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Singular values, Ramanujan modular equations, and Landen transformations

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Abstract. A new connection between geometric function theory and number theory is derived from Ramanujan's work on modular equations. This connection involves the function $\varphi_K(r)$ recurrent in the theory of plane quasiconformal maps. Ramanujan's modular identities yield numerous new functional identities for $\varphi_{1/p}(r)$ for various primes p.

1. Introduction. The argument $r \in (0,1)$ of the complete elliptic integral

$$\mathcal{K}(r) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}} \tag{1.1}$$

is often called the *modulus* of $\mathcal{K}$, and the equation

$$\frac{\mathcal{K}(s')}{\mathcal{K}(s)} = p\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad p > 0, \tag{1.2}$$

where $r' := \sqrt{1-r^2}$ is the *complement* of r, is called the *modular equation of degree* p. Modular equations occur in number theory [BB], [B], [SC]. In such contexts, p is usually an integer or a rational number. Note that (1.2) makes sense for all positive real numbers p, since $\mathcal{K}(r')/\mathcal{K}(r)$ is a homeomorphism of $(0,1)$ onto $(0,\infty)$. If we use the notation [LV, Sect. II.3]

$$\varphi_K(r) := \mu^{-1}(\mu(r)/K), \qquad \mu(r) := \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)},\ K > 0, \tag{1.3}$$

the solution of (1.2) is

$$s = \varphi_{1/p}(r). \tag{1.4}$$

Since modular equations occur in different contexts such as number theory [BB] and geometric function theory [LV], it is of interest to find those values of p for which the function $\varphi_{1/p}(r)$ reduces to a simpler, algebraic function. As far as we know, there are neither systematic studies of these

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cases nor a list of positive numbers p nor even positive integers p for which the solution $s = \varphi_{1/p}(r)$ can be explicitly given. In case $p = 2^n$, $n = 0, 1, \ldots,$ then $\varphi_{1/p}(r)$ is the algebraic function obtained by iterating the descending Landen transformation n times (cf. Section 3). Note that for all $K > 0$ and $r \in (0,1)$, $\varphi_K(r)$ can be expressed as an infinite Jacobi product (see (2.7) and [J], [VV], [AV]).

The purpose of this note is to study some other particular cases of the function (1.4), using Ramanujan's results on modular equations as a tool. Ramanujan's work on modular equations remained rather inaccessible until 1991, when B. C. Berndt published Vol. III in his series of edited versions of Ramanujan's notebooks [B]. The currently known original notebooks do not contain proofs, but in [B] reconstructed proofs are given. Many of the reconstructed proofs of modular equations in [B] are due to G. N. Watson, as Berndt points out. As in [B], we mean by a modular equation not only the transcendental equation (1.2), but also an algebraic equation that follows from (1.2) and involves r and s. An example is the classical Legendre modular equation of degree 3,

$$\sqrt{rs} + \sqrt{r's'} = 1, \qquad s = \varphi_{1/3}(r). \tag{1.5}$$

Although it is unorthodox to write the Legendre modular equation in terms of $\varphi_K(r)$, this notation is suitable for our purpose. An equivalent form of (1.5) in Ramanujan's (α, β)-notation is

$$\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1, \tag{1.5$'$}$$

with $\alpha = r^2$, $\beta = \varphi_{1/3}(r)^2$. Dozens of algebraic modular equations of various degrees were given by Ramanujan in his notebooks, and a helpful chart of his work on this topic appears in [B, pp. 8–9]. Three proofs of (1.5) are given in [H, pp. 214–218]. In some cases these algebraic modular equations reduce to solvable polynomial equations, as does (1.5). In such cases one can obtain the solution s of (1.2) as an explicit algebraic function of r. In particular, $\varphi_{1/3}(r)$ can be explicitly determined from (1.5) (the solution was recently worked out in [KZ]) and thus, by the symmetric formula (2.1) and the composition property (2.2) and (3.6), we have explicit expressions for $\varphi_{2^p3^q}(r)$, $p, q \in \mathbb{Z}$. However, even the formula for $\varphi_{1/3}(r)$ is so long that it would be difficult to write down. Long formulas like that can be manipulated reasonably only in computer symbolic computation programs. Theorem 2.3 below summarizes from [B] those results of Ramanujan that we shall use.

