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Generalized limits and a mean ergodic theorem

by

YUAN-CHUAN LI and SEN-YEN SHAW (Chung-Li)

Abstract. For a given linear operator L on ℓ^∞ with $\|L\| = 1$ and $L(1) = 1$, a notion of limit, called the L-limit, is defined for bounded sequences in a normed linear space X. In the case where L is the left shift operator on ℓ^∞ and $X = \ell^\infty$, the definition of L-limit reduces to Lorentz's definition of σ-limit, which is described by means of Banach limits on ℓ^∞. We discuss some properties of L-limits, characterize reflexive spaces in terms of existence of L-limits of bounded sequences, and formulate a version of the abstract mean ergodic theorem in terms of L-limits. A theorem of Sinclair on the form of linear functionals on a unital normed algebra in terms of states is also generalized.

1. Introduction. In [7] Lorentz defined the so-called σ-limits for bounded sequences in ℓ^∞, the Banach algebra of all bounded sequences in $\mathbb{C}$, with the supremum norm $\|\cdot\|_\infty$. It has been studied by many authors (see [1], [7]–[11]). This notion of limit can be generalized to bounded sequences in a general normed linear space in the following way.

Let $\mathbf{A}$ be a complex normed algebra with unit element 1. The *state space* of $\mathbf{A}$ is the set

$$D(1) \equiv D(1, \mathbf{A}) := \{\phi \in \mathbf{A}^* : \|\phi\| = \phi(1) = 1\}.$$

For a bounded linear operator U on $\mathbf{A}$ with $U1 = 1$ and $\|U\| = 1$, we denote by π_U the set $\{\phi \in D(1) : U^*\phi = \phi\}$.

Definition. Let L be a linear operator on ℓ^∞ (in notation, $L \in B(\ell^\infty)$) such that $L1 = 1$ and $\|L\| = 1$. A bounded sequence $\{x_n\}$ in a normed linear space X is said to have an L-*limit* x ($x \in X$), written as $L\text{-}\lim x_n = x$, if

$$F\{\langle f, x_n\rangle\} = \langle f, x\rangle \quad \text{for all } f \in X^* \text{ and } F \in \pi_L.$$

Let ψ be a map from $\mathbb{N}_0$ to $\mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all nonnegative integers, and let $\widehat{\psi} \in B(\ell^\infty)$ be the operator defined as $(\widehat{\psi}x)(n) = x(\psi(n))$,

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$n = 0, 1, \ldots$ ($x \in \ell^\infty$). Then $\widehat{\psi}1 = 1$ and $\|\widehat{\psi}\| = 1$. In particular, if σ is the map $n \to n+1$ ($n \in \mathbb{N}_0$), then $\widehat{\sigma}$ is the left shift operator defined by $(\widehat{\sigma}x)(n) = x(n+1)$ for $n \in \mathbb{N}_0$ ($x \in \ell^\infty$), and $\pi_{\widehat{\sigma}}$ is the set of all Banach limits on ℓ^∞. Thus, when $X = \ell^\infty$ and $L = \widehat{\sigma}$, the above definition of L-limit reduces to Lorentz's definition of σ-limit for bounded sequences in ℓ^∞ (see [7]).

The main purpose of this paper is to present some results concerning L-limits. Section 2 has independent interest. Theorem 2.1 generalizes a theorem of Sinclair [13], which shows that the states of a Banach algebra span the dual space. A corollary (Corollary 2.3) is used in Section 3, which is concerned with basic properties of L-limits. Theorem 3.4 characterizes reflexive spaces as those with the property that every bounded sequence has an L-limit for some $L \in B(\ell^\infty)$ satisfying $L(1) = 1$, $\|L\| = 1$, and $L^m\{a_n\} \to 0$ for all $\{a_n\} \in c_0$. And, in Proposition 3.5 we use L-limits to characterize closed operators and bounded operators.

In Theorem 3.2, it is shown in particular that

$$\sigma\text{-}\lim x_n = x \Leftrightarrow w\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} x_{k+n} = x \text{ uniformly in } n.$$

Therefore we have

$$\sigma\text{-}\lim x_n = y \Rightarrow w\text{-}\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} x_k = y. \tag{1.1}$$

In general, the converse of the above implication does not hold. For instance, consider the sequence $\{x_n\}$ in $\mathbb{R}$ which is defined as $x_n = 0$ when $\sum_{k=1}^{2m} k \le n < \sum_{k=1}^{2m+1} k$ for some integer $m \ge 1$, and $x_n = 1$ otherwise. It is easy to see that for any fixed $m > 1$, $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} x_{k+m} = 1/2$. But for any positive integer n the set $\{a_m, a_{m+1}, \ldots, a_{m+n}\}$ is equal to $\{0\}$ for infinitely many m. Hence the last convergence is not uniform in m, and so $\sigma\text{-}\lim x_n = 1/2$ is not true.

