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Complex Unconditional Metric Approximation Property for $\mathcal{C}_\Lambda(\mathbb{T})$ spaces

by

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Abstract. We study the Complex Unconditional Metric Approximation Property for translation invariant spaces $\mathcal{C}_\Lambda(\mathbb{T})$ of continuous functions on the circle group. We show that although some "tiny" (Sidon) sets do not have this property, there are "big" sets Λ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP); though these sets are such that $L^\infty_\Lambda(\mathbb{T})$ contains functions which are not continuous, we show that there is a linear invariant lifting from these $L^\infty_\Lambda(\mathbb{T})$ spaces into the Baire class 1 functions.

Introduction. The translation invariant subspaces of continuous functions on $\mathbb{T}$ all have the Metric Approximation Property (MAP). We study in this paper the spaces $\mathcal{C}_\Lambda(\mathbb{T})$ which satisfy a stronger approximation property, the Complex Unconditional Metric Approximation Property ($\mathbb{C}$-UMAP).

The (Real) Unconditional Approximation Property (UMAP) was introduced in 1989 by P. Casazza and N. Kalton as an extreme possibility of approximation ([3], Th. 3.5), and they showed ([3], Th. 3.8) that it actually coincides for a separable Banach space X with the existence for every $\varepsilon > 0$ of an unconditional expansion of the identity of X with constant $1+\varepsilon$, which means, by a result of A. Pełczyński and P. Wojtaszczyk ([21], Th. 1.1) that for every $\varepsilon > 0$, X may be isometrically embedded in a Banach space Y with a $(1+\varepsilon)$-FDD for which there is a projection $P : Y \to X$ with $\|P\| \le 1+\varepsilon$. Its complex version was defined and studied in ([7], §§8 and 9).

To begin with, we construct subsets $\Lambda \subseteq \mathbb{Z}$ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP). They are of two kinds: the first contain arbitrarily long arithmetical progressions, so that they are not $\Lambda(1)$-sets, but their pace tends to infinity; the second are Sidon sets, but have a pace which does not tend to infinity.

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Next, we show that the ($\mathbb{C}$-UMAP) can always be achieved by convolution operators. This implies that if $\mathcal{C}_\Lambda$ has ($\mathbb{C}$-UMAP), then so do $\mathcal{C}_{\Lambda_0}$ for all $\Lambda_0 \subseteq \Lambda$, as well as all the spaces L^p_Λ, $1 \le p < \infty$.

When $\mathcal{C}_\Lambda$ has ($\mathbb{C}$-UMAP), we remark that Λ is a Rosenthal set if (and only if) $\mathcal{C}_\Lambda$ contains no subspace isomorphic to c_0. We show that this is not always the case: $\mathcal{C}_\Lambda$ can have ($\mathbb{C}$-UMAP) when Λ is a Hilbert set, and then $\mathcal{C}_\Lambda$ has subspaces isomorphic to c_0. However, we show that Λ cannot contain any IP-set, nor the sum of two infinite sets. We also show that the uniform density of Λ must be less than or equal to $1/2$, though it is likely that it is null. Finally, we show that for such a set there exists a linear invariant lifting from $L^\infty_\Lambda(\mathbb{T})$ into the Baire class 1 functions.

The notation is classical. $\mathbb{T}$ is the quotient $\mathbb{R}/2\pi\mathbb{Z}$ and for every $n \in \mathbb{Z}$ we denote by e_n the character defined by $e_n(x) = e^{inx}$ for $x \in \mathbb{T}$.

We recall that a complex (separable) Banach space X has ($\mathbb{C}$-UMAP) if there is a sequence of finite rank operators $R_n : X \to X$ such that

$$\|R_n x - x\| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } x \in X, \qquad \sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)R_n\| \underset{n\to\infty}{\longrightarrow} 1.$$

When λ belongs only to $\mathbb{R}$, instead of $\mathbb{C}$, X is said to have (real) (UMAP).

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I. Construction of sets Λ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP). In this section, we give some examples of subsets $\Lambda \subseteq \mathbb{Z}$ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP).

It is worth mentioning that there is no Sidon subset $\Lambda \subseteq \mathbb{Z}$ with constant 1 whenever card $\Lambda \ge 3$ ([2], p. 532).

LEMMA 1. *For every finite subset $F \subseteq \mathbb{N}$ and every $\varepsilon > 0$, there are L intervals $I_1, \ldots, I_L$ of length $1/L$ such that for each finite subset $G \subseteq \mathbb{N}$ there is an integer $n > \max F$ such that for every $f \in \mathcal{C}_F(\mathbb{T})$ and every $g \in \mathcal{C}_G(\mathbb{T})$, we have*

$$\|f + e_n g\|_{\mathcal{C}(\mathbb{T})} \ge \frac{1}{1+\varepsilon} \sup_{1\le l\le L} (\|f\|_{\mathcal{C}(I_l)} + \|g\|_{\mathcal{C}(I_l)}).$$

Proof. We may and do suppose that $\varepsilon \le 3/4$.

Let $\pi_F : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ be the projection which associates with $f \in \mathcal{C}(\mathbb{T})$ the trigonometric polynomial

$$\pi_F(f) = \sum_{k\in F} \widehat{f}(k) e_k.$$

Note that $\pi_F(f + e_n g) = f$ for every $f \in \mathcal{C}_F(\mathbb{T})$ and every $g \in \mathcal{C}_G(\mathbb{T})$ whenever $n > \max F$ and $G \subseteq \mathbb{N}$. Since $\mathcal{C}_F(\mathbb{T})$ is finite-dimensional, there exists a number $\varrho > 0$ such that, for every $f \in \mathcal{C}_F(\mathbb{T})$, we have:

$$|x - y| \le \varrho \Rightarrow |f(x) - f(y)| \le \frac{\varepsilon}{8\|\pi_F\|} \|f\|_\infty.$$

We then divide $\mathbb{T}$ in L intervals $I_1, \ldots, I_L$ of length $1/L \le \varrho$.

SUBLEMMA. *Let $M = (8/\varepsilon)(1 + \|\pi_F\|)$. Then there is an integer $n > \max F$ such that for each $h \in M \cdot B_{\mathcal{C}_G(\mathbb{T})}$ and $1 \le l \le L$, we have*

$$\|1 + e_n h\|_{\mathcal{C}(I_l)} \ge (1 - \varepsilon/2)(\|h\|_{\mathcal{C}(I_l)} + 1).$$

Proof. Choose an $(\varepsilon/8)$-net $h_1, \ldots, h_K$ of $M \cdot B_{\mathcal{C}_G(\mathbb{T})}$. We have the following:

CLAIM. *Given p non-void open intervals $L_1, \ldots, L_p$, there exist $y_q \in L_q$, $1 \le q \le p$, such that $1, y_1, \ldots, y_p$ are $\mathbb{Q}$-independent.*

Indeed, if $1, t_1, \ldots, t_p$ are $\mathbb{Q}$-independent real numbers, we can choose $r_q \in \left(\frac{1}{t_q} L_q\right) \cap \mathbb{Q}^*$, $1 \le q \le p$, and then the numbers $y_q = r_q t_q$ are suitable.