In this note we confine our attention to a particular case of the problem of finding $\varphi_{1/p}(r)$ explicitly, namely to the problem of finding the pth *singular value* r_p of the complete elliptic integral $\mathcal{K}(r)$. The singular value r_p is defined by ([SC], [BB])

$$\mu(r_p) = \frac{\pi}{2}\sqrt{p}, \qquad p > 0. \tag{1.6}$$

Since $\mu(1/\sqrt{2}) = \pi/2$, we see that the solution of (1.6) is

$$r_p = \varphi_{1/\sqrt{p}}(1/\sqrt{2}). \tag{1.7}$$

Thus if we can solve (1.2) for a given $\sqrt{p} > 0$ we also obtain the solution r_p of (1.6). For several integral values of p the solutions of (1.6) are given in [BB, p. 298]. The arithmetic character of r_p was investigated in [SC].

The main observation of this note is that the quasiconformal distortion function (1.3) satisfies numerous identities given by Ramanujan. We present some applications of these identities. In Theorem 2.6 we give some singular values (cf. [BB, p. 139, p. 298]) and also give an evaluation of an infinite product (cf. Corollary 2.8 below). We also provide some inequalities for the function $s = \varphi_K(r)$ for all $K > 0$ in terms of the Landen transformation. Recently similar inequalities were obtained in [AVV2], [P1–3], and [QVV]. The many modular identities of Ramanujan in [B] and in Section 2 may lead to function-theoretic applications since $\varphi_K(r)$ measures the extremal radial distortion of a normalized K-quasiconformal map of the unit disk into itself [LV]. Ramanujan's work on modular equations was based on theta functions, see [B] or the recent papers [BBG], [BC], [FK] and [S1], [S2]. At the end of the paper we give a conjecture.

We use the notation $\operatorname{ch} x$ and $\operatorname{th} x$ for the hyperbolic cosine and tangent, respectively, and $\operatorname{arch} x$, $\operatorname{arth} x$ for their inverse functions.

2. Ramanujan modular equations. From the identity $\mu(r)\mu(r') = \pi^2/4$ it follows that the function φ_K satisfies the following symmetric identity:

$$\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1 \tag{2.1}$$

for $K > 0$, $r \in (0,1)$. From (1.3) it follows trivially that φ_K has the composition property

$$\varphi_{AB}(r) = \varphi_A(\varphi_B(r)), \qquad A, B > 0,\ r \in (0,1), \tag{2.2}$$

which also implies that $\varphi_{1/K}(r) = \varphi_K^{-1}(r)$.

In the next theorem we state those modular equations of Ramanujan that we shall use later. There are many more similar results of Ramanujan in [B] which probably could be used for the same purpose, and we hope to stimulate further work by proving just a few such results. Note that Theorem 2.3(3) is due to Schröter [B, p. 352] and was discovered independently by Ramanujan.

2.3. THEOREM. *The function φ_K satisfies the following identities*:

(1) *For* $\alpha = r^2$, $\beta = \varphi_{1/5}(r)^2$, *we have*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1.$$

(2) *For* $\alpha = r^2$, $\beta = \varphi_{1/7}(r)^2$, *we have*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1.$$

(3) *For* $\alpha = r^2$, $\beta = \varphi_{1/3}(r)^2$, $\gamma = \varphi_{1/9}(r)^2$ *we have*

$$\{\alpha(1-\gamma)\}^{1/8} + \{\gamma(1-\alpha)\}^{1/8} = 2^{1/3}\{\beta(1-\beta)\}^{1/24}.$$

(4) *For* $\alpha = r^2$, $\beta = \varphi_{1/23}(r)^2$, *we have*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1.$$

(5) *For* $\alpha = r^2$, $\beta = \varphi_{1/7}(r)^2$, *or for* $\alpha = \varphi_{1/3}(r)^2$, $\beta = \varphi_{1/5}(r)^2$, *we have*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = \left\{\frac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})\right\}^{1/2}.$$

Proof. All of these identities are from [B]: (1) is [B, p. 280, Entry 13(i)]; (2) is p. 314, Entry 19(i); (3) is p. 352, Entry 3(vi); (4) is p. 411, Entry 15(i); and (5) is p. 435, Entry 21(i). ∎