A natural question to ask is: when is the converse to (1.1) true? By the mean ergodic theorem, if $x_n = T^n x$ with T a power bounded operator on a Banach space X and $x \in X$, then the existence of a weak cluster point x of the sequence $\{\frac{1}{n+1}\sum_{k=0}^{n} x_k\}$ implies $s\text{-}\lim_{n\to\infty} x_n = x$. Does this remain true if one replaces the strong limit by the weaker notion of σ-limit? This leads us to consider in Section 4 mean ergodic theorems for L-lim. Theorem 4.1 is an L-limit analogue of the abstract mean ergodic theorem of [12]. Corollary 4.3 is an equivalent version of the classical Cesàro mean ergodic theorem. In particular, we have the equivalences:

$$\sigma\text{-}\lim T^n x = y \Leftrightarrow y \text{ is a weak cluster point of } \{T_n x : n = 0, 1, \ldots\}$$
$$\Leftrightarrow s\text{-}\lim_{n\to\infty} T_n x = y \Leftrightarrow \sigma\text{-}\lim T_n x = y,$$

where $T_n := \frac{1}{n+1}\sum_{k=0}^{n} T^k$.

2. A generalized Sinclair theorem. Let $\mathbf{A}$ be a complex normed algebra with unit element 1. The *numerical radius* of an element $a \in \mathbf{A}$ is $v(a) := \sup\{|\phi(a)| : \phi \in D(1)\}$. $v(\cdot)$ is a norm on $\mathbf{A}$ satisfying $\|a\|/e \le v(a) \le \|a\|$ ($a \in \mathbf{A}$) (see [2, Theorem 5.1]). It was proved by Sinclair (see [3, §31] or [13]) that the dual space $\mathbf{A}^*$ of $\mathbf{A}$ is the linear span of $D(1)$. More precisely, for each $\phi \in \mathbf{A}^*$ there exist $\alpha_k \ge 0$ and $\phi_k \in D(1)$, $k = 1, 2, 3, 4$, such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \le \sqrt{2}\, v(\phi) \text{ and } \phi = \alpha_1\phi_1 - \alpha_2\phi_2 + i(\alpha_3\phi_3 - \alpha_4\phi_4), \tag{2.1}$$

where $v(\phi) := \sup\{|\phi(a)| : a \in \mathbf{A}, v(a) \le 1\}$. This is the special case $U = I$ of the following generalization.

THEOREM 2.1. *Let $\mathbf{A}$ be a complex unital normed algebra and $U \in B(\mathbf{A})$ with $U1 = 1$ and $\|U\| = 1$. If ϕ is a fixed point of U^*, then there exist $\alpha_k \ge 0$ and $\phi_k \in \pi_U$, $k = 1, 2, 3, 4$, such that* (2.1) *holds.*

To prove Theorem 2.1 we need the following lemma.

LEMMA 2.2. *Let U be as in Theorem 2.1. If $\{\phi_n\}$ is a sequence in $D(1)$, then π_U contains all w^*-cluster points of $\{U_n^*\phi_n\}$, where $U_n = \frac{1}{n+1}\sum_{k=0}^{n} U^k$.*

Proof. Let ψ be an arbitrary w^*-cluster point of $\{U_n^*\phi_n\}$. Since $\|U\| \le 1$, we have $\|U_n\| \le 1$, $\|\psi\| \le 1$, and $U_n(U - I) = \frac{1}{n+1}(U^{n+1} - I) \to 0$ as $n \to \infty$ in uniform operator norm. Fix any $x \in \mathbf{A}$. Then there is a subsequence $\{U_{n_j}^*\phi_{n_j}\}$ so that $\lim_{j\to\infty} U_{n_j}^*\phi_{n_j}(Ux - x) = \psi(Ux - x)$. Hence

$$|\psi(Ux - x)| = \lim_{j\to\infty} |U_{n_j}^*\phi_{n_j}(Ux - x)| = \lim_{j\to\infty} |\phi_{n_j}[U_{n_j}(Ux - x)]|$$
$$\le \lim_{j\to\infty} \|U_{n_j}(Ux - x)\| = 0.$$

Therefore $U^*\psi = \psi$. Since $U_n^*\phi(1) = 1$ for all $\phi \in D(1)$ and $n \ge 1$, we must have $\psi(1) = 1$, so that $\psi \in D(1)$. Hence $\psi \in \pi_U$.

Proof of Theorem 2.1. Let $\phi \in N(U^* - I)$. It follows from Sinclair's theorem that there exist $\alpha_k \ge 0$ and $\psi_k \in D(1)$ ($k = 1, 2, 3, 4$) such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \le \sqrt{2}\, v(\phi), \quad \phi = \alpha_1\psi_1 - \alpha_2\psi_2 + i(\alpha_3\psi_3 - \alpha_4\psi_4).$$