We can then choose $y_1^l, \ldots, y_K^l \in I_l$ such that

$$|h_k(y_k^l)| \ge (1 - \varepsilon/8)\|h_k\|_{\mathcal{C}(I_l)}$$

and such that the set $\{1\} \cup \{y_k^l : 1 \le l \le L,\ 1 \le k \le K\}$ is $\mathbb{Q}$-independent. Then Kronecker's theorem ([10], Th. 442) enables us to find an integer $n > \max F$ such that, for $1 \le l \le L$ and $1 \le k \le K$, we have

$$\left| e^{iny_k^l} - \frac{|h_k(y_k^l)|}{h_k(y_k^l)} \right| \le \frac{\varepsilon}{8}.$$

Then $\left| e^{iny_k^l} h_k(y_k^l) - |h_k(y_k^l)| \right| \le \frac{\varepsilon}{8} |h_k(y_k^l)|$, and so

$$\begin{aligned}\|1 + e_n h_k\|_{\mathcal{C}(I_l)} &\ge |1 + e^{iny_k^l} h_k(y_k^l)| \\ &\ge 1 + |h_k(y_k^l)| - \left| e^{iny_k^l} h_k(y_k^l) - |h_k(y_k^l)| \right| \\ &\ge 1 + |h_k(y_k^l)| - \frac{\varepsilon}{8}|h_k(y_k^l)| \ge 1 + \left(1 - \frac{\varepsilon}{8}\right)^2 \|h_k\|_{\mathcal{C}(I_l)}.\end{aligned}$$

Now, for each $h \in M \cdot B_{\mathcal{C}_G(\mathbb{T})}$, there is an index k such that

$$\|h - h_k\|_{\mathcal{C}(I_l)} \le \|h - h_k\|_{\mathcal{C}(\mathbb{T})} \le \varepsilon/8;$$

hence

$$\begin{aligned}\|1 + e_n h\|_{\mathcal{C}(I_l)} &\ge \|1 + e_n h_k\|_{\mathcal{C}(I_l)} - \|h - h_k\|_{\mathcal{C}(I_l)} \\ &\ge (1 - \varepsilon/8)^2 \|h_k\|_{\mathcal{C}(I_l)} + 1 - \varepsilon/8 \\ &\ge (1 - \varepsilon/8)^2 (\|h\|_{\mathcal{C}(I_l)} - \varepsilon/8) + 1 - \varepsilon/8 \\ &\ge (1 - \varepsilon/8)^2 \|h\|_{\mathcal{C}(I_l)} + (1 - \varepsilon/8)^3 \\ &\ge (1 - \varepsilon/2)(\|h\|_{\mathcal{C}(I_l)} + 1). \ \blacksquare\end{aligned}$$

In the sequel we use the n given by the sublemma. Let $f \in \mathcal{C}_F(\mathbb{T})$ and $g \in \mathcal{C}_G(\mathbb{T})$, and fix $l_0 \in \{1, \dots, L\}$ such that $\|f\|_{\mathcal{C}(I_{l_0})} + \|g\|_{\mathcal{C}(I_{l_0})} = \sup_{1\le l\le L}(\|f\|_{\mathcal{C}(I_l)} + \|g\|_{\mathcal{C}(I_l)})$.

We have two cases.

Assume first that $\|f\|_{\mathcal{C}(I_{l_0})} \le (\varepsilon/8)\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}$. Then we observe first that since l_0 gives the maximum value, we have $\|f\|_{\mathcal{C}(I_{l_0})} + \|g\|_{\mathcal{C}(I_{l_0})} \ge \|f+e_ng\|_{\mathcal{C}(\mathbb{T})}$, and so

$$\|g\|_{\mathcal{C}(I_{l_0})} \ge \left(1-\frac{\varepsilon}{8}\right)\|f+e_ng\|_{\mathcal{C}(\mathbb{T})} \ge \frac{2+\varepsilon}{8}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})};$$

hence

$$\begin{aligned}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})} &\ge \|f+e_ng\|_{\mathcal{C}(I_{l_0})} \ge \|g\|_{\mathcal{C}(I_{l_0})} - \|f\|_{\mathcal{C}(I_{l_0})}\\ &\ge \|g\|_{\mathcal{C}(I_{l_0})} - \frac{\varepsilon}{8}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}\\ &= \left(\frac{\|g\|_{\mathcal{C}(I_{l_0})}}{\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}} - \frac{\varepsilon}{8}\right)\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}\\ &\ge \frac{1}{1+\varepsilon}\left(\frac{\|g\|_{\mathcal{C}(I_{l_0})}}{\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}} + \frac{\varepsilon}{8}\right)\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}\\ &\qquad\qquad \text{since } \frac{\|g\|_{\mathcal{C}(I_{l_0})}}{\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}} \ge \frac{2+\varepsilon}{8}\\ &= \frac{1}{1+\varepsilon}\left(\|g\|_{\mathcal{C}(I_{l_0})} + \frac{\varepsilon}{8}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}\right)\\ &\ge \frac{1}{1+\varepsilon}(\|g\|_{\mathcal{C}(I_{l_0})} + \|f\|_{\mathcal{C}(I_{l_0})}).\end{aligned}$$

Suppose now that $\|f\|_{\mathcal{C}(I_{l_0})} \ge (\varepsilon/8)\|f+e_ng\|_{\mathcal{C}(\mathbb{T})}$. Let $x_0 \in I_{l_0}$ be such that $|f(x_0)| = \|f\|_{\mathcal{C}(I_{l_0})}$. For every $x \in I_{l_0}$, we have

$$\begin{aligned}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})} &\ge |f(x_0)+e^{inx}g(x)| - |f(x_0)-f(x)|\\ &\ge \|f\|_{\mathcal{C}(I_{l_0})}\left|1+e^{inx}\frac{g(x)}{f(x_0)}\right| - \frac{\varepsilon}{8\|\pi_F\|}\|f\|_{\mathcal{C}(\mathbb{T})}\\ &\ge \|f\|_{\mathcal{C}(I_{l_0})}|1+e^{inx}h(x)| - \frac{\varepsilon}{8}\|f+e_ng\|_{\mathcal{C}(\mathbb{T})},\end{aligned}$$

where $h = \frac{1}{f(x_0)}g \in (8/\varepsilon)(1+\|\pi_F\|)\cdot B_{\mathcal{C}_G(\mathbb{T})}$, since

$$\|g\|_\infty = \|e_ng\|_\infty = \|(\mathrm{Id}-\pi_F)(f+e_ng)\|_\infty \le (1+\|\pi_F\|)\|f+e_ng\|_\infty.$$