2.4. COROLLARY. *The function φ_K satisfies the following identities for* $s \in (0,1)$:

(1) $xy + x'y' + 2^{5/3}\{xyx'y'\}^{1/3} = 1$, *where* $x = \varphi_{\sqrt{5}}(s)$, $y = \varphi_{1/\sqrt{5}}(s)$,

(2) $(xy)^{1/4} + (x'y')^{1/4} = 1$, *where* $x = \varphi_{\sqrt{7}}(s)$, $y = \varphi_{1/\sqrt{7}}(s)$,

(3) $(xy)^{1/4} + (x'y')^{1/4} = 2^{1/3}(s^2(1-s^2))^{1/24}$, *where* $x = \varphi_3(s)$, $y = \varphi_3(s')$,

(4) $(xy)^{1/4} + (x'y')^{1/4} + 2^{2/3}(xx'yy')^{1/12} = 1$, *where* $x = \varphi_{1/\sqrt{23}}(s)$, $y = \varphi_{\sqrt{23}}(s)$,

(5) $(xy)^{1/4} + (x'y')^{1/4} - \{xx'yy'\}^{1/4} = \{\frac{1}{2}(1+xy+x'y')\}^{1/2}$, *where* $x = \varphi_{\sqrt{5/3}}(s)$, $y = \varphi_{\sqrt{3/5}}(s)$.

Proof. All of these identities follow from Theorem 2.3 and (2.1)–(2.2) in the same way as soon as r is chosen appropriately. For this reason we give here the details only for (5). Set $r = \varphi_{\sqrt{15}}(s)$. By (2.1) and (2.2) we see that

$$\alpha = \varphi_{1/3}(r)^2 = \varphi_{\sqrt{5/3}}(s)^2, \quad 1-\alpha = \varphi_{\sqrt{3/5}}(s')^2,$$

and thus the assertion follows from Theorem 2.3(5). ∎

2.5. COROLLARY. *We have the following identities*:

(1) $2uu' + 2^{5/3}(uu')^{2/3} = 1$; $u = \varphi_{\sqrt{5}}(1/\sqrt{2})$,

(2) $2(uu')^{1/4} = 1$; $u = \varphi_{\sqrt{7}}(1/\sqrt{2})$,

(3) $\sqrt{u} + \sqrt{u'} = 2^{1/4}$; $u = \varphi_3(1/\sqrt{2})$,

(4) $2(uu')^{1/4} + 2^{2/3}(uu')^{1/6} = 1$; $u = \varphi_{\sqrt{23}}(1/\sqrt{2})$,

(5) $2(uu')^{1/4} - (uu')^{1/2} = \{\frac{1}{2}(1+2uu')\}^{1/2}$; $u = \varphi_{\sqrt{5/3}}(1/\sqrt{2})$.

Proof. All of these identities follow from Corollary 2.4 and (2.1)–(2.2) in the same way. We give here the details only for (5). Set $s = 1/\sqrt{2}$ in Corollary 2.4(5) and observe that then $x' = y$, $y' = x$, and thus (5) follows. ∎

Parts (1) and (2) of the next theorem are given on page 139 of [BB] in a slightly different form, whereas part (3) is probably new.

2.6. THEOREM. *The function φ has the following special values*:

$$\varphi_{1/\sqrt{3}}(1/\sqrt{2}) = \frac{\sqrt{2-\sqrt{3}}}{2}, \tag{1}$$

$$\varphi_{1/\sqrt{7}}(1/\sqrt{2}) = \frac{\sqrt{8-\sqrt{63}}}{4}, \tag{2}$$

$$\varphi_{1/\sqrt{23}}(1/\sqrt{2}) = \sqrt{\frac{1-\sqrt{1-4z^{24}}}{2}}, \tag{3}$$

where

$$z = -\frac{1}{3\sqrt[3]{2}} + \frac{1}{3(25+\sqrt{621})^{1/3}} + \frac{\sqrt[3]{25+\sqrt{621}}}{3\cdot 2^{2/3}}.$$

Proof. (1) We write (1.5′) as

$$\sqrt{r\varphi_{1/3}(r)} + \sqrt{r'\varphi_3(r')} = 1$$

and then set $r = \varphi_{\sqrt{3}}(1/\sqrt{2})$. Solving this yields the desired formula.