Since $U^*\phi = \phi$, we have

$$\phi = \alpha_1 U_n^*\psi_1 - \alpha_2 U_n^*\psi_2 + i(\alpha_3 U_n^*\psi_3 - \alpha_4 U_n^*\psi_4), \tag{2.2}$$

and, since, for $k = 1, 2, 3, 4$, $\{U_n^*\psi_k\}$ is a sequence in the w^*-compact set $D(1)$, there is a subnet $\{U_{n_j}\}$ of $\{U_n\}$ such that $\{U_{n_j}\psi_k\}$ converges weakly* for $k = 1, 2, 3, 4$. Let $\phi_k := w^*\text{-}\lim_j U_{n_j}\psi_k$ for $k = 1, 2, 3, 4$. It follows from Lemma 2.2 that $\phi_k \in \pi_U$ for $k = 1, 2, 3, 4$. From this and (2.2) we obtain

$$\phi = \alpha_1\phi_1 - \alpha_2\phi_2 + i(\alpha_3\phi_3 - \alpha_4\phi_4).$$

This completes the proof.

COROLLARY 2.3. *Let* $\mathbf{A}$ *be a complex normed algebra and* $U \in B(\mathbf{A})$ *with* $U1 = 1$ *and* $\|U\| = 1$. *Then* $\overline{R(U-I)} = \bigcap_{\phi\in\pi_U} \ker\phi$.

Proof. Let $M := \bigcap_{\phi\in\pi_U} \ker\phi$. It follows from the definition of π_U that $\phi \in N(U^*-I^*) = \overline{R(U-I)}^{\perp}$ for all $\phi \in \pi_U$. Hence $\overline{R(U-I)} \subset M$. Let $\phi \in N(U^*-I^*)$. From Theorem 2.1 we have

$$\phi = \alpha_1\phi_1 - \alpha_2\phi_2 + i(\alpha_3\phi_3 - \alpha_4\phi_4)$$

for some $\alpha_k \geq 0$ and $\phi_k \in \pi_U$ $(k = 1,2,3,4)$. Hence $M \subset \bigcap_{k=1}^4 \ker\phi_k \subset \ker\phi$. Since $\phi \in N(U^*-I^*)$ is arbitrary, we must have

$$M \subset \bigcap\{\ker\phi : \phi \in N(U^*-I^*)\} = {}^{\perp}N(U^*-I^*) = \overline{R(U-I)}.$$

The result follows.

3. Some properties of L-limits. We consider the following condition:

$$(*) \qquad \lim_{m\to\infty} L^m\{a_n\} = 0 \quad \text{for all } \{a_n\} \in c_0,$$

where c_0 is the space of all null sequences. If $\|L\| = 1$, it is clear that the condition $(*)$ is equivalent to

$$\lim_{m\to\infty} L^m e_k = 0 \quad \text{in } \ell^\infty \text{ for all } k \in \mathbb{N}_0.$$

Clearly, condition $(*)$ is satisfied if $c_0 \subset \ker L$. But, in general, an operator L which satisfies condition $(*)$ need not have the property $c_0 \subset \ker L$. For instance, $\frac{1}{2}(I+\widehat{\sigma})$ and $\widehat{\sigma}$ satisfy $(*)$ but their kernels do not contain c_0. A consequence of condition $(*)$ is that

$$\begin{aligned} F\{a_n\} &= (L^*)^m F\{a_n\} = FL^m\{a_n\} = \lim_{m\to\infty} FL^m\{a_n\} \\ &= F\lim_{m\to\infty} L^m\{a_n\} = 0 \end{aligned}$$

for all $F \in \pi_L$ and $\{a_n\} \in c_0$.

Let $\mathbf{K} := \{L \in B(\ell^\infty) : L1 = 1,\ \|L\| = 1, \text{ and } L \text{ satisfies } (*)\}$. Some useful basic properties of the L-limit are as follows.

PROPOSITION 3.1. *Let* $L \in \mathbf{K}$ *and let* $\{x_n\}$ *be a bounded sequence in a normed linear space* X. *Then*

(a) *If* $x_n \to x$ *weakly, then* $L\text{-}\lim x_n = x$.
(b) *If* $\{x_n\}$ *has an* L*-limit, then it is unique.*
(c) *If* $f \in X^*$ *and* $L\text{-}\lim x_n = x$, *then* $L\text{-}\lim f(x_n) = f(x)$.
(d) *If* $L\text{-}\lim x_n = x$, *then* $\|x\| \leq \limsup \|x_n\|$.
(e) *If* $\{y_n\}$ *is a bounded sequence in* X, *and* $L\text{-}\lim y_n = y$ *and* $L\text{-}\lim x_n = x$ *for some* $y \in X$, *then*

$$ax + by = L\text{-}\lim(ax_n + by_n).$$

(f) *If* $L\text{-}\lim x_n = x$ *and* $L\text{-}\lim y_n = y$, *then* $L\text{-}\lim(x_n, y_n) = (x, y)$, *where both* (x_n, y_n) *and* (x, y) *are in* $X \oplus X$.
(g) *If* $L\text{-}\lim x_n = x$ *and* $T \in B(X)$, *then* $L\text{-}\lim Tx_n = Tx$.