The previous inequalities can be read as

$$\|f+e_ng\|_\infty \ge \|f\|_{\mathcal{C}(I_{l_0})}\|1+e_nh\|_{\mathcal{C}(I_{l_0})} - \frac{\varepsilon}{8}\|f+e_ng\|_\infty,$$

and so the sublemma gives

$$\begin{aligned}\|f+e_ng\|_\infty &\ge \frac{1}{1+\varepsilon/8}\|f\|_{\mathcal{C}(I_{l_0})}\|1+e_nh\|_{\mathcal{C}(I_{l_0})}\\ &\ge \frac{1}{1+\varepsilon/8}\|f\|_{\mathcal{C}(I_{l_0})}(1-\varepsilon/2)(1+\|h\|_{\mathcal{C}(I_{l_0})})\\ &= \frac{1-\varepsilon/2}{1+\varepsilon/8}(\|f\|_{\mathcal{C}(I_{l_0})} + \|g\|_{\mathcal{C}(I_{l_0})})\\ &\ge \frac{1}{1+\varepsilon}(\|f\|_{\mathcal{C}(I_{l_0})} + \|g\|_{\mathcal{C}(I_{l_0})})\\ &= \frac{1}{1+\varepsilon}\sup_{1\le l\le L}(\|f\|_{\mathcal{C}(I_l)} + \|g\|_{\mathcal{C}(I_l)}). \ \blacksquare\end{aligned}$$

THEOREM 2. *There exist subsets $\Lambda \subseteq \mathbb{Z}$ such that Λ contains arbitrarily long arithmetical progressions and such that $C_\Lambda(\mathbb{T})$ has* ($\mathbb{C}$-UMAP).

Remark. Then Λ is not a $\Lambda(1)$-set ([25], Th. 4.1), that is to say, L^1_Λ is not reflexive (by [13] and [1]; see also [22]).

In order to prove this theorem, we need the following lemma.

LEMMA 3. *For every $\varepsilon > 0$ and every finite set $F \subseteq \mathbb{N}$, there is an $L \ge 1$ such that for every finite set $G \subseteq L\mathbb{N}^*$, there is $n > \max F$ such that*

$$\|f+e_ng\|_\infty \ge \frac{1}{1+\varepsilon}(\|f\|_\infty + \|g\|_\infty)$$

for every $f \in \mathcal{C}_F$ and every $g \in \mathcal{C}_G$.

Proof. This follows from Lemma 1, since $G \subseteq L\mathbb{N}^*$ implies that every $g \in \mathcal{C}_G(\mathbb{T})$ has period $1/L$ and so $\|g\|_\infty = \|g\|_{\mathcal{C}(I_l)}$ for every $l \in \{1, \dots, L\}$; in particular, Lemma 3 is obtained by considering l_0 such that $\|f\|_\infty = \|f\|_{\mathcal{C}(I_{l_0})}$.

Proof of Theorem 2. The construction of Λ will be done by induction. First, we apply Lemma 3 with $\varepsilon = \varepsilon_1$ and $F = \{0\}$; we find a suitable $L_1 \ge 1$ and with $G = \{L_1\}$, we now find an $n_1 \ge 1$; we set $\Lambda_1 = \{0, L_1+n_1\}$.

Suppose now that we have constructed a set $\Lambda_k = H_0 \cup \ldots \cup H_k \subseteq \mathbb{N}$ where $H_0 = \{0\}$ and $H_j = n_j + \{L_j, 2L_j, \dots, jL_j\}$, $1 \le j \le k$, are disjoint sets such that for $1 \le j \le k$,

$$(1)\qquad \Big\|\sum_{l=0}^k f_l\Big\|_\infty \ge \frac{1}{(1+\varepsilon_j)\dots(1+\varepsilon_k)}\Big(\Big\|\sum_{l=0}^{j-1} f_l\Big\|_\infty + \sum_{l=j}^k \|f_l\|_\infty\Big)$$

where $f_l \in \mathcal{C}_{H_l}$, $0 \le l \le k$, and where $\varepsilon_1, \dots, \varepsilon_k > 0$ have been chosen in such a way that

$$\prod_{l=j}^{k}(1+\varepsilon_l) \le 1 + \left(1 + \ldots + \frac{1}{2^{k-j-1}}\right)\varepsilon_j, \quad 1 \le j \le k-1.$$

We apply Lemma 3 again with $\varepsilon = \varepsilon_{k+1}$ chosen such that

$$\prod_{l=j}^{k+1}(1+\varepsilon_l) \le 1 + \left(1 + \ldots + \frac{1}{2^{k-j}}\right)\varepsilon_j, \quad 1 \le j \le k, \tag{2}$$

and with $F = \Lambda_k$; we find L_{k+1} and, taking

$$G = G_{k+1} = \{L_{k+1}, 2L_{k+1}, \ldots, (k+1)L_{k+1}\},$$

we obtain $n_{k+1} > \max \Lambda_k$. We then set

$$\Lambda_{k+1} = \Lambda_k \cup (n_{k+1} + G_{k+1}).$$

The formulas (1) and (2) show that

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_j}\|_{\mathcal{L}(\mathcal{C}_{\Lambda_{k+1}})} \le 1 + \left(1 + \ldots + \frac{1}{2^{k-j}}\right)\varepsilon_j.$$

Now $\Lambda = \bigcup_{k\ge1}\Lambda_k$ is a set such that $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP) since

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_j}\|_{\mathcal{L}(\mathcal{C}_{\Lambda})} \le 1 + 2\varepsilon_j. \ ■$$

Remark. This construction, though different, follows the same idea as H. P. Rosenthal's ([24]). We obtain a Rosenthal set of "ℓ_1-sum" type.