Parts (2) and (3) follow from Corollary 2.5(2) and (4), respectively. We have used the Mathematica computer program for part (3). ∎

We next recall Jacobi's product formula [J], [VV] for $r \in (0,1)$, $K > 0$:

$$\varphi_K(r) = \prod_{j=1}^{\infty} \mathrm{th}^4((2j-1)K\mu(r')). \tag{2.7}$$

2.8. COROLLARY. *We have*

$$\prod_{j=1}^{\infty} \mathrm{th}^4\left((2j-1)\frac{\pi}{2\sqrt{3}}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Proof. Combine (2.7) and Theorem 2.6(1). ∎

3. Approximation by the Landen transformation

3.1. *Landen transformation.* For $r \in (0,1)$ the elliptic integral $\mathcal{K}(r)$ satisfies the Landen identity [AB], [BB], [L]

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \tag{3.2}$$

which yields the functional identity

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right) \tag{3.3}$$

for $r \in (0,1)$. The transformation $r \mapsto 2\sqrt{r}/(1+r)$ is called the *ascending Landen transformation*, and its inverse $r \mapsto (r/(1+r'))^2$ is called the *descending Landen transformation*. To study these and their iterates we set $L(r,0) = r$ and define

$$\begin{aligned} L(r,-k) &= \left(\frac{L(r,-k+1)}{1+L(r,-k+1)'}\right)^2, \\ L(r,k) &= \frac{2\sqrt{L(r,k-1)}}{1+L(r,k-1)}, \end{aligned} \tag{3.4}$$

for $k = 1, 2, \ldots$ It follows easily from (3.3) that

$$\mu(L(r,-k)) = 2^k \mu(r), \tag{3.5}$$

for all integers k, and also that

$$\varphi_{2^k}(r) = L(r,k) \tag{3.6}$$

for all integers k and $r \in (0,1)$. We also see that the Landen transformation satisfies the following composition property:

$$L(L(r,m),n) = L(r,m+n). \tag{3.7}$$

We next apply these identities to prove inequalities for the function $\varphi_K(r)$.

3.8. THEOREM. *For $r \in (0,1)$ and $p = 1, 2, \ldots$, $\varphi_K(L(r,p)) = L(\varphi_K(r), p)$ for $K > 0$. For $K > 1$ and $r \in (0,1)$,*

$$L(r,-p)^{1/K} \le L(\varphi_K(r), -p) \le 4^{1-1/K} L(r,-p)^{1/K}, \tag{1}$$

$$4^{1-K} L(r,-p)^K \le L(\varphi_{1/K}(r), -p) \le L(r,-p)^K. \tag{2}$$

Proof. The identity follows from (3.6) and (2.2). For (1) observe first that $\varphi_K(r) > r$ for $K > 1$, and similarly $L(\varphi_K(r), -p) > L(r,-p)$. Next, since $\mu(r)/\log(4/r)$ and $\mu(r)/\log(1/r)$ are decreasing and increasing, respectively [AVV1, Lemma 4.2], we have, for $t \in (0,1)$,

$$\begin{aligned} \mu(s_p)/\log(4/s_p) &\le \mu(t_p)/\log(4/t_p), \\ \mu(t_p)/\log(1/t_p) &\le \mu(s_p)/\log(1/s_p), \end{aligned}$$

where $s_p = L(\varphi_K(t), -p)$ and $t_p = L(t,-p)$. The proof of (1) now follows, since $\mu(s_p) = \mu(t_p)/K$. The proof of (2), which is similar, uses the relation $\varphi_{1/K}(r) < r$ for $K > 1$. ∎

The next corollary is similar to a result of D. Partyka [P2, Theorem 1.3].

3.9. COROLLARY. *For $r \in (0,1)$ and $K > 1$, $p = 1, 2, \ldots$,*

$$\varphi_K(r) \ge L(L(r,-p)^{1/K}, p). \tag{1}$$

Next, if $K > 1$, $r \in (0,1)$, and p_0 is so large that $4^{1-1/K} L(r,-p_0)^{1/K} < 1$, then, for $p = p_0, p_0+1, \ldots$,

$$\varphi_K(r) \le L(4^{1-1/K} L(r,-p)^{1/K}, p). \tag{2}$$

Further, for $K > 1$, $r \in (0,1)$, $p = 1, 2, \ldots$,

$$L(4^{1-K} L(r,-p)^K, p) \le \varphi_{1/K}(r) \le L(L(r,-p)^K, p). \tag{3}$$

Finally, for $K > 1$, $r \in (0,1)$,

$$\sqrt{1 - L(L(r',-p)^K, p)^2} \le \varphi_K(r) \le \sqrt{1 - L(4^{1-K} L(r',-p)^K, p)^2}. \tag{4}$$