Proof. (a) If $f \in X^*$ and $w\text{-}\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty}\langle f, x_n - x\rangle = 0$. Therefore we have for $F \in \pi_L$,

$$F\{\langle f, x_n\rangle\} = F\{\langle f, x_n - x\rangle\} + F\{\langle f, x\rangle\} = 0 + \langle f, x\rangle.$$

This proves $L\text{-}\lim x_n = x$. (b)–(e) follow from the definition of L-limit and (f) follows from $(X\oplus X)^* = X^* \oplus X^*$. If $L\text{-}\lim x_n = x$ and $T \in B(X)$, then for $F \in \pi_L$ and $f \in X^*$ we have

$$F\{\langle f, Tx_n\rangle\} = F\{\langle T^*f, x_n\rangle\} = \langle T^*f, x\rangle = \langle f, Tx\rangle.$$

Therefore $L\text{-}\lim Tx_n = Tx$. This proves (g).

Using Corollary 2.3 we now prove the following theorem.

THEOREM 3.2. *Let* $L \in B(\ell^\infty)$ *with* $L1 = 1$ *and* $\|L\| = 1$. *Let* $\{x_n\}$ *be a bounded sequence in a normed linear space* X. *Then*

(a) $L\text{-}\lim x_n = x$ *if and only if* $\{\langle f, x_n - x\rangle\} \in \overline{R(L-I)}$ *for all* $f \in X^*$.
(b) *If* $L\text{-}\lim x_n = x$, *then* $x \in \overline{\mathrm{co}}\{x_k : k \geq 0\}$.
(c) *If, in addition,* L *satisfies the condition* $(*)$, *i.e.* $L \in \mathbf{K}$, *then* $L\text{-}\lim x_n = x$ *implies* $x \in \overline{\mathrm{co}}\{x_k : k \geq n\}$ *for all* $n \geq 0$.
(d) *If* $L = \widehat{\psi}$ *for some mapping* $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ *then* $\psi\text{-}\lim x_n = x$ *if and only if*

$$w\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} x_{\psi^k(n)} = x \quad \textit{uniformly in } n.$$

In particular, if $L = \widehat{\sigma}$, *where* $\sigma(n) = n+1$ *for* $n = 0,1,2,\ldots$, *then* $\sigma\text{-}\lim x_n = x$ *if and only if*

$$w\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} x_{k+n} = x \quad \textit{uniformly in } n.$$

Proof. We show (a). Since $L^k(1) = 1$ and $\|L^k\| = 1$, L^k can be represented as an infinite matrix $(b_{ij}^{(k)})$ such that $\sum_{j=1}^\infty b_{ij}^{(k)} = 1$ for all i. Let $\{y_{kn}\} := (b_{ij}^{(k)})\{x_n\}$ for $k \in \mathbb{N}_0$. Then $y_{kn} = \sum_{j=1}^\infty b_{nj}^{(k)} x_j \in \overline{\mathrm{co}}\{x_k : k \geq 0\}$, $k \in \mathbb{N}_0$. We have

$$
\begin{aligned}
&L\text{-}\lim x_n = x\\
&\Leftrightarrow F\{\langle f, x_n - x\rangle\} = 0 \text{ for all } f \in X^* \text{ and } F \in \pi_L\\
&\Leftrightarrow \{\langle f, x_n - x\rangle\} \in \overline{R(L-I)} \text{ for all } f \in X^* \text{ (by Corollary 2.3)}\\
&\Leftrightarrow \lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} L^k\{\langle f, x_n - x\rangle\} = 0 \text{ in } \ell^\infty \text{ for all } f \in X^*\\
&\Leftrightarrow \lim_{m\to\infty} \Big\{\Big\langle f, \frac{1}{m+1}\sum_{k=0}^{m} y_{kn}\Big\rangle\Big\} = \langle f, x\rangle \text{ in } \ell^\infty \text{ for all } f \in X^*.
\end{aligned}
$$

This means $x = w\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} y_{kn}$ uniformly in n and hence $x \in \overline{\mathrm{co}}\{x_k : k \ge 0\}$. When $L = \widehat{\psi}$ for some mapping $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ we have $y_{mn} = x_{\psi^m(n)}$. This proves (a), (b), and (d).

We show (c). Let m be any positive integer. We define

$$y_n = \begin{cases} x_m - x_n & \text{if } n \le m,\\ 0 & \text{if } n \ge m+1.\end{cases}$$

Since $\{y_n\}$ is a null sequence, we must have $L\text{-}\lim y_n = 0$. It follows from Proposition 3.1(e) that $L\text{-}\lim(x_n + y_n) = x$. It follows from part (b) that

$$x \in \overline{\mathrm{co}}\{x_n + y_n : n \ge 0\} = \overline{\mathrm{co}}\{x_n : n \ge m\}.$$

This proves (c) and completes the proof.

COROLLARY 3.3. *Let $\{x_n\}$ be a bounded sequence in a normed linear space X. If $y = \sigma\text{-}\lim x_n$, then $y = \sigma\text{-}\lim \frac{1}{n+1}\sum_{k=0}^{n} x_k$.*

Proof. Let $y_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$. It follows from Theorem 3.2(d) that $y_n \to y$ weakly as $n \to \infty$. Hence $\sigma\text{-}\lim y_n = y$ by Proposition 3.1(a).