THEOREM 4. *For every finite set $F_0 \subseteq \mathbb{N}$, there is an increasing sequence $(n_k)_{k\ge0}$ of integers such that $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP) with $\Lambda = \bigcup_{k\ge0}(n_k + F_0)$. In particular, the pace of Λ does not tend to infinity.*

Proof. We first remark that for every $u, v \in \mathcal{C}(\mathbb{T})$, we have

$$\sup_{|\lambda|=1} \|u + \lambda v\|_\infty = \sup_{x\in\mathbb{T}}(|u(x)| + |v(x)|).$$

Starting from $n_0 = 0$, we apply Lemma 1 with $F = G = F_0$. We obtain an integer $n_1 > \max F_0 = N_0$ such that for $f, g \in \mathcal{C}_{F_0}(\mathbb{T})$ and $|\lambda| = 1$, we have

$$\|-\lambda f + e_{n_1} g\|_\infty \le (1+\varepsilon_1)\|f + e_{n_1} g\|_\infty.$$

Hence, on setting $\Lambda_0 = F_0$ and $\Lambda_1 = \Lambda_0 \cup (n_1 + \Lambda_0)$, we have

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_0}\|_{\mathcal{L}(\mathcal{C}_{\Lambda_1})} \le 1 + \varepsilon_1.$$

Suppose now we have already constructed integers $n_1, \ldots, n_k$ such that

$$N_0 + n_{j-1} < n_j, \quad 1 \le j \le k,$$

and $\varepsilon_1 > \ldots > \varepsilon_k > 0$ such that

$$\prod_{l=j}^{k}(1+\varepsilon_l) \le 1 + \left(1 + \ldots + \frac{1}{2^{k-j-1}}\right)\varepsilon_j$$

and such that, for $1 \le j \le k$, we have

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_{j-1}}\|_{\mathcal{L}(\mathcal{C}_{\Lambda_k})} \le (1+\varepsilon_j)\ldots(1+\varepsilon_k),$$

where we have set $\Lambda_k = \bigcup_{j=0}^{k}(n_j + F_0)$.

We now apply Lemma 1 with $F = \Lambda_k$ and $G = F_0$, and $\varepsilon = \varepsilon_{k+1} > 0$ chosen in such a way that $\varepsilon_{k+1} < \varepsilon_k$ and such that for $1 \le j \le k$,

$$\left(1 + \varepsilon_j + \frac{\varepsilon_j}{2} + \ldots + \frac{\varepsilon_j}{2^{k-j-1}}\right)(1+\varepsilon_{k+1}) \le 1 + \varepsilon_j + \frac{\varepsilon_j}{2} + \ldots + \frac{\varepsilon_j}{2^{k-j-1}} + \frac{\varepsilon_j}{2^{k-j}}.$$

We then find an integer $n_{k+1} > \max \Lambda_k = N_0 + n_k$ and L intervals $I_1, \ldots, I_L$ of length $1/L$ such that

$$\|(u+v) + w\|_\infty \ge \frac{1}{1+\varepsilon_{k+1}} \sup_{1\le l\le L}(\|u+v\|_{\mathcal{C}(I_l)} + \|w\|_{\mathcal{C}(I_l)})$$

for $u \in \mathcal{C}_{\Lambda_j}$, $v \in \mathcal{C}_{\Lambda_k\setminus\Lambda_j}$ and $w \in \mathcal{C}_{n_{k+1}+F_0}$.

In particular, we have

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_k}\|_{\mathcal{L}(\mathcal{C}_{\Lambda_{k+1}})} \le (1+\varepsilon_j)\ldots(1+\varepsilon_{k+1}).$$

Writing now $w = e_{n_{k+1}} u_1$ with $u_1 \in \mathcal{C}_{F_0}(\mathbb{T})$, we have

$$\begin{aligned}
\|(u+v) + w\|_\infty &\ge \frac{1}{1+\varepsilon_{k+1}} \sup_{x\in\mathbb{T}}(|u(x) + v(x)| + |u_1(x)|) \\
&\ge \frac{1}{1+\varepsilon_{k+1}} \sup_{x\in\mathbb{T}} \sup_{|\sigma|=1} (|u(x) + \sigma u_1(x) + v(x)|) \\
&= \frac{1}{1+\varepsilon_{k+1}} \sup_{|\sigma|=1} \|(u + \sigma u_1) + v\|_\infty \\
&\ge \frac{1}{1+\varepsilon_{k+1}} \cdot \frac{1}{(1+\varepsilon_j)\ldots(1+\varepsilon_k)} \sup_{|\sigma|=1} \sup_{|\lambda|=1} \|-\lambda(u + \sigma u_1) + v\|_\infty \\
&\qquad \text{by induction hypothesis since } F_0 \subseteq \Lambda_j \\
&= \frac{1}{(1+\varepsilon_j)\ldots(1+\varepsilon_k)(1+\varepsilon_{k+1})} \sup_{x\in\mathbb{T}}(|u(x)| + |u_1(x)| + |v(x)|) \\
&\ge \frac{1}{(1+\varepsilon_j)\ldots(1+\varepsilon_{k+1})} \sup_{x\in\mathbb{T}}(|u(x)| + |v(x) + e^{in_{k+1}x}u_1(x)|) \\
&\ge \frac{1}{(1+\varepsilon_j)\ldots(1+\varepsilon_{k+1})} \sup_{|\lambda|=1} \|-\lambda u + (v+w)\|_\infty;
\end{aligned}$$

hence

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_j}\|_{\mathcal{L}(\mathcal{C}_{\Lambda_{k+1}})} \le (1+\varepsilon_j)\dots(1+\varepsilon_{k+1}).$$

Setting now

$$\Lambda = \bigcup_{k\ge 0} \Lambda_k = \bigcup_{k\ge 0} (n_k + F_0)$$

we obviously have for all $j \ge 0$,

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_j}\|_{\mathcal{L}(\mathcal{C}_\Lambda)} \le 1 + 2\varepsilon_j.$$

Since $\lim_{j\to\infty} \varepsilon_j = 0$, the space $\mathcal{C}_\Lambda(\mathbb{T})$ does have ($\mathbb{C}$-UMAP). ■

Remark. The proof shows that $\{n_k\}_{k\ge 1}$ is a Sidon set with constant $1+2\varepsilon_1$. Since F_0 is a finite set, Drury's theorem implies that the above constructed set Λ is itself a Sidon set, as a finite union of Sidon sets.

II. Properties implied by ($\mathbb{C}$-UMAP) for $\mathcal{C}_\Lambda(\mathbb{T})$ spaces. Most of the properties below do not depend on the particular nature of the group $\mathbb{T}$, so we state them in the abstract setting of a compact metrizable abelian group G.

First, we are going to see that the approximation can be achieved with convolution operators.

LEMMA 5. *If $\mathcal{C}_\Lambda(G)$ has ($\mathbb{C}$-UMAP), there exists a sequence of finite rank convolution operators $C_n : \mathcal{C}_\Lambda(G) \to \mathcal{C}_\Lambda(G)$ such that*

$$\|f - C_n f\|_\infty \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)C_n\| \underset{n\to\infty}{\longrightarrow} 1.$$

Proof. Let $R_n : \mathcal{C}_\Lambda \to \mathcal{C}_\Lambda$ be finite rank operators such that $\|R_n f - f\|_\infty \to 0$ for every $f \in \mathcal{C}_\Lambda(G)$ and $\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)R_n\| \to 1$ as $n \to \infty$.

Let $(K_l)_{l\ge 1}$ be an approximate unit, and $S_l : \mathcal{C}_\Lambda \to \mathcal{C}_\Lambda$ be the convolution operator associated with K_l. Since R_n has finite rank, by using a $2^{-(n+2)}$-net in $R_n(B_{\mathcal{C}_\Lambda})$, we can find an index l_n such that

$$\|R_n - S_{l_n} R_n\| \le 2^{-n}.$$

The operator $T_n = S_{l_n} R_n$ satisfies

$$\|f - T_n f\|_\infty \le \|f - R_n f\|_\infty + \|R_n f - S_{l_n} R_n f\|_\infty \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\begin{aligned}\|\mathrm{Id} - (1+\lambda)T_n\| &\le \|\mathrm{Id} - (1+\lambda)R_n\| + 2\|R_n - S_{l_n} R_n\| \\ &\le \|\mathrm{Id} - (1+\lambda)R_n\| + 2^{-n+1},\end{aligned}$$

so that $\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)T_n\| \to 1$ as $n \to \infty$. Moreover, $T_n(\mathcal{C}_\Lambda) \subseteq \mathcal{P}_\Lambda$ (where $\mathcal{P}_\Lambda$ is the set of trigonometric polynomials with spectrum in Λ).

Now, set

$$\widetilde{T}_n(f) = \int_G [T_n(f_x)]_{-x}\, dx.$$

Since $T_n(\mathcal{C}_\Lambda)$ is finite-dimensional and $T_n(\mathcal{C}_\Lambda) \subseteq \mathcal{P}_\Lambda$, there is a finite collection of characters $\Lambda_n = \{\gamma_1, \dots, \gamma_N\}$ such that $T_n(\mathcal{C}_\Lambda) \subseteq \mathcal{P}_{\Lambda_n}$, so that $\widetilde{T}_n(\mathcal{C}_\Lambda) \subseteq \mathcal{P}_{\Lambda_n}$ is also finite-dimensional. Moreover, $\|\widetilde{T}_n f - f\|_\infty \to 0$ for each $f \in \mathcal{C}_\Lambda$, and $\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\widetilde{T}_n\| \to 1$ as $n \to \infty$. A direct computation gives that $C_n = \widetilde{T}_n$ is the convolution operator associated with the trigonometric polynomial $P_n = \sum_{\gamma\in\Lambda_n} \widehat{T_n(\gamma)}(\gamma)\gamma$. ■

COROLLARY 6. *If $\mathcal{C}_\Lambda(G)$ has ($\mathbb{C}$-UMAP), then so does $\mathcal{C}_{\Lambda_0}(G)$ for every $\Lambda_0 \subseteq \Lambda$.*

We now give some consequences of this result.

THEOREM 7. *If $\mathcal{C}_\Lambda(G)$ has ($\mathbb{C}$-UMAP), then so do all the spaces $L^p_\Lambda(G)$, $1 \le p < \infty$.*

Proof. That follows from Lemma 5 and from the following observation. Let P be a trigonometric polynomial, and C be the associated convolution operator. Then we have

$$(\forall g \in \mathcal{P}_\Lambda) \qquad \|g - P * g\|_p \le \|g - P * g\|_\infty,$$

and

$$\|\mathrm{Id} - (1+\lambda)C\|_{\mathcal{L}(L^p_\Lambda)} \le \|\delta_0 - (1+\lambda)P\|_{\mathcal{M}/\mathcal{M}_{\Lambda^c}} = \|\mathrm{Id} - (1+\lambda)C\|_{\mathcal{L}(\mathcal{C}_\Lambda)}$$

(see [11], Th. 2). ■

Remark. By ([6], Prop. 2.4) and ([8], Th. II.2), if L^1_Λ has ($\mathbb{C}$-UMAP) and so (UMAP), then Λ is a Shapiro set ([4], Def. 1.6); in particular, it is a Riesz set ([4], Th. 1.9). Moreover, it then follows from ([7], Th. 9.2) that the predual $\mathcal{C}/\mathcal{C}_{\Lambda'}$ of L^1_Λ also has ($\mathbb{C}$-UMAP), and more precisely, there is a sequence of finite rank operators $A_n : \mathcal{C}/\mathcal{C}_{\Lambda'} \to \mathcal{C}/\mathcal{C}_{\Lambda'}$ such that

$$\begin{aligned}&\|A_n\varphi - \varphi\| \underset{n\to\infty}{\longrightarrow} 0, \quad \forall \varphi \in \mathcal{C}/\mathcal{C}_{\Lambda'},\\ &\|A_n^* f - f\|_1 \underset{n\to\infty}{\longrightarrow} 0, \quad \forall f \in L^1_\Lambda,\\ &\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)A_n\| \underset{n\to\infty}{\longrightarrow} 1.\end{aligned}$$

Now, given an approximate unit $(K_n)_{n\ge 1}$, by ([14], Lemma 1) we have $A_n - K_n \to 0$ weakly in $\mathcal{K}(\mathcal{C}/\mathcal{C}_{\Lambda'})$ as $n \to \infty$ so that there are finite convex blocks C_n of the K_n's such that the above three conditions are also true with C_n instead of A_n. That gives:

PROPOSITION 8. *Let G be an abelian compact metrizable group and Γ its countable dicrete dual group. The class $\mathcal{G} = \{\Lambda \subseteq \Gamma : L^1_\Lambda$ has (ℂ-UMAP)$\}$ is an $F_{\sigma\delta}$ in $\mathcal{P}(\Gamma)$.*

The topology on $\mathcal{P}(\Gamma)$ is the product topology of $\{0,1\}^\Gamma$, where $\mathcal{P}(\Gamma)$ and $\{0,1\}^\Gamma$ are identified by the map $\Lambda \mapsto \mathbf{1}_\Lambda$.

Proof. Let $(K_n)_{n\geq 1}$ be a given approximate unit, and $\mathcal{B}$ the set of all the rational finite convex blocks of the K_n's; it is a countable set, and we have

$$\mathcal{G} = \bigcap_{n\geq 1} \bigcup_{R\in\mathcal{B}} \mathcal{G}_{n,R}$$

where

$$\mathcal{G}_{n,R} = \{\Lambda \subseteq \Gamma : \sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)R\|_{\mathcal{L}(L^1_\Lambda)} \leq 1 + 1/n\}.$$

We thus obtain the proposition since this last set is closed in $\mathcal{P}(\Gamma)$: if $\Lambda_j \to \Lambda$ in $\mathcal{P}(\Gamma)$ as $n \to \infty$, with $\Lambda_j \in \mathcal{G}_{n,R}$, let $P \in \mathcal{P}_\Lambda$; since $\mathrm{spec}(P) \subseteq \Lambda$ is a finite set, there is an index j such that $\mathrm{spec}(P) \subseteq \Lambda_j$, and so $\sup_{|\lambda|=1} \|P - (1+\lambda)R * P\|_1 \leq (1+1/n)\|P\|_1$. ■

It is proved in ([6], Prop. 2.8) that if a Banach space with (UMAP) does not contain any subspace isomorphic to c_0, then it is isometric to a (separable) dual Banach space; in particular, such a space has the Radon–Nikodym Property. From the characterization of F. Lust-Piquard ([16]), we obtain:

PROPOSITION 9. *If $\mathcal{C}_\Lambda(G)$ has* (UMAP) *and contains no subspace isomorphic to c_0, then Λ is a Rosenthal set.*

Let us recall that a *Rosenthal set* is a set Λ for which $\mathcal{C}_\Lambda = L^\infty_\Lambda$ and that no example is known of a non-Rosenthal set Λ for which $\mathcal{C}_\Lambda$ does not contain any subspace isomorphic to c_0.

However, there are non-Rosenthal sets Λ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has (ℂ-UMAP):

THEOREM 10. *There are Hilbert sets $\Lambda \subseteq \mathbb{Z}$ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has* (ℂ-UMAP).

Let us recall that a *Hilbert set* Λ is defined by

$$\Lambda = \bigcup_{n\geq 1} \Big\{q_n + \sum_{k=1}^{n} \varepsilon_k p_k : \varepsilon_k = 0 \text{ or } 1\Big\},$$

where $(q_n)_{n\geq 1}$ and $(p_n)_{n\geq 1}$ are two sequences in $\mathbb{Z}$, $p_n \neq 0$, and that $\mathcal{C}_\Lambda(\mathbb{T})$ has subspaces isomorphic to c_0 for any Hilbert set Λ ([15], Th. 2); in particular, Hilbert sets are never Rosenthal sets.

Proof of Theorem 10. It is a consequence of the following

THEOREM (Y. Meyer ([19], Ch. VIII, § 5.1, Th. IV, p. 247). *Let $(t_k)_{k\geq 1}$ be an increasing sequence of positive numbers such that*

(a) $t_{k+1} > s_k = t_1 + \ldots + t_k$,
(b) $\sum_{k=1}^{\infty} (t_k/t_{k+1})^2 < \infty$.

Then the IP-set

$$\Lambda_\infty = \Big\{\sum_{k=1}^{n} \varepsilon_k t_k : \varepsilon_k = 0 \text{ or } 1,\ n \geq 1\Big\}$$

has the following property: for every $\varepsilon > 0$, there is an integer $m_0 \in \mathbb{N}$ for which

$$\|P_{m_0}\|_\infty \leq (1+\varepsilon)\|P\|_\infty,$$

where P is any trigonometric polynomial with spectrum in Λ_∞:

$$P(x) = \sum_{\varepsilon_k = 0,1} a_{(\varepsilon_1,\ldots,\varepsilon_n)} e^{2\pi i (\sum_{k=1}^{n} \varepsilon_k t_k) x}$$

and

$$P_m(x, x_{m+1}, \ldots, x_n) = \sum_{\varepsilon_k=0,1} a_{(\varepsilon_1,\ldots,\varepsilon_n)} e^{2\pi i [(\sum_{k=1}^{m} \varepsilon_k t_k) x + \sum_{k=m+1}^{n} \varepsilon_k x_k]}.$$

It is then clear that, for integers t_k, the Hilbert set $\Lambda \subseteq \Lambda_\infty$ defined by

$$\Lambda = \bigcup_{n\geq 1} \Big\{t_{2n} + \sum_{k=1}^{n} \varepsilon_k t_{2k-1} : \varepsilon_k = 0 \text{ or } 1\Big\}$$

satisfies

$$\sup_{x\in\mathbb{T}} [|(\pi_{\Lambda_m} P)(x)| + |P(x) - (\pi_{\Lambda_m} P)(x)|] \leq \|P_{2m}\|_\infty \leq (1+\varepsilon)\|P\|_\infty$$

for every $m \geq m_0$, where

$$\Lambda_m = \bigcup_{n=1}^{m} \Big\{t_{2n} + \sum_{k=1}^{n} \varepsilon_k t_{2k-1} : \varepsilon_k = 0 \text{ or } 1\Big\};$$

and that means that for $m \geq m_0$,

$$\sup_{|\lambda|=1} \|\mathrm{Id} - (1+\lambda)\pi_{\Lambda_m}\| \leq 1 + \varepsilon,$$

and so $\mathcal{C}_\Lambda(\mathbb{T})$ has (ℂ-UMAP).

Indeed, we can first see that assumption (a) implies that the sets $\Lambda_{m+1} \setminus \Lambda_m$ are disjoint and that we can write

$$P(x) = (\pi_{\Lambda_m} P)(x) + \sum_{k=1}^{[n/2]-m+1} e^{2\pi i t_{2m+2k} x} Q_k(x)$$

where Q_k are trigonometric polynomials with spectrum in $\{\sum_{j=1}^{m+k} \varepsilon_j t_{2j-1} : \varepsilon_j = 0 \text{ or } 1\}$; therefore

$$P_{2m}(x, x_{2m+1}, x_{2m+2}, \ldots, x_n) = (\pi_{\Lambda_m} P)(x) + \sum_{k=1}^{[n/2]-m+1} e^{2\pi i x_{2m+2k}} Q_k(x, x_{2m+1}, x_{2m+3}, \ldots, x_{2m+2k-1}),$$

so, taking the supremum over $x_{2m+2}, x_{2m+4}, \ldots, x_{2[n/2]+2}$, and setting $x_{2m+1} = x_{2m+3} = \ldots = x$, we get

$$\|P_{2m}\|_\infty \geq \sup_{x\in\mathbb{T}} \Big[|(\pi_{\Lambda_m} P)(x)| + \sum_{k=1}^{[n/2]-m+1} |Q_k(x)|\Big] \geq \sup_{x\in\mathbb{T}} [|(\pi_{\Lambda_m} P)(x)| + |P(x) - (\pi_{\Lambda_m} P)(x)|]. \ \blacksquare$$

Although it is possible that $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP) when Λ is a Hilbert set, we have, however,

PROPOSITION 11. *If $\mathcal{C}_\Lambda(\mathbb{T})$ has* (UMAP), *then Λ cannot contain any IP-set.*

Let us recall that an *IP-set* is the set

$$\Big\{\sum_{j=1}^{n} \varepsilon_j p_j : \varepsilon_j = 0 \text{ or } 1,\ n \geq 1\Big\}$$

of all the finite sums of a given sequence $(p_n)_{n\geq 1}$ in $\mathbb{N}^*$.

Proof. This follows from the lemma below, by choosing $k_n = \sum_{j=1}^n p_{2j}$ and $l_n = \sum_{j=1}^n p_{2j+1}$.

LEMMA 12. *If there are two sequences $(k_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ in $\mathbb{Z}$ such that*

$$|k_n|, |l_n|, |k_n + l_n| \xrightarrow[n\to\infty]{} \infty$$

and such that $\Lambda \supseteq \{0, k_n, l_n, k_n + l_n : n \geq 1\}$, then $\mathcal{C}_\Lambda(\mathbb{T})$ cannot have (UMAP).

Proof. Assume that $\mathcal{C}_\Lambda(\mathbb{T})$ has (UMAP). From Lemma 5, there is a sequence $(A_n)_{n\geq 1}$ of trigonometric polynomials such that

$$\forall f \in \mathcal{C}_\Lambda \quad \|A_n * f - f\|_\infty \xrightarrow[n\to\infty]{} 0,$$

$$\|\mathrm{Id} - 2C_{A_n}\| = 1 + \varepsilon_n, \quad \text{where} \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0,$$

and where C_{A_n} is the convolution operator associated with A_n.

We choose $u(x) \equiv -1$ and n_0 large enough to have $\|A_{n_0} * u - u\|_\infty \leq 1/2$ and $\varepsilon_{n_0} < 1/33$.

We also choose n large enough in order that $k_n, l_n, k_n + l_n \notin \operatorname{spec}(A_{n_0})$. Then the functions v and w defined by

$$v(x) = e^{ik_n x} \quad \text{and} \quad w(x) = e^{il_n x}(1 + e^{ik_n x})$$

are in $\mathcal{C}_\Lambda$ and satisfy $v * A_{n_0} = 0$ and $w * A_{n_0} = 0$. We then have

$$\|u + v + w\|_\infty = \sup_{x\in\mathbb{T}} |u(x) + v(x) + w(x)| \leq \sup_{x\in\mathbb{T}} (|u(x) + v(x)| + |w(x)|) = \sup_{x\in\mathbb{T}} (|-1 + e^{ik_n x}| + |1 + e^{ik_n x}|) = 2\sqrt{2}.$$

On the other hand, since $A_{n_0} * (v + w) = 0$, we have

$$\begin{aligned}\|(\mathrm{Id} - 2C_{A_{n_0}})(u + v + w)\|_\infty &= \|v + w + [u - 2(A_{n_0} * u)]\|_\infty \\ &\geq \|v + w - u\|_\infty - 2\|u - A_{n_0} * u\|_\infty \\ &\geq \sup_{x\in\mathbb{T}} |v(x) + w(x) - u(x)| - 1 \\ &= \sup_{x\in\mathbb{T}} (|e^{ik_n x} + e^{il_n x}(1 + e^{ik_n x}) + 1|) - 1 = 3.\end{aligned}$$

We must then have $3 \leq (1 + \varepsilon_{n_0}) 2\sqrt{2}$, which is not possible since $\varepsilon_{n_0} < 1/33$. $\blacksquare$

Remark. Actually, the proof gives $\liminf \varepsilon_n \geq \sqrt{2} - 1$.

As a consequence of Lemma 12, we have:

COROLLARY 13. *If $\Lambda \supseteq \Lambda_1 + \Lambda_2$ for two infinite sets $\Lambda_1, \Lambda_2 \subseteq \mathbb{N}$, then $\mathcal{C}_\Lambda(\mathbb{T})$ does not have* (UMAP).

Indeed, by translation, we may assume that $0 \in \Lambda_1$ and $0 \in \Lambda_2$.

Remark. Y. Meyer showed ([18], Th. 3a, p. 558) that $\Lambda = \{n_k + n_l : k, l \geq 1\}$ is a $\Lambda(p)$-set for all $p \geq 1$ if $(n_k)_{k\geq 1}$ is a Hadamard sequence. However, $\mathcal{C}_\Lambda(\mathbb{T})$ does not have (UMAP) by Corollary 13.

Moreover, S. Neuwirth pointed out to me that Lemma 12 shows that the same holds for the Sidon set $\Lambda = \{0\} \cup \{2^n : n \in \mathbb{N}\}$. Hence (UMAP) seems to be connected to the rapidity of the growth to infinity.

Having ($\mathbb{C}$-UMAP) is a strong hypothesis on a Banach space, and we might expect that Λ has null uniform density whenever $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP). Indeed, this density is null when $\mathcal{C}_\Lambda(\mathbb{T})$ has no subspace isomorphic to c_0 ([17], Th. 3), or merely when Λ contains no Hilbert subset ([15], Cor. 8). We have not been able to solve this question, but Lemma 12 provides a partial answer.

Let us recall the definition of the *uniform density* of a subset $\Lambda \subseteq \mathbb{Z}$:

$$d^*(\Lambda) = \lim_{h\to\infty} \Big[\sup_{a\in\mathbb{Z}} \frac{\operatorname{card}(\Lambda \cap]a, a + h])}{h}\Big].$$

PROPOSITION 14. *$d^*(\Lambda) \le 1/2$ as soon as $\mathcal{C}_\Lambda(\mathbb{T})$ has* (UMAP).

Proof. Let Λ be such that $d^*(\Lambda) > 1/2$. We may suppose that $0 \in \Lambda$.

Then the proposition is a consequence of Lemma 12 and of the following result of N. Hindman ([12], Th. 3.4): if $d^*(\Lambda) > 1/2$, then for each $n \in \mathbb{N}$ we have $d^*(\Lambda \cap (\Lambda - n)) \ge 2d^*(\Lambda) - 1$. We should note, however, that the definition of $d^*(\Lambda)$ given by N. Hindman is (at least formally) slightly different from ours, and he assumes that $\Lambda \subseteq \mathbb{N}$, but his proof works as well in our setting.

Indeed, choosing $k_1 \in \Lambda$ with $k_1 \ne 0$, we have, by Hindman's result, $d^*(\Lambda \cap (\Lambda - k_1)) = d^*(\Lambda \cap (\Lambda - |k_1|)) > 0$, and so we can find an $l_1 \in \Lambda \cap (\Lambda - k_1)$ with $|l_1| \ge |k_1| + 1$; then $k_1, l_1, k_1 + l_1 \in \Lambda$. We iterate the same process with $\Lambda_n = \Lambda \setminus [-n, +n]$ instead of Λ; we still have $d^*(\Lambda_n) > 1/2$ and we choose $k_n \in \Lambda_n$ and find $l_n \in \Lambda_n \cap (\Lambda_n - k_n)$ such that $|l_n| \ge |k_n| + n$. ■

Our last result is the existence of a linear invariant lifting for L^∞_Λ.

PROPOSITION 15. *If $\mathcal{C}_\Lambda(G)$ has* (UMAP), *then there exists a linear invariant lifting $\mathcal{R} : L^\infty_\Lambda \to \mathrm{Ba}(G)$, where $\mathrm{Ba}(G)$ is the space of first Baire class functions. In particular, Λ has the Godefroy–Lust-Piquard's property* (ϱ) (see [9], Def. V.1).

Proof. Let $(A_n)_{n\ge1}$ be a sequence of trigonometric polynomials such that

$$\forall f \in \mathcal{C}_\Lambda \quad \|A_n * f - f\|_\infty \xrightarrow[n\to\infty]{} 0,$$
$$\|\mathrm{Id} - 2C_{A_n}\| = 1 + \varepsilon_n, \quad \text{where} \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0,$$

and where C_{A_n} is the convolution operator associated with A_n.

For all $\Phi \in \mathcal{C}^{**}_\Lambda$, the limit

$$T(\Phi) = w^*\text{-}\lim_{n\to\infty} C^{**}_{A_n}(\Phi)$$

exists (see [5], proof of Th. IV.1, or from [3], Th. 3.8, we may suppose that the series $\sum(C_{A_{n+1}-A_n})$ is weakly unconditionally convergent). Hence, for every $\mu \in \mathcal{M}$ (the space of Radon measures on G), we have $\langle T(\Phi), \mu\rangle = \lim_{n\to\infty}\langle \Phi, C^*_{A_n}(\mu)\rangle$. But $C^*_{A_n}(\mu) = \check{A}_n * \mu \in L^1$ and so the above limit depends only on $\Phi_{|L^1}$ which is an element of L^∞_Λ. Since every element of L^∞_Λ can be written in this way, we can then define for every $g \in L^\infty_\Lambda$ an element $\widetilde{T}(g) \in \mathcal{C}^{**}_\Lambda$ which satisfies, for every $\mu \in \mathcal{M}$,

$$\langle \widetilde{T}(g), \mu\rangle = \lim_{n\to\infty}\langle g, \check{A}_n * \mu\rangle = \lim_{n\to\infty}\langle A_n * g, \mu\rangle.$$

In particular, we can define, for every $x \in G$,

$$(\mathcal{R}g)(x) = \lim_{n\to\infty}\langle A_n * g, \delta_x\rangle,$$

which defines an element of the first Baire class since $A_n * g$ is a continuous function.

Moreover, by dominated convergence, for every $\varphi \in L^1$ we have

$$\int_G (\mathcal{R}g)(x)\varphi(x)\,dx = \lim_{n\to\infty}\int_G (A_n * g)(x)\varphi(x)\,dx = \lim_{n\to\infty}(A_n * g * \check{\varphi})(0) = (g * \check{\varphi})(0) = \langle g, \varphi\rangle$$

since $g * \check{\varphi} \in \mathcal{C}_\Lambda$. Hence $\mathcal{R}g$ is a representative of g.

Now, the linear form $\varrho : L^\infty_\Lambda \to \mathbb{C}$ defined by $\varrho(g) = (\mathcal{R}g)(0)$ is of the first Baire class in the w^*-topology of L^∞_Λ and $\varrho(f) = f(0)$ for every $f \in \mathcal{C}_\Lambda$, so that Λ has Godefroy–Lust-Piquard's property (ϱ). ■

As a corollary of Theorem 10 and Proposition 15, we have

THEOREM 16. *There is a set $\Lambda \subseteq \mathbb{Z}$ which has Godefroy–Lust-Piquard's lifting property (ϱ), but which is not a Rosenthal set.*

This answers negatively a question of [9], where it was conjectured that the only possibility to have property (ϱ) was that all the elements of L^∞_Λ were already continuous.

Remark. As recalled at the beginning, there is no Sidon set with constant 1 in $\mathbb{Z}$. However, it is not known if the ($\mathbb{C}$-UMAP) can be realized with an approximating sequence $(R_n)_{n\ge1}$ such that $\|\mathrm{Id} - (1+\lambda)R_n\| = 1$. Nevertheless, there is no increasing sequence of finite subsets $F_n \subseteq \mathbb{Z}$ such that $\bigcup_{n\ge1} F_n = \Lambda$ and $\|\mathrm{Id} - (1+\lambda)\pi_{F_n}\| = 1$ for all λ with $|\lambda| = 1$.

Indeed, if such a sequence existed, we could pick $n_1 \in F_1$ and $n_2 \notin F_1$, and then F_{n_2} such that $n_2 \in F_{n_2}$, and finally $n_3 \notin F_1 \cup F_{n_2}$. Then we would have

$$\begin{aligned}
\|a_1e_{n_1} + a_2e_{n_2} + a_3e_{n_3}\|_\infty
&\ge \sup_{|\lambda|=1}\|(\mathrm{Id} - (1+\lambda)\pi_{F_1})(a_1e_{n_1} + a_2e_{n_2} + a_3e_{n_3})\|_\infty\\
&= \sup_{|\lambda|=1}\|-\lambda a_1e_{n_1} + a_2e_{n_2} + a_3e_{n_3}\|_\infty\\
&= |a_1| + \|a_2e_{n_2} + a_3e_{n_3}\|_\infty\\
&\ge |a_1| + \sup_{|\lambda|=1}\|(\mathrm{Id} - (1+\lambda)\pi_{F_{n_2}})(a_2e_{n_2} + a_3e_{n_3})\|_\infty\\
&= |a_1| + \sup_{|\lambda|=1}\|-\lambda a_2e_{n_2} + a_3e_{n_3}\|_\infty = |a_1| + |a_2| + |a_3|.
\end{aligned}$$

Therefore, we would conclude that $\{n_1, n_2, n_3\}$ is a Sidon set with constant 1, which is not possible.

For the sake of completeness, we now sketch a proof of this last fact, different from that of [2], which was given to us by G. Pisier. We may assume that $n_1 = 0$ and write $n_2 = k$, $n_3 = l$. Let θ_j $(j = 1, 2)$ be the character of $\mathbb{T}^2$ which associates with (x_1, x_2) the value e^{ix_j}. If $\{0, k, l\}$ where a Sidon set with constant 1, then for every $a, b \in \mathbb{C}$ we would have $\|1 + ae_k + be_l\|_\infty = \|1 + a\theta_1 + b\theta_2\|_\infty$, and then also $\|1 + ae_k + be_l\|_p = \|1 + a\theta_1 + b\theta_2\|_p$ for every $p \in [1, \infty[$ (see [23], or [20], Th. 1). But then, from ([26], Th. I), (e_k, e_l) and (θ_1, θ_2) would have the same distribution, and this is false.

Added in proof. S. Neuwirth showed that $C_\Lambda(\mathbb{T})$ does not have ($\mathbb{C}$-UMAP) for $\Lambda = \{q^n : n \geq 1\}$, and $q \geq 2$ an integer, but $C_\Lambda(\mathbb{T})$ does have ($\mathbb{C}$-UMAP) if $\Lambda = \{n_k : k \geq 1\}$ with $n_{k+1}/n_k \to \infty$ as $k \to \infty$.

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