Proof. These inequalities follow from Theorem 3.8, (3.7), and (2.1). ∎

Note that from (2.1) and (3.7) we obtain

$$L(r,p)^2 + L(r',-p)^2 = 1 \tag{3.10}$$

for $r \in (0,1)$ and $p \in \mathbb{Z}$. The idea of using the Landen transformations for the estimation of $\varphi_K(r)$ was indicated in [AVV2, Remark 4.23], and it was also independently studied by D. Partyka [P1, P2]. He has shown, for instance, that these upper and lower bounds converge quite quickly to the value of the function $\varphi_K(r)$; see Remark 3.22 below.

The function $\mathcal{K}(r)$, and thus also $\mu(r)$, can be most efficiently computed with the help of the well-known arithmetic-geometric mean iteration [BB]. The function $\varphi_K(r) = \mu^{-1}(\mu(r)/K)$ is more tedious to compute because of the presence of μ^{-1}. We shall next discuss these functions in more detail.

First, Jacobi's inversion formula gives ([J], [L], [VV])

$$\mu^{-1}(y) = \sqrt{1 - \prod_{n=1}^{\infty} \operatorname{th}^8((2n-1)y)} = \prod_{n=1}^{\infty} \operatorname{th}^4\left((2n-1)\frac{\pi^2}{4y}\right), \tag{3.11}$$

for all $y \in (0,\infty)$. From (3.11) we obtain

$$\sqrt{1 - \operatorname{th}^8 y} < \mu^{-1}(y) < \operatorname{th}^4\left(\frac{\pi^2}{4y}\right) \tag{3.12}$$

and

$$\mu(r) > \operatorname{arth} \sqrt[4]{r'}. \tag{3.13}$$

Combining (3.13) with $\mu(r) < \log(2(1+r')/r)$ [LV, p. 62] we obtain

$$\operatorname{arth} \sqrt[4]{s'} < \mu(s) < \log 2 + \operatorname{arch} \frac{1}{s}, \tag{3.14}$$

and putting $s = L(r, -p)$ we get, by (3.5),

$$\operatorname{arth} \sqrt[4]{L(r,-p)'} < 2^p \mu(r) < \log 2 + \operatorname{arch} \frac{1}{L(r,-p)}. \tag{3.15}$$

Substituting $y = \mu(r)$ and solving for $r = \mu^{-1}(y)$ yields, for all $p = 1, 2, \ldots$,

$$L\Big(\sqrt{1-\operatorname{th}^8(2^p y)}, p\Big) < \mu^{-1}(y) < L(1/\operatorname{ch}((2^p y - \log 2)_+), p), \tag{3.16}$$

where $a_+ = \max\{a, 0\}$. Note that in (3.16) both bounds for $\mu^{-1}(y)$ are elementary functions.

Finally we remark (cf. (3.15)) that Ramanujan derived a very close approximation for $\mu(r)$. In [B, p. 91] this approximation is given in the form

$$F(1-e^{-x}) \approx \frac{x}{10 + \sqrt{36 + x^2}} \tag{3.17}$$

with $F(t) = \exp\big(-2\mu(\sqrt{t})\big)$. Solving (3.17) for $\mu(r)$ we get a close approximation in terms of elementary functions for small $r \in (0, 0.5)$.

3.18. *Asymptotic behavior of singular values.* Computer experiments led us to the conjecture that for $p \to \infty$ we have the asymptotic formula

$$\varphi_{1/p}(r) \sim 4\exp(-p\mu(r)). \tag{3.19}$$

The following analytic proof of (3.19) was kindly provided by G. D. Anderson. We have

$$\begin{aligned}\exp(p\mu(r))\varphi_{1/p}(r) &= \exp(p\mu(r))\mu^{-1}(p\mu(r)) \\ &= \exp(s)\mu^{-1}(s) = x\exp(\mu(x)) \to 4, \quad x \to 0,\end{aligned}$$

where $s = p\mu(r)$, $x = \mu^{-1}(s)$, and the last limit follows from [LV, (2.11), p. 62].

Note that (3.19) provides information about the asymptotic behavior of singular values (1.7):

$$r_{p^2} = \varphi_{1/p}(1/\sqrt{2}) \sim 4\exp\Big(-\frac{\pi}{2}p\Big) \quad (p \to \infty). \tag{3.20}$$

3.21. CONJECTURE. We have seen in Theorem 2.6 that for some algebraic numbers $K > 0$ the values $\varphi_K(1/\sqrt{2})$ are also algebraic. (Note that $\mu(1/\sqrt{2}) = \pi/2$ is not algebraic.) The following question arises: is it true that $\varphi_K(r)$ is an algebraic number whenever both r and K are? We conjecture that this is the case.

3.22. Remark. In [P2, Theorem 1.5] the rate of convergence of the Landen approximation is studied, and it is shown for instance that, for all $0 < x < 1$,

$$\begin{aligned} L(\varphi_K(x), -n) &\le \min\{4^{1-1/K} L^{1/K}(x,-n), 1\} \\ &\le L(\varphi_K(x)(1 - x^{2^{n+1}/K})^{-1}, -n) \end{aligned}$$

for $K \ge 1$, $n = 2, 3, \ldots$, and

$$L(\varphi_K(x)(1-x^{2^{n+1}})^{1/K2^n}, -n) \le 4^{1-1/K} L^{1/K}(x,-n) \le L(\varphi_K(x), -n)$$

for $0 < K \le 1$, $n = 1, 2, \ldots$ Further results appear in [P1, Theorem 1.2].

D. Partyka [P3] has also proved, for every $K > 1$, the identity

$$\max_{0 \le r \le 1} (\varphi_K^2(\sqrt{r}) - r) = M(K) := 1 - 2\bigg(\mu^{-1}\Big(\frac{\pi}{2}\sqrt{K}\Big)\bigg)^2 = 1 - 2r_K^2,$$

as well as an analogous result for the case $K \in (0, 1)$.

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Complex Unconditional Metric Approximation Property for $\mathcal{C}_\Lambda(\mathbb{T})$ spaces

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Abstract. We study the Complex Unconditional Metric Approximation Property for translation invariant spaces $\mathcal{C}_\Lambda(\mathbb{T})$ of continuous functions on the circle group. We show that although some "tiny" (Sidon) sets do not have this property, there are "big" sets Λ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP); though these sets are such that $L^\infty_\Lambda(\mathbb{T})$ contains functions which are not continuous, we show that there is a linear invariant lifting from these $L^\infty_\Lambda(\mathbb{T})$ spaces into the Baire class 1 functions.

Introduction. The translation invariant subspaces of continuous functions on $\mathbb{T}$ all have the Metric Approximation Property (MAP). We study in this paper the spaces $\mathcal{C}_\Lambda(\mathbb{T})$ which satisfy a stronger approximation property, the Complex Unconditional Metric Approximation Property ($\mathbb{C}$-UMAP).

The (Real) Unconditional Approximation Property (UMAP) was introduced in 1989 by P. Casazza and N. Kalton as an extreme possibility of approximation ([3], Th. 3.5), and they showed ([3], Th. 3.8) that it actually coincides for a separable Banach space X with the existence for every $\varepsilon > 0$ of an unconditional expansion of the identity of X with constant $1+\varepsilon$, which means, by a result of A. Pełczyński and P. Wojtaszczyk ([21], Th. 1.1) that for every $\varepsilon > 0$, X may be isometrically embedded in a Banach space Y with a $(1+\varepsilon)$-FDD for which there is a projection $P : Y \to X$ with $\|P\| \le 1+\varepsilon$. Its complex version was defined and studied in ([7], §§8 and 9).

To begin with, we construct subsets $\Lambda \subseteq \mathbb{Z}$ for which $\mathcal{C}_\Lambda(\mathbb{T})$ has ($\mathbb{C}$-UMAP). They are of two kinds: the first contain arbitrarily long arithmetical progressions, so that they are not $\Lambda(1)$-sets, but their pace tends to infinity; the second are Sidon sets, but have a pace which does not tend to infinity.

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