfying*

$$\langle A, \mathcal{N} \rangle = 0, \quad \operatorname{supp} A \subset [1, s], \quad \operatorname{supp} A \setminus \operatorname{supp}\mathcal{N} \neq \emptyset.$$

Moreover, these schemes are orthogonal to each other and are of the following form:

$$A = \frac{1}{\lambda}\mathcal{N} + \chi_p, \quad p \in \{1, \ldots, s\} \setminus \operatorname{supp}\mathcal{N}. \tag{2.33}$$

(The orthogonality of the set (2.33) follows from the equality $\langle \mathcal{N} \rangle = \lambda\langle E_q \rangle = 0$.)

In the next theorem we consider the problem of extension of the set $\Omega_{\mathcal{N}}$ in the canonical decomposition and its maximal possible cardinality.

THEOREM 2.12. *Let* $\{\mathcal{N}\} \cup \mathcal{A}$ *be an orthogonal set of schemes with* $\mathcal{N} \in \mathrm{BS}^*$, $\mathcal{A} \subset \mathrm{PNC}$ *and*

$$\operatorname{supp}\mathcal{A} \subset \operatorname{supp}\mathcal{N} \quad \text{if } \mathcal{N}^b = \lambda E_q. \tag{2.34}$$

Then

$$\#\mathcal{A} \le d(\mathcal{N}) \tag{2.35}$$

$$:= \begin{cases} \#\operatorname{supp}\mathcal{N} - 1 & \text{if } \mathcal{N} \sim \{m_1, m_2; m_1 + m_2\},\ m_1 m_2 \neq 0; \\ \#\operatorname{supp}\mathcal{N} - \#\operatorname{supp}\mathcal{N}^b & \text{otherwise.} \end{cases}$$

Moreover, there exists $\mathcal{A}' \subset \mathrm{PNC}$ *with (2.34) such that* $\mathcal{A} \subset \mathcal{A}'$, $\{\mathcal{N}\} \cup \mathcal{A}'$ *is orthogonal and* $\#\mathcal{A}' = d(\mathcal{N})$.

Remark 2.13. Note that if $\mathcal{N}^b \neq \lambda E_q$, then the inclusion in (2.34) holds by Lemma 2.10.

Proof of Theorem 2.12. Suppose $\mathcal{N}^b = T_k \ldots T_1\mathcal{N}$, where $\mathcal{N}^b \in$ BS and T_i are positive quadratic transformations. Then

$$n_i \neq 0 \quad \text{for } i \in I := \operatorname{supp} T_1. \tag{2.36}$$

We use induction on k. The case $k = 0$, i.e. $\mathcal{N} = \mathcal{N}^b \in$ BS, follows from Corollary 2.8.

Suppose now the theorem holds for the scheme $T_1\mathcal{N} = T_2 \ldots T_k\mathcal{N}^b$. Then $(T_1\mathcal{N})^b = \mathcal{N}^b$ and we denote

$$T_1\mathcal{N} = \{n_1^*, \ldots, n_s^*; n^*\}.$$

For $i \in I$ we set $\{p_i, q_i\} := I \setminus \{i\}$ and

$$I_0 := \{i \in I : \langle \mathcal{N}, E_{p_i,q_i} \rangle = 0\} = \{i \in I : n_i^* = 0\}.$$

Thus, we have

$$d(T_1\mathcal{N}) + \#I_0 = d(\mathcal{N}). \tag{2.37}$$

Consider the following families of schemes:

$$\mathcal{A}_0 := \bigcup_{i \in I_0} E_{p_i,q_i}, \quad \mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0.$$

According to Theorem 1.16 the transformation T_1 is applicable to each scheme $A \in \mathcal{A}_1$ and in view of Lemma 1.11(ii) the set $\{T_1\mathcal{N}\} \cup T_1(\mathcal{A}_1)$ is orthogonal, where

$$T_1(\mathcal{A}_1) := \mathcal{B}_1 = \{T_1 A : A \in \mathcal{A}_1\}.$$

Suppose first that $\mathcal{N} \not\sim \lambda E_q$. Then using the induction hypothesis and (2.37) we obtain (2.35):

$$\#\mathcal{A} \le \#T_1(\mathcal{A}_1) + \#I_0 \le d(T_1\mathcal{N}) + \#I_0 = d(\mathcal{N}).$$

On the other hand, there exists a set $\mathcal{B}_1'$ of prime numerical curves such that $\mathcal{B}_1 \subset \mathcal{B}_1'$, $\{T_1\mathcal{N}\} \cup \mathcal{B}_1'$ is orthogonal and $\#\mathcal{B}_1' = d(T_1\mathcal{N})$. Consider the set

$$\mathcal{A}' := T_1(\mathcal{B}_1') \cup \mathcal{A}_0 \subset \mathrm{PNC}.$$

(T_1 is applicable to any scheme from $\mathcal{B}_1'$, since $E_{p_i,q_i} \in \mathcal{B}_1'$ with $i \in I_0$ implies $n_{p_i}^* + n_{q_i}^* = n^*$, i.e. $n_i = 0$, which contradicts (2.36).)

It is easy to see that $\mathcal{A} \subset \mathcal{A}'$ and $\#\mathcal{A}' = d(\mathcal{N})$. Let us check the orthogonality of $\{\mathcal{N}\} \cup \mathcal{A}'$. In view of Lemma 1.11(ii) we only need to prove that

$$\langle E_{p_i,q_i}, TB \rangle = 0 \quad \text{if } i \in I_0,\ B = \{\beta_1, \ldots, \beta_s; \beta\} \in \mathcal{B}_1'. \tag{2.38}$$

For $i \in I_0$ we have $n_i^* = 0$ and since $\langle B, T_1\mathcal{N}\rangle = 0$, Lemma 2.10 implies that $\operatorname{supp} B \subset \operatorname{supp}(T_1\mathcal{N})$, i.e. $\beta_i = 0$, which is equivalent to (2.38).

Consider now the case $\mathcal{N}^b = \lambda E_q$ for some λ and q. We define

$$\mathcal{B}_{1,1} = \{B \in \mathcal{B}_1 : \operatorname{supp} B \subset \operatorname{supp}(T_1\mathcal{N})\}, \quad \mathcal{B}_{1,2} = \mathcal{B}_1 \setminus \mathcal{B}_{1,1}.$$

It follows from Lemma 2.10 and (2.34) that

$$\mathcal{B}_{1,2} = \{B_i : i \in I_1\}, \quad \text{where} \quad B_i := \frac{1}{\lambda} T_1\mathcal{N} + \chi_i,\ I_1 \subset I_0. \tag{2.39}$$

The set $\{T_1\mathcal{N}\} \cup \mathcal{B}_{1,1}$ of schemes is orthogonal and according to the induction hypothesis there exists a set $\mathcal{B}'_{1,1}$ of prime numerical curves such that $\mathcal{B}_{1,1} \subset \mathcal{B}'_{1,1}$, $\{T_1\mathcal{N}\} \cup \mathcal{B}'_{1,1}$ is orthogonal, $\#\mathcal{B}'_{1,1} = d(T_1\mathcal{N})$ and

$$\operatorname{supp} \mathcal{B}'_{1,1} \subset \operatorname{supp}(T_1\mathcal{N}). \tag{2.40}$$

Consider the set

$$\mathcal{A}'' := \bigcup_{i \in I_0 \setminus I_1} E_{p_i,q_i} \cup T_1(\mathcal{B}'_{1,1}) \cup T_1(\mathcal{B}_{1,2}).$$

From (2.37) and (2.39) we have (2.34) for $\mathcal{A}'$ and

$$\#\mathcal{A}'' = d(T_1\mathcal{N}) + \#I_0 = d(\mathcal{N}).$$

The orthogonality of the set $\mathcal{N} \cup \mathcal{A}''$ can be checked as in the previous case, using (2.39) and (2.40).

It remains to prove that $\mathcal{A} \subset \mathcal{A}''$. Since $\mathcal{A}_1 = T_1(\mathcal{B}_1) \subset T_1(\mathcal{B}_{1,1}) \cup T_1(\mathcal{B}_{1,2})$, we only need to show that $E_{p_i,q_i} \in \mathcal{A} \cap \mathcal{A}_0$ implies $i \in I_0 \setminus I_1$. Indeed, for any $A \in \mathcal{A}_1$ we have $\langle E_{p_i,q_i}, A\rangle = 0$ and hence $i \notin \operatorname{supp} T_1(A)$. Since the ith member of B_i equals one (see (2.39)), this means that $B_i \notin \mathcal{B}$, i.e. $i \in I_0 \setminus I_1$. Theorem 2.12 is proved.

We define

$$\operatorname{supp} \mathcal{A} := \bigcup_{A \in \mathcal{A}} \operatorname{supp} A,$$

where $\mathcal{A}$ is a set of schemes.

THEOREM 2.14. *Let $\mathcal{A}$ be an orthogonal set of prime numerical curves with $\operatorname{supp} \mathcal{A} \subset U \subset \mathbb{Z}_+$ and*

$$d_U(\mathcal{A}) := sup\{\#\mathcal{A}' : \mathcal{A} \subset \mathcal{A}',\ \operatorname{supp} \mathcal{A}' \subset U,\ \mathcal{A}' \textit{ is orthogonal}\}.$$

Then either $d_U(\mathcal{A}) = \#U - 1$ or $d_U(\mathcal{A}) = \#U$.

Proof. We use induction on $\#U$. It is easy to see that $d_U(\mathcal{A}) = 1, 3, 3$ if $\#U = 2, 3, 4$ respectively. Suppose $\#U \geq 5$. Two cases are possible:

Case 1: *There are $p, q \in U$ such that*

$$E_{p,q} \perp \mathcal{A} \quad or \quad E_{p,q} \in \mathcal{A}. \tag{2.41}$$

Let Q be an arbitrary quadratic transformation with $\{p, q\} \subset I := \operatorname{supp} Q \subset U$. Define

$$\mathcal{A}_0 := \{E_{p_i,q_i} : i \in I_0\}, \quad \mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0,$$

where $I_0 := \{i \in I : E_{p_i,q_i} \perp \mathcal{A} \text{ or } E_{p,q} \in \mathcal{A}\}$, $\#I_0 \leq 3$. The quadratic transformation Q is applicable to any scheme from $\mathcal{A}_1$ and according to Lemma 1.11(ii) the set

$$\mathcal{B} = Q(\mathcal{A}_1) := \{Q(A) : A \in \mathcal{A}_1\}$$

is orthogonal. On the other hand, we have

$$\operatorname{supp} \mathcal{B} \subset U_1 := U \setminus I_0, \quad \#U_1 = \#U - \#I_0 < \#U.$$

Therefore the induction hypothesis implies that $d_{U_1}(\mathcal{B})$ is either $\#U_1 - 1$ or $\#U$ and there exists an orthogonal set $\mathcal{B}'$ such that

$$\mathcal{B} \subset \mathcal{B}' \subset \mathrm{PNC}, \quad \operatorname{supp} \mathcal{B}' \subset U_1, \quad \#\mathcal{B}' = d_{U_1}(\mathcal{B}). \tag{2.42}$$

Define

$$\mathcal{A}' := Q(\mathcal{B}') \cup \mathcal{A}_0.$$

Since $I_0 \cap \operatorname{supp} \mathcal{B}' = \emptyset$, it follows that $\mathcal{A}_0 \perp Q(\mathcal{B}')$, which combined with Lemma 1.11(ii) means that $\mathcal{A}'$ is an orthogonal set. On the other hand, we have

$$\mathcal{A} \subset \mathcal{A}', \quad \#\mathcal{A}' = \mathcal{B}' + \#\mathcal{A}_0 = d_{U_1}(\mathcal{B}) + \#I_0,$$

hence $\#\mathcal{A}'$ is either $\#U - 1$ or $\#U$. To complete the proof in Case 1 it remains to show that there is no prime numerical curve A with

$$A \perp \mathcal{A}', \quad \operatorname{supp} A \subset U. \tag{2.43}$$

Indeed, if (2.43) holds, then $A \notin \mathcal{A}_0$, hence Q is applicable to A and

$$Q(A) \perp \mathcal{B}', \quad \operatorname{supp} Q(A) \subset U_1,$$

which contradicts (2.42).

Case 2: *There are no $p, q \in U$ satisfying* (2.41). Then any quadratic transformation Q with $\operatorname{supp} Q \subset U$ is applicable to any scheme from $\mathcal{A}$ and $\operatorname{supp} Q(\mathcal{A}) = \operatorname{supp} \mathcal{A}$. Moreover, as in Case 1, we can prove that $\mathcal{A}$ is maximal on U if and only if $Q(\mathcal{A})$ is, i.e. $d_U(\mathcal{A}) = d_U(Q(\mathcal{A}))$. This means that using a finite number of quadratic transformations we can reduce Case 2 to Case 1 (since any prime numerical curve is quadratically equivalent to $\{1, 1; 1\}$). Theorem 2.14 is proved.

DEFINITION 2.15. A scheme $\mathcal{N}$ with canonical decomposition (2.9) is called *maximal* if $\mathcal{N}^{\downarrow} \not\sim \lambda E_q$ and $\#\Omega_{\mathcal{N}} = d(\mathcal{N})$.

Using Theorem 2.12 and following the proof of the analog of Bezout's theorem for numerical curves ([GHS95], Theorem 5.7) we can improve it as follows:

THEOREM 2.16. *Let $\mathcal{N}, \mathcal{M}$ be numerical curves.*

(i) *If $\langle \mathcal{N}, \mathcal{M}\rangle > 0$, then $\mathcal{N}$ and $\mathcal{M}$ have a common prime numerical curve in their canonical decompositions.*

(ii) *If $\langle \mathcal{N}, \mathcal{M}\rangle \geq 0$ and one of $\mathcal{N}, \mathcal{M}$ is maximal, then $\mathcal{N}$ and $\mathcal{M}$ have a common scheme in their canonical decompositions.*

3. Standard and general quadratic transformations of polynomials. We will call the points

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

of $\mathbb{R}^3$ the *standard fundamental points* and the coordinate planes the *standard fundamental planes.*

Let $\overline{\pi}_n(\mathbb{R}^3)$ be the space of homogeneous polynomials of degree n in three variables and let $P \in \overline{\pi}_n(\mathbb{R}^3)$:

$$P(x) = P(x_1, x_2, x_3) = \sum_{i_1+i_2+i_3=n} a_{i_1 i_2 i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}. \tag{3.1}$$

It is not hard to check the following

Remark 3.1. The order of the zero of the polynomial P at the fundamental point e_i is the difference of $\deg P$ and the highest degree of x_i in (3.1).

Suppose now that P has zeros of order $\geq n_i$ at e_i, $i = 1,2,3$. Then P can be represented in the following form (with fixed i, $\{j,k\} = \{1,2,3\} \setminus \{i\}$):

$$P(x) = \sum_{\nu=n_i}^{n} x_i^{n-\nu} A_\nu(x_j, x_k),$$

where A_ν is a bivariate homogeneous polynomial of order ν. Therefore we have

$$P(x_2x_3, x_1x_3, x_1x_2) = x_1^{n_1} x_2^{n_2} x_3^{n_3} P^*(x_1, x_2, x_3) \tag{3.2}$$

for some $P^* \in \overline{\pi}_{n-t}(\mathbb{R}^3)$, where

$$t = n_1 + n_2 + n_3 - n, \qquad t \in \mathbb{Z}.$$

Now we have the following (cf. [W50] and [GHS92]):

DEFINITION 3.2. The polynomial $Q(P) = P^*$ defined by (3.2) is called the *standard* (n_1, n_2, n_3)*-quadratic transformation* of P.

Setting $x_1 = x_2x_3$, $x_2 = x_1x_3$, $x_3 = x_1x_2$ in (3.2) we get

$$x_1^{n_1^*} x_2^{n_2^*} x_3^{n_3^*} P(x_1, x_2, x_3) = P^*(x_2x_3, x_1x_3, x_1x_2), \tag{3.3}$$

where $n_i^* := n_i - t$. Using Remark 3.1, relations (3.2), (3.3) and the fact that the degree of x_i, $i = 1,2,3$, on the left hand side of (3.2) is $\leq n$, and the one on the right hand side of (3.3) is $\leq n^* := n - t$, we obtain

Remark 3.3. The standard quadratic transformation has the following properties:

1. The polynomial $Q(P)$ has a zero of order not less than n_i^* at e_i, $i = 1,2,3$.
2. For every $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ with $u_1u_2u_3 \neq 0$ the order of the zero of P at u is the same as the order of the zero of $Q(P)$ at $Q(u) := (u_2u_3, u_1u_3, u_1u_2)$.
3. The order of the zero of P at e_i is greater than n_i if and only if $Q(P)$ has a factor x_i ($i = 1,2,3$).
4. The order of the zero of $Q(P)$ at e_i is greater than n_i^* if and only if P has a factor x_i ($i = 1,2,3$).
5. The polynomial P has a factor different from x_i ($i = 1,2,3$) if and only if $Q(P)$ has such a factor.
6. The (n_1^*, n_2^*, n_3^*)-quadratic transformation of $Q(P)$ is P.

Suppose now v_1, v_2, v_3 are arbitrary points from $\mathbb{R}^3$ with $\mathrm{vol}_3[0, v_1, v_2, v_3] \neq 0$ and L is a linear transformation of $\mathbb{R}^3$ with $L(e_i) = v_i$, $i = 1,2,3$, $(\det L \neq 0)$.

DEFINITION 3.4. The polynomial

$$Q_{v_1,v_2,v_3}(P) := L^{-1}Q(LP) \quad \text{ with } LP(x) := P(L(x))$$

is called the (general) (n_1, n_2, n_3)*-quadratic transformation* of P with respect to the points v_1, v_2, v_3.

The points v_1, v_2, v_3 and the planes $H_{i,j}$ passing through $0, v_i, v_j$, $1 \leq i < j \leq 3$, are called *fundamental* for the transformation Q_{v_1,v_2,v_3}.

Let the fundamental planes be given by the equalities

$$h_{i,j}(x_1, x_2, x_3) = 0, \qquad 1 \leq i < j \leq 3.$$

It is not hard to see that

(i) A polynomial P has a factor $h_{i,j}$ iff LP has a factor x_k, $\{i,j,k\} = \{1,2,3\}$.

(ii) A point x belongs to a general fundamental plane iff $L(x)$ belongs to the corresponding standard fundamental plane.

(iii) The order of the zero of LP at x equals the order of the zero of P at $L(x)$.

Using these properties we easily get from Remark 3.3

Remark 3.5. The general quadratic transformation has the following properties:

1. The polynomial $Q_{v_1,v_2,v_3}(P)$ has a zero of order not less than $n_i^* := n_i - t$ at v_i, $i = 1, 2, 3$.

2. For every $u = (u_1, u_2, u_3)$ outside the fundamental planes the order of the zero of P at u is the same as that of $Q_{v_1,v_2,v_3}(P)$ at $Q_{v_1,v_2,v_3}(u) := L^{-1}Q(L(u))$.

3. The order of the zero of P at v_i is greater than n_i if and only if $Q_{v_1,v_2,v_3}(P)$ has a factor $h_{j,k}$, $\{i,j,k\} = \{1,2,3\}$.

4. The order of the zero of $Q_{v_1,v_2,v_3}(P)$ at v_i is greater than n_i^* if and only if P has a factor $h_{j,k}$.

5. The polynomial P has a factor different from $h_{i,j}$ if and only if $Q_{v_1,v_2,v_3}(P)$ has such a factor.

6. The (n_1^*, n_2^*, n_3^*)-quadratic transformation of $Q_{v_1,v_2,v_3}(P)$ with respect to v_1, v_2, v_3 is P.

4. The Hilbert function of schemes. For every scheme $\mathcal{N} = \{n_1, \ldots, n_s; n\}$ and $\tau = 0, 1, \ldots$ we define

$$\mathcal{N}_{+\tau} := \{n_1, \ldots, n_s, n_{s+1}, \ldots, n_{s+\tau}; n\},$$

where $n_{s+1} = \ldots = n_{s+\tau} = 1$, and set

$$R(\mathcal{N}) := \min\{\tau : \mathcal{N}_{+\tau} \text{ is regular}\}, \quad 0 \le R(\mathcal{N}) \le \overline{n+1}.$$

It is easy to see that $R(\mathcal{N})$ coincides with the minimal (with respect to the node set $\mathcal{T}$) dimension of the space of algebraic curves of degree $\le n$ passing through $\mathcal{T}$ with multiplicity $\ge \mathcal{N}'$.

Definition 4.1. The *Hilbert function* $h(\mathcal{N})$ is defined by the equality

$$h(\mathcal{N}) := \overline{n+1} - R(\mathcal{N}), \quad \mathcal{N} \in S.$$

We have $R(\mathcal{N}) = 0$ and $h(\mathcal{N}) = \overline{n+1}$ for a regular scheme $\mathcal{N}$.

Note that $h(\mathcal{N})$ coincides with the maximal (with respect to $\mathcal{T}$) number of independent conditions, given by $\mathcal{N}$, i.e. in (1.3).

Let us denote by $\operatorname{mult}_x C$ the multiplicity of $x \in \mathbb{R}^3$ in an algebraic curve C and by $\operatorname{mult}_{\mathcal{T}} C$ the sequence $\{\operatorname{mult}_x C : x \in \mathcal{T}\}$, where $\mathcal{T}$ is a node set.

Remark 4.2. Note that $R(\mathcal{N}) = 1$ means that there is a set $U_{\mathcal{N}} \subset \mathbb{R}^{3s}$ whose complement has measure zero such that for every node set $\mathcal{T} \in U_{\mathcal{N}}$ there is a unique algebraic curve $\mathcal{N}^{\mathcal{T}}$ of degree n such that

$$\operatorname{mult}_{\mathcal{T}} \mathcal{N}^{\mathcal{T}} \ge \mathcal{N}'. \tag{4.1}$$

Indeed, if V is the set of $\mathcal{T}$'s for which there are at least two distinct curves satisfying (4.1), then $(\mathcal{N}_{+1}, \mathcal{T} \cup \{w\})$ is singular for $\mathcal{T} \in V$ and all $w \in \mathbb{R}^3$. Since the scheme $\mathcal{N}_{+1}$ is regular, therefore $\operatorname{vol}_{3s} V = 0$.

Theorem 4.3. *For every scheme $\mathcal{N} = \{n_1, \ldots, n_s; n\} \in \mathrm{BS}^{\times *}$ we have*

$$h(\mathcal{N}) = \overline{n+1} - R(\mathcal{N}^{\downarrow}) = \overline{n+1} - \overline{n^{\downarrow}+1} + \sum_{\nu=1}^{s} \overline{n_\nu^{\downarrow}} \tag{4.2}$$

$$= \sum_{\nu=1}^{s} \overline{n_\nu} - \sum_{A \in \Omega_{\mathcal{N}}} \overline{\mu_A - 1};$$

the second equality holds provided that Conjecture 1.10 *is true.*

The proof of the theorem follows from the next two lemmas.

Lemma 4.4. *If $\mathcal{N} \to \mathcal{M}$, then $R(\mathcal{N}) = R(\mathcal{M})$.*

Proof. According to Theorem 1.8, if $\mathcal{N}$ reduces to $\mathcal{M}$, then they are both singular or both regular. On the other hand, we have $\mathcal{N}_\tau \to \mathcal{M}_\tau$ for all $\tau \ge 0$.

Lemma 4.5. *If $\mathcal{N} \in \mathrm{BS}^*$ and Conjecture* 1.10 *is true, then*

$$h(\mathcal{N}) = \sum_{\nu=1}^{s} \overline{n}_\nu. \tag{4.3}$$

Proof. Note first that if (4.3) holds and $\mathcal{M} = \{m_1, \ldots, m_s; m\} \sim \mathcal{N}$, then

$$h(\mathcal{M}) = \sum_{\nu=1}^{s} \overline{m}_\nu.$$

Indeed,

$$h(\mathcal{M}) = \overline{m+1} - R(\mathcal{M}) = \overline{m+1} - R(\mathcal{N}) = \overline{m+1} + h(\mathcal{N}) - \overline{n+1}$$

$$= \overline{m+1} + \sum_{\nu=1}^{s} \overline{n}_\nu - \overline{n+1} = \overline{m+1} + \overline{\mathcal{N}} - 1$$

$$= \overline{m+1} + \overline{\mathcal{M}} - 1 = \sum_{\nu=1}^{s} \overline{m}_\nu.$$

Therefore it is enough to prove (4.3) for the basic scheme $\mathcal{N} \in \mathrm{BS}$, provided that Conjecture 1.10 is true. This follows from the fact that in this case the minimal τ for which $\mathcal{N}_{+\tau}$ is regular is determined by $\overline{\mathcal{N}}_{+\tau} = 1$.

Indeed, except for the schemes of the form (4.4) below, $\mathcal{N} \in \mathrm{BS}$ implies $\mathcal{N}_{+\tau} \in \mathrm{BS}$. And if $\mathcal{N}$ is of the form

$$1)\ \{n; n\}, \quad 2)\ \{n-1, ; n\}, \quad 3)\ \{n_1, n_2; n_1 + n_2\},\ n_1 n_2 \ne 0, \tag{4.4}$$

then we get this using reductions in the first case, quadratic transformations in the second case and one quadratic transformation in the third case (see Theorem 1.8).

5. Some geometrical results. In this section we consider applications of numerical curves to algebraic curves. We start with establishing the connection between reducibility of polynomials and schemes.

THEOREM 5.1. *Suppose a scheme $\mathcal{N}$ is singular and for every node set $\mathcal{T} \in F \subset \mathbb{R}^{3s}$, where $\mathrm{vol}_{3s} F > 0$, there is a reducible polynomial $P = P_\mathcal{T}$ satisfying* (1.3). *Then*

$$\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2, \tag{5.1}$$

where $\mathcal{N}_i \neq 0$ is singular.

Proof. Denote by D the set of all decompositions of $\mathcal{N}$ of the form (5.1). Obviously D is a finite set. Let $\mathcal{T} \in F$ and $P_\mathcal{T} = P_1 P_2$. Since $\deg P = \deg P_1 + \deg P_2$ and the multiplicity of P at each point equals the sum of the corresponding multiplicities of P_1 and P_2, we have $\mathcal{N} = \mathcal{N}_{1,\mathcal{T}} + \mathcal{N}_{2,\mathcal{T}}$, where $(\mathcal{N}_{i,\mathcal{T}}, \mathcal{T})$ is singular. This means that

$$F \subset \bigcup_{(\mathcal{N}_1,\mathcal{N}_2)\in D} (S_{\mathcal{N}_1} \cap S_{\mathcal{N}_2}), \tag{5.2}$$

where

$$S_\mathcal{M} := \{\mathcal{T} \in \mathbb{R}^{3s} : (\mathcal{M}, \mathcal{T}) \text{ is singular}\}.$$

On the other hand, $\mathrm{vol}_{3s} S_\mathcal{M} = 0$ if $\mathcal{M}$ is regular. Thus, if each couple of D contains a regular scheme, then

$$\mathrm{vol}_{3s}(S_{\mathcal{N}_1} \cap S_{\mathcal{N}_2}) = 0$$

for all $(\mathcal{N}_1, \mathcal{N}_2) \in D$, which contradicts (5.2), since $vol_{3s} F > 0$. Theorem 5.1 is proved.

We will say that a quadratic transformation Q is *applicable* for the node set $\mathcal{T} = \{t_1, \ldots, t_s\} \subset \mathbb{R}^3$ if there is no plane containing 0 and the nodes $\{t_\nu : \nu \in \mathrm{supp}\, Q\}$. In this case we define

$$Q(\mathcal{T}) := \{Q(t_1), \ldots, Q(t_s)\}.$$

For $A \in \mathrm{PNC}$ with $A = Q_1 \ldots Q_l E_{p,q}$ we denote by G_A the collection of node sets $\mathcal{T}$ such that

(i) Q_i is applicable for $Q_{i-1} \ldots Q_1 \mathcal{T}$, $i = 1, \ldots, l$;

(ii) the plane passing through the pth and qth nodes of the node set $Q_l \ldots Q_1(\mathcal{T})$ and through 0 does not contain any other node of that set.

Obviously, the complement of G_A in $\mathbb{R}^{3s}$ has measure zero (in fact, it is a union of surfaces).

THEOREM 5.2. *Suppose $A = \{\alpha_1, \ldots, \alpha_s; \alpha\} \in \mathrm{PNC}$ and $\mathcal{T} \in G_A$. Then there exists a unique algebraic curve $A^\mathcal{T}$ of degree α with*

$$\mathrm{mult}_\mathcal{T} A^\mathcal{T} \geq A'. \tag{5.3}$$

Moreover,

1) $\mathrm{mult}_\mathcal{T} A^\mathcal{T} = A'$.
2) *There is no singular point of $A^\mathcal{T}$ outside $\mathcal{T}$.*
3) *$A^\mathcal{T}$ is quadratically equivalent to a line and hence is irreducible.*

Proof. Suppose that $A = Q_1 \ldots Q_l E_{p,q}$. We use induction on l. The case $l = 0$ is obvious. Suppose the theorem holds for $Q_1 A$ and $Q_1(\mathcal{T}) \in G_{Q_1 A}$.

Consider the algebraic curve C of degree α satisfying (5.3), i.e.

$$\mathrm{mult}_\mathcal{T} C \geq A'$$

(the existence of C follows from $A \in \mathrm{LC}$). If 1) does not hold, then in view of Remark 3.5(3) we will have a reducible curve with multiplicity $(Q_1 A)'$ on the node set $Q_1(\mathcal{T})$, which contradicts the induction hypothesis.

On the other hand, C cannot be reducible by Remark 3.5(4, 5) and the induction hypothesis. In a similar way we obtain the uniqueness of C.

To check 2), note that in view of (1.10) we have

$$\sum_{\nu=1}^{s} \alpha_i(\alpha_i - 1) = (\alpha - 2)(\alpha - 1). \tag{5.4}$$

Now, if the algebraic curve C has a singular point outside $\mathcal{T}$ with multiplicity $\alpha_0 \geq 2$, then from (5.4) we get

$$\sum_{\nu=0}^{s} \alpha_i(\alpha_i - 1) > (\alpha - 2)(\alpha - 1),$$

which means that C is reducible (see [W50], Theorem 4.4). Theorem 5.2 is proved.

In the next theorem we generalize this result for an orthogonal subset of PNC.

THEOREM 5.3. *Suppose $\mathcal{A}$ is an orthogonal set of prime numerical curves,*

$$\mathcal{T} \in G_\mathcal{A} := \bigcap_{A\in\mathcal{A}} G_A, \quad \mathcal{M} = \sum_{A\in\mathcal{A}} \lambda_A A,$$

where $\lambda_A \in \mathbb{Z}_+$. Then

(i) *There exists a unique algebraic curve $\mathcal{M}^\mathcal{T}$ with*

$$\mathrm{mult}_\mathcal{T} \mathcal{M}^\mathcal{T} \geq \mathcal{M}'. \tag{5.5}$$

(ii) *For this unique curve we have equality in* (5.5), *and the curve consists of the components $A^\mathcal{T}$ with multiplicities λ_A, i.e.*

$$\mathcal{M}^\mathcal{T} = \sum_{A\in\mathcal{A}} \lambda_A A^\mathcal{T}. \tag{5.6}$$

(iii) *If $x \in \mathcal{M}^{\mathcal{T}} \setminus \mathcal{T}$ then there exists a unique $A \in \mathcal{A}$ such that $x \in A^{\mathcal{T}}$; moreover*,

$$\mathrm{mult}_x \mathcal{M}^{\mathcal{T}} = \mathrm{mult}_x A^{\mathcal{T}} = \lambda_A.$$

Proof. For algebraic curves C_1, C_2 and a node set $\mathcal{T}$ we define

$$\langle C_1, C_2\rangle_{\mathcal{T}} := \sum_{x\in\mathcal{T}} \mathrm{mult}_x C_1 \mathrm{mult}_x C_2 - \deg C_1 \deg C_2.$$

Suppose $\mathcal{T} \in G_{\mathcal{A}}$. Since $\mathcal{M}$ is a numerical curve (and hence a singular scheme), there exists an algebraic curve C of degree $\deg \mathcal{M}$ such that $\mathrm{mult}_{\mathcal{T}} C \geq \mathcal{M}'$. Then for fixed $B \in \mathcal{A}$ with $\lambda_B \geq 1$ we have $\langle C, B^{\mathcal{T}}\rangle_{\mathcal{T}} \geq \langle \mathcal{M}, B\rangle = \lambda_B$. Therefore, $B^{\mathcal{T}}$, which is irreducible, is a component of C with multiplicity $\lambda \geq 1$ by Bezout's theorem.

Let us now prove that

$$\lambda \geq \lambda_B. \tag{5.7}$$

Suppose $\lambda < \lambda_B$ and $C_1 := C - \lambda B^{\mathcal{T}}$. Then Theorem 5.2 implies that

$$\mathrm{mult}_{\mathcal{T}} C_1 \geq \sum_{A\in\mathcal{A}} \lambda_A A' - \lambda B',$$

and hence

$$\langle C_1, B^{\mathcal{T}}\rangle_{\mathcal{T}} \geq \Big\langle \sum_{A\in\mathcal{A}} \lambda_A A - \lambda B, B\Big\rangle = \Big\langle \sum_{A\in\mathcal{A}} \lambda_A A, B\Big\rangle - \lambda = \lambda_B - \lambda > 0.$$

By Bezout's theorem this contradicts the fact that $B^{\mathcal{T}}$ is not a component of C_1.

Using the equality

$$\deg C = \deg \mathcal{M} = \sum_{A\in\mathcal{A}} \lambda_A \deg A,$$

we deduce from (5.7) that for every $A \in \mathcal{A}$ the multiplicity of $A^{\mathcal{T}}$ in the algebraic curve C equals λ_A, this curve is unique and satisfies (5.6). Equality in (5.5) follows from Theorem 5.2.

Statement (iii) of the theorem follows from the fact that $\langle A_1^{\mathcal{T}}, A_2^{\mathcal{T}}\rangle_{\mathcal{T}} = \langle A_1, A_2\rangle = 0$ for $A_1, A_2 \in \mathcal{A}$, and hence no two of the curves $A^{\mathcal{T}}$ for $A \in \mathcal{A}$ have an intersection point outside $\mathcal{T}$. Theorem 5.3 is proved.

Remark 5.4. Note that if $\lambda_A \leq 1$ for all $A \in \mathcal{A}$, then $\mathcal{M}^{\mathcal{T}}$ has no singular point outside $\mathcal{T}$.

In the following theorem we discuss the problem of irreducibility of algebraic curves with given singularities.

THEOREM 5.5. *Suppose Conjecture* 1.10 *is true and $\mathcal{N} \in \mathrm{BS}^*$, $\overline{\mathcal{N}} = 0$. Then there is a set $G_{\mathcal{N}} \subset \mathbb{R}^{3s}$ whose complement has measure zero such that for every node set $\mathcal{T} \in G_{\mathcal{N}}$*:

(i) *There is a unique algebraic curve $\mathcal{N}^{\mathcal{T}}$ of degree n with $\mathrm{mult}_{\mathcal{T}} \mathcal{N}^{\mathcal{T}} \geq \mathcal{N}'$.*

(ii) *The curve $\mathcal{N}^{\mathcal{T}}$ has a unique irreducible component with multiplicity λ if $\mathcal{N} \sim \lambda C_0$, $\lambda \geq 2$, and is irreducible otherwise.*

Proof. Note first that the assumptions of the theorem imply $R(\mathcal{N}) = 1$. Hence, according to Remark 4.2 we have (i) for every $\mathcal{T}$ from a set $U_{\mathcal{N}}$ whose complement in $\mathbb{R}^{3s}$ has measure zero. In the case $\mathcal{N} \not\sim \lambda C_0$ for $\lambda \geq 2$ it follows from Theorems 2.1 and 5.1 that the set $V_{\mathcal{N}}$ of $\mathcal{T}$ for which $\mathcal{N}^{\mathcal{T}}$ is reducible has measure zero and it remains to take $G_{\mathcal{N}} = U_{\mathcal{N}} \setminus V_{\mathcal{N}}$.

If $\mathcal{N} \sim \lambda C_0$ with $\lambda \geq 2$, then in view of $\mathcal{N} = \lambda \mathcal{N}^{\Downarrow}$ and $\mathcal{N}^{\Downarrow} \sim C_0$ the uniqueness of the curve $\mathcal{N}^{\mathcal{T}}$ implies that it has a unique component $(\mathcal{N}^{\Downarrow})^{\mathcal{T}}$ with multiplicity λ. To complete the proof we set $G_{\mathcal{N}} = U_{\mathcal{N}} \setminus V_{\mathcal{N}^{\Downarrow}}$.

The next theorem concerns the (exact) number of components of algebraic curves with given singularities.

THEOREM 5.6. *Suppose $\mathcal{N}$ is a singular scheme with canonical decomposition*

$$\mathcal{N} = \sum_{A\in\Omega_{\mathcal{N}}} \mu_A A + \mu^{\Downarrow}\mathcal{N}^{\Downarrow}, \qquad \mathcal{N}^{\Downarrow} \neq 0,$$

and let $\mathcal{T} \in G_{\Omega_{\mathcal{N}}}$. Then each curve C of degree $\deg \mathcal{N}$ with $\mathrm{mult}_{\mathcal{T}} C \geq \mathcal{N}'$ contains all the components $A^{\mathcal{T}}$, $A \in \Omega_{\mathcal{N}}$, with multiplicities μ_A and at least one additional component (not necessarily irreducible) $C^{\Downarrow}$ of degree $\deg \mathcal{N}^{\Downarrow}$ with multiplicity $\mu^{\Downarrow}$ such that

$$\mathrm{mult} C^{\Downarrow} \geq (\mathcal{N}^{\Downarrow})'. \tag{5.8}$$

In addition,

(i) $\mathrm{mult}_x C = \mathrm{mult}_x C^{\Downarrow}$ *if $x \in C^{\Downarrow} \setminus \mathcal{T}$.*

(ii) $\mathrm{mult}_x C = \mathrm{mult}_x A^{\mathcal{T}}$ *if $x \in C \setminus C^{\Downarrow}$, where $A \in \Omega_{\mathcal{N}}$ with $x \in A^{\mathcal{T}}$.*

Moreover, if $R(\mathcal{N}) = 1$ and Conjecture 1.10 *is true, then for every $\mathcal{T} \in G_{\Omega_{\mathcal{N}}} \cap U_{\mathcal{N}^{\Downarrow}}$ there is exactly one additional component $C^{\Downarrow}$ (with multiplicity $\mu^{\Downarrow}$) which is irreducible.*

Proof. As in the proof of Theorem 5.3 we see that the curve C has all the components $A^{\mathcal{T}}$, $A \in \Omega_{\mathcal{N}}$, with multiplicities μ_A. Denote the product of the remaining components by $C^{\Downarrow}$:

$$C^{\Downarrow} := C : \sum_{A\in\Omega_{\mathcal{N}}} \mu_A A^{\mathcal{T}}.$$

The relation (5.8) follows from Theorem 5.3. The orthogonality of the canonical decomposition and Bezout's theorem imply that the curves $C^{\Downarrow}$ and $A^{\mathcal{T}}$ have no common point outside $\mathcal{T}$, which means that (i) holds. The statement (ii) follows from the fact that two distinct curves $A^{\mathcal{T}}, B^{\mathcal{T}}$ with $A, B \in \Omega_{\mathcal{N}}$

cannot have a common point outside $C^{\Downarrow}$, which follows from Lemma 2.10. To end the proof it remains to note that $R(\mathcal{N}^{\Downarrow}) = R(\mathcal{N})$ and use Theorem 5.5.

Finally, we consider the following problem: can the size of $\overline{\mathcal{N}}$ guarantee the regularity of $\mathcal{N}$?

THEOREM 5.7. *If Conjecture* 1.10 *is true, then all the schemes* $\mathcal{N} = \{n_1, \ldots, n_s; n\}$ *satisfying* $\overline{\mathcal{N}} > \overline{n-1}$ *are regular, while for the singular scheme* $\mathcal{N} = n\{1,1;1\}$ *we have*

$$\overline{\mathcal{N}} = \overline{n-1}. \tag{5.9}$$

Proof. According to Remark 1.15 we can assume that $\mathcal{N} \in \mathrm{BS}^{\times *}$. We have the following inequalities for the coefficients of the canonical decomposition of $\mathcal{N}$:

$$\sum_{A\in\Omega_{\mathcal{N}}} \overline{\mu_A - 1} \le \overline{\sum_{A\in\Omega_{\mathcal{N}}} \mu_A - k} \le \overline{n-k} \le \overline{n-1}, \tag{5.10}$$

where $k = \#\Omega_{\mathcal{N}}$. This, in view of (1.12), implies that $\overline{\mathcal{N}^{\downarrow}} > 0$. Therefore, if the conjecture is true, the scheme $\mathcal{N}^{\downarrow}$ and hence $\mathcal{N}$ is regular.

Remark 5.8. Note that we have equalities in (5.10) if and only if $\mathcal{N} = n\{1,1;1\}$, hence these schemes are the only singular schemes satisfying (5.9).

References

[GHS90] H. Gevorgian, H. Hakopian and A. Sahakian, *On bivariate Hermite interpolation*, Mat. Zametki 48 (1990), 137–139 (in Russian).

[GHS92] —, —, —, *On the bivariate polynomial interpolation*, Mat. Sb. 183 (1992), 111–126 (in Russian); English transl.: Russian Acad. Sci. Sb. Math. 76 (1993), 211–223.

[GHS92b] —, —, —, *On the bivariate Hermite interpolation problem*, Constr. Approx. 11 (1995), 23–35.

[GHS93] —, —, —, *Bivariate Hermite interpolation and numerical curves*, in: Proc. of the International Conference on Open Problems in Approximation Theory, Voneshta Voda, 1993, 85–89.

[GHS95] —, —, —, *Bivariate Hermite interpolation and numerical curves*, J. Approx. Theory 85 (1996), 297–317.

[H94] H. Hakopian, *Multivariate Polynomial Interpolation*, Habilitation survey, Mathematical Institute of Polish Acad. Sci., Warszawa, 1994.

[JS91] R-Q. Jia and A. Sharma, *Solvability of some multivariate interpolation problems*, J. Reine Angew. Math. 421 (1991), 73–81.

[LL84] G. G. Lorentz and R. A. Lorentz, *Multivariate interpolation*, in: Lecture Notes in Math. 1105, Springer, Berlin, 1984, 136–144.

[P92] S. H. Paskov, *Singularity of bivariate interpolation*, J. Approx. Theory 75 (1992), 50–67.

[W50] R. Walker, *Algebraic Curves*, Princeton Univ. Press, Princeton, N.J., 1950.

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