An infinite matrix $R = (a_{ij})$ is said to be an *R-matrix* (see [5, p. 222]) if it satisfies the following conditions:

(i) $\sum_{j=0}^{\infty} a_{ij}$ converges for all $i = 0, 1, \ldots$ and $\sum_{j=0}^{\infty} a_{ij} \not\to 0$ as $i \to \infty$;
(ii) $a_{ij} \to 0$ as $i \to \infty$ for $j = 0, 1, \ldots$

THEOREM 3.4. *Let X be a Banach space. Then the following statements are equivalent:*

(a) *X is reflexive.*
(b) *For every bounded sequence $\{x_n\}$ in X there is an $L \in \mathbf{K}$ such that $L\text{-}\lim x_n$ exists.*
(c) *For every bounded sequence $\{x_n\}$ in X there is an R-matrix $R = (r_{ij})$ such that $\{\sum_{j=0}^{\infty} r_{ij} x_j\}$ converges weakly as $i \to \infty$.*

Proof. The equivalence of (a) and (c) is known (see [4], [5, Proposition 19.6]).

(a)⇒(b). Let $\{x_n\}$ be a bounded sequence in X. Since X is reflexive, any closed ball of X is weakly compact. Hence it follows from the Eberlein–Shmul'yan theorem that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$, say $x = w\text{-}\lim_{k\to\infty} x_{n_k}$. We may assume that $n_k > k$ for all $k = 0, 1, \ldots$ Let $\psi(k) = n_k$ for $k \in \mathbb{N}_0$. Then $\widehat{\psi}1 = 1$, $\|\widehat{\psi}\| = 1$, and $\widehat{\psi}$ satisfies the condition $(*)$. Since $\widehat{\psi} = (a_{k,j})$ with $a_{k,j} = \delta_{\psi(k),j}$, we have for $f \in X^*$,

$$\widehat{\psi}\{\langle f, x_n\rangle\} = \Big\{\Big\langle f, \sum_{j=0}^{\infty} \delta_{\psi(n),j} x_j\Big\rangle\Big\} = \{\langle f, x_{\psi(n)}\rangle\}.$$

Since $w\text{-}\lim_{k\to\infty} x_{\psi(k)} = w\text{-}\lim_{k\to\infty} x_{n_k} = x$, by Proposition 3.1(a) we have for $f \in X^*$ and $F \in \pi_{\widehat{\psi}}$,

$$F\{\langle f, x_n\rangle\} = F\{\widehat{\psi}\{\langle f, x_n\rangle\}\} = F\{\langle f, x_{\psi(n)}\rangle\} = \langle f, x\rangle.$$

Therefore $L\text{-}\lim x_n = x$.

(b)⇒(c). Let $L \in \mathbf{K}$. Write $L^k = (b_{ij}^{(k)})$ for $k = 0, 1, \ldots$ Then $L^k 1 = 1$ and $\|L^k\| = 1$ for all $k \ge 0$. From this fact, it is clear that $b_{ij}^{(k)} \ge 0$ for all $i, j, k = 0, 1, \ldots$, and $\sum_{j=0}^{\infty} b_{ij}^{(k)} = 1$ for all $i, k = 0, 1, \ldots$ Since L satisfies the condition $(*)$, we have for every fixed $j = 0, 1, \ldots$,

$$\frac{1}{m+1}\sum_{k=0}^{m} b_{nj}^{(k)} = \frac{1}{m+1}\sum_{k=0}^{m} L^k e_j \to 0 \quad \text{in } \ell^\infty \text{ as } m \to \infty. \tag{1.4}$$

Let $r_{mj} = \frac{1}{m+1}\sum_{k=0}^{m} b_{1j}^{(k)}$ for $m, j = 0, 1, \ldots$, and let $R = (r_{mj})$. Then R is an R-matrix. Indeed, since $\sum_{j=0}^{\infty} b_{1j}^{(k)} = 1$, we have for $m = 0, 1, \ldots$,

$$
\begin{aligned}
\sum_{j=0}^{\infty} r_{mj} &= \sum_{j=0}^{\infty} \frac{1}{m+1}\sum_{k=0}^{m} b_{1j}^{(k)} = \sum_{k=0}^{m} \frac{1}{m+1}\sum_{j=0}^{\infty} b_{1j}^{(k)}\\
&= \frac{1}{m+1}\sum_{k=0}^{m} 1 = 1,
\end{aligned}
$$

and, by (1.4),

$$0 \le r_{mj} \le \Big\|\frac{1}{m+1}\sum_{k=0}^{m} L^k e_j\Big\| \to 0 \quad \text{as } m \to \infty \text{ for all } j.$$

Now suppose $\{x_n\}$ is a bounded sequence in X such that $x = L\text{-}\lim x_n$ exists. We show the weak convergence of the sequence $\{\sum_{j=0}^{\infty} r_{nj} x_j\}$. It follows from Theorem 3.2(a) that $\{\langle f, x_n - x\rangle\} \in \overline{R(L-I)}$ for all $f \in X^*$, which is equivalent to

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} L^k\{\langle f, x_n - x\rangle\} = 0 \quad \text{in } \ell^\infty \text{ for all } f \in X^*.$$

Since

$$\frac{1}{m+1}\sum_{k=0}^{m} L^k\{\langle f, x_n - x\rangle\} = \left\{\left\langle f, \sum_{j=0}^{\infty}\frac{1}{m+1}\sum_{k=0}^{m} b_{nj}^{(k)} x_j\right\rangle - \langle f, x\rangle\right\},$$

we see that

$$w\text{-}\lim_{m\to\infty}\sum_{j=0}^{\infty}\frac{1}{m+1}\sum_{k=0}^{m} b_{nj}^{(k)} x_j = x \quad \text{uniformly in } n.$$

In particular, for $n = 1$ we have

$$x = w\text{-}\lim_{m\to\infty}\sum_{j=0}^{\infty}\frac{1}{m+1}\sum_{k=0}^{m} b_{1j}^{(k)} x_j = w\text{-}\lim_{m\to\infty}\sum_{j=0}^{\infty} r_{mj} x_j.$$

This proves (c) and completes the proof.

Remark. It follows from Theorems 3.2 and 3.4 that if $\{x_n\}$ is a bounded sequence in a reflexive Banach space, then

$$\{\langle f, x_n\rangle\} \in \bigcup_{L\in\mathbf{K}} (\overline{R(L-I)} \oplus \mathbb{C}\cdot 1)$$

for all $f \in X^*$. In particular, by setting $X = \mathbb{C}$ we obtain

$$\ell^\infty = \bigcup_{L\in\mathbf{K}} (\overline{R(L-I)} \oplus \mathbb{C}\cdot 1).$$

PROPOSITION 3.5. *Let $L \in \mathbf{K}$ and let X and Y be two normed linear spaces.*

(i) *A linear operator $A : X \supset D(A) \to Y$ is closed if and only if the graph $G(A)$ of A is closed with respect to L-limit, i.e. whenever $\{x_n\} \subset D(A)$ and $L\text{-}\lim(x_n, Ax_n) = (x, y)$ for some $x \in X$, $y \in Y$, one has $x \in D(A)$ and $y = Ax$.*

(ii) *For a linear operator $A : X \to Y$, the following statements are equivalent*:

(a) *A is bounded.*

(b) *$L\text{-}\lim Ax_n = 0$ whenever $\{x_n\}$ is a sequence in X such that $L\text{-}\lim x_n = 0$.*

Moreover, when X is complete, we also have the next equivalent condition:

(c) *$L\text{-}\lim Ax_n = 0$ whenever $\{x_n\}$ is a sequence in X such that $L\text{-}\lim x_n = 0$ and such that $\{Ax_n\}$ is bounded.*

Proof. (i) To show the sufficiency, let $\{(x_n, Ax_n)\}$ be a sequence in $G(A)$ such that

$$s\text{-}\lim_{n\to\infty}(x_n, Ax_n) = (x, y) \qquad \text{for some } x, y \in X.$$

Then, by Proposition 3.1(a), we have $L\text{-}\lim(x_n, Ax_n) = (x, y)$, which implies that $(x, y) \in G(A)$ and $Ax = y$. Therefore $G(A)$ is closed.

For the converse, let $\{(x_n, Ax_n)\}$ be a sequence in $G(A)$ such that

$$L\text{-}\lim_{n\to\infty}(x_n, Ax_n) = (x, y) \text{ for some } x, y \in X.$$

It follows from Theorem 3.2(b) and the closedness of A that

$$(x, y) \in \overline{\mathrm{co}}\{(x_n, Ax_n) : n \ge 0\} \subset \overline{G(A)} = G(A).$$

Therefore we have $x \in D(A)$ and $Ax = y$.

(ii) (a)⇒(b) follows from Proposition 3.1(g). To show (b)⇒(a), we suppose that A is unbounded. Then there is a sequence $\{u_n\}$ in X such that $\|u_n\| = 1$ for all $n = 0, 1, \ldots$ and $\lim_{n\to\infty}\|Au_n\| = \infty$. We may assume $Au_n \ne 0$ for all n. Let $x_n = \|Au_n\|^{-1/2}u_n$ for $n = 0, 1, \ldots$. Then $\|x_n\| \to 0$ and $\|Ax_n\| \to \infty$. But, by Proposition 3.1(a), we have $L\text{-}\lim y_n = 0$, and so it follows from the assumption that $\{Ax_n\}$ is bounded. This is a contradiction.

(b)⇒(c) is obvious. To show (c)⇒(a) for the case where X is complete, it suffices to show that A is closed. For this we use (i). Let $\{x_n\}$ be a sequence in X such that $L\text{-}\lim_{n\to\infty}(x_n, Ax_n) = (x, y)$ for some $x \in X$ and $y \in Y$. Then $L\text{-}\lim(x_n - x, A(x_n - x)) = (0, y - Ax)$. By (c), we have $y - Ax = 0$. Hence A is closed.

4. An abstract ergodic theorem for L-limit. Let T be a power bounded linear operator on a Banach space and let $T_n := \frac{1}{n+1}\sum_{k=0}^{n} T^k$. The well-known Cesàro ergodic theorem (see e.g. [6], [12], [14]) states that $s\text{-}\lim T_n x = y$ if and only if y is the weak cluster point of $\{T_n x\}$, and that the map $P : x \to \lim T_n x$ defines a linear projection with range $R(P) = N(T-I)$ and null space $N(P) = \overline{R(T-I)}$. This is a specialization of the abstract mean ergodic theorem established in [12, Theorem 1.1]. In this section we prove the following version of that theorem for the L-limit.

THEOREM 4.1. *Let $\{A_n\}$ be a sequnce of bounded operators on a normed linear space X and let A be a closed operator on X. Assume that $L \in B(\ell^\infty)$ satisfies $L1 = 1$, $\|L\| = 1$, and the following conditions*:

(L1) *$\|A_n\| \le M$ for all $n \ge 0$ and some constant $M > 0$;*

(L2) *If $x \in N(A)$ then $L\text{-}\lim A_n x = x$;*

(L3) *$R(A_n - I) \subset \overline{R(A)}$ for all n;*

(L4) *$R(A_n) \subset D(A)$, $L\text{-}\lim AA_n x = 0$ for all $x \in X$, and $L\text{-}\lim A_n Ax = 0$ for all $x \in D(A)$.*

Let Q be the operator defined by $Qx := L\text{-}\lim A_n x$ for those x for which the limit exists. Then Q is a bounded, closed linear projection with $\|Q\| \le M$, $R(Q) = N(A)$, $N(Q) = \overline{R(A)}$, and $D(Q) = N(A) \oplus \overline{R(A)}$.

Proof. $\|Q\| \le M$ follows from (L1) and Proposition 3.1(d). Next, we show that Q is closed. Let $\{x_m\}$ be a sequence in $D(Q)$ such that

$$s\text{-}\lim_{m\to\infty}(x_m, Qx_m) = (x,y) \quad \text{for some } x, y \in X.$$

Then for $F \in \pi_L$ and $f \in X^*$ we have

$$\begin{aligned}|F\{\langle f, A_n x\rangle\} - \langle f, y\rangle| &= |F\{\langle f, A_n(x-x_m)\rangle\} + F\{\langle f, A_n x_m\rangle\} - \langle f, y\rangle| \\ &\le \|F\|\cdot\|f\|\cdot\sup_n\|A_n\|\cdot\|x-x_m\| + |\langle f, Qx_m - y\rangle| \\ &\le M\|f\|\cdot\|x-x_m\| + \|f\|\cdot\|Qx_m - y\| \\ &\to 0 \quad \text{as } m\to\infty.\end{aligned}$$

Therefore we have $x \in D(Q)$ and $y = Qx$.

That $N(A) \subset R(Q)$ and $Q|_{N(A)} = I|_{N(A)}$ follows from (L2). $R(Q) \subset N(A)$ follows from the first part of (L4), Proposition 3.5(i), and the assumption that A is closed. Hence $Q^2 = Q$ and $R(Q) = N(A)$. The second part of (L4) and the closedness of Q imply that $\overline{R(A)} \subset N(Q)$. Let $x \in N(Q)$. It follows from (L3) that $L\text{-}\lim(A_n - I)x = -x \in \overline{R(A)}$. This proves $N(Q) = \overline{R(A)}$. The proof is complete.

Proposition 4.2. *Let $\{A_n\}$ be a sequence of operators on a normed linear space X and let A be a closed operator on X. Suppose $\{A_n\}$ and A satisfy the conditions* (L1)–(L4) *for $L = \widehat{\sigma}$. Let $T_n = \frac{1}{n+1}\sum_{k=0}^n A_k$, $n = 0, 1, \ldots$ Then for given $x, y \in X$ the following statements are equivalent*:

(a) $\sigma\text{-}\lim A_n x = y$.

(b) $\sigma\text{-}\lim T_n x = y$.

(c) *There is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that $w\text{-}\lim T_{n_k} x = y$.*

(c′) *There is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that $\sigma\text{-}\lim T_{n_k} x = y$.*

(d) *There is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that*

$$w\text{-}\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^m T_{n_j} x = y.$$

(d′) *There is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that*

$$\sigma\text{-}\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^m T_{n_j} x = y.$$

Proof. (a)⇒(b) follows from Corollary 3.3, (a)⇒(c) and (b)⇒(d) follow from Theorem 3.2(c). (b)⇒(c′), (c)⇒(c′), and (d)⇒(d′) are obvious. So, it remains to show (c′)⇒(a) and (d′)⇒(a).

(c′)⇒(a). Suppose $\{T_{n_k}\}$ is a subsequence of $\{T_n\}$ such that $\sigma\text{-}\lim T_{n_k} x = y$. Since

$$(T_{n_k} - I)x = \frac{1}{n_k+1}\sum_{j=0}^{n_k}(A_j - I)x \in \overline{R(A)},$$

it follows from Theorem 3.2(a) that $y - x \in \overline{R(A)}$. Since $\sigma\text{-}\lim AA_n x = 0$, it follows from Theorem 3.2(c) that $w\text{-}\lim AT_n x = 0$. Hence we must have $w\text{-}\lim AT_{n_k} x = 0$ and hence $\sigma\text{-}\lim AT_{n_k} x = 0$ by Proposition 3.1(a). It follows from the closedness of A that $y \in N(A)$. Therefore we have $x = (x-y) + y \in \overline{R(A)} \oplus N(A) \equiv D(Q)$ and

$$\sigma\text{-}\lim A_n x = Qx = Qy = y.$$

Here Q is the operator as defined in Theorem 4.1. This proves (a).

Finally, we show that (d′)⇒(a). Since $\sigma\text{-}\lim AA_n x = 0$, we have $w\text{-}\lim AT_n x = 0$. Hence $w\text{-}\lim_{k\to\infty} AT_{n_k} x = 0$ and

$$w\text{-}\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^m AT_{n_k} x = 0.$$

So $\sigma\text{-}\lim \frac{1}{m+1}\sum_{k=0}^m AT_{n_k} x = 0$ by Proposition 3.1(a). It follows from the closedness of A that $y \in N(A)$. Since $\frac{1}{m+1}\sum_{j=0}^m T_{n_j} x - x \in \overline{R(A)}$, we must have $y - x \in \overline{R(A)}$ (by Theorem 3.2(a)). Therefore we have $x = (x-y) + y \in \overline{R(A)} \oplus N(A) \equiv D(Q)$ and

$$\sigma\text{-}\lim A_n x = Qx = Qy = y.$$

This proves (a) and completes the proof.

If T is a power bounded operator, it is clear that (L1)–(L3) hold for $A = T - I$ and $A_n = T^n$. If $F \in \pi_{\widehat{\sigma}}$, then F is a Banach limit on ℓ^∞ so that

$$\begin{aligned}F\{\langle f, AA_n x\rangle\} &= F\{\langle f, (T^{n+1} - T^n)x\rangle\} \\ &= F\{\langle f, T^{n+1}x\rangle\} - F\{\langle f, T^n x\rangle\} = 0\end{aligned}$$

for $x \in X$ and $f \in X^*$. Therefore (L4) also holds. Hence from Theorem 4.1 and Proposition 4.2 we can deduce the following mean ergodic theorem for σ-limit.

Corollary 4.3. *Let T be a power bounded operator on a normed linear space X and let Q be the linear operator defined by $Qx = \sigma\text{-}\lim T^n x$ for those x for which the σ-limit exists. Then Q is a bounded, closed linear projection with $\|Q\| \le \limsup\|T^n\|$, $R(Q) = N(T-I)$, $N(Q) = \overline{R(T-I)}$, and $D(Q) = N(T-I) \oplus \overline{R(T-I)}$. Moreover, the following statements are equivalent*:

(a) $\sigma\text{-}\lim T^n x = y$.

(b) $\sigma\text{-}\lim \frac{1}{n+1}\sum_{j=0}^{n} T^j x = y$.

(c) *There is a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$w\text{-}\lim_{k\to\infty} \frac{1}{n_k+1}\sum_{j=0}^{n_k} T^j x = y.$$

(c′) *There is a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$\sigma\text{-}\lim_{k\to\infty} \frac{1}{n_k+1}\sum_{j=0}^{n_k} T^j x = y.$$

(d) *There is a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$w\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} \frac{1}{n_k+1}\sum_{j=0}^{n_k} T^j x = y.$$

(d′) *There is a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$\sigma\text{-}\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} \frac{1}{n_k+1}\sum_{j=0}^{n_k} T^j x = y.$$

Remark. It follows from Corollary 4.3 and the classical Cesàro ergodic theorem that the two projections P and Q coincide, and so we have the equivalence relations:

$$\sigma\text{-}\lim T^n x = y$$

$$\Leftrightarrow y \text{ is a weak cluster point of } \left\{\frac{1}{n+1}\sum_{j=0}^{n} T^j x : n = 0, 1, \ldots\right\}$$

$$\Leftrightarrow s\text{-}\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} T^j x = y$$

$$\Leftrightarrow \sigma\text{-}\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} T^j x = y.$$

References

[1] Z. U. Ahmad and Mursaleen, *An application of Banach limits*, Proc. Amer. Math. Soc. (1) 103 (1988), 244–246.

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Ser. 2, Cambridge Univ. Press, 1971.

[3] —, —, *Numerical Ranges II*, London Math. Soc. Lecture Ser. 10, Cambridge Univ. Press, 1973.

[4] A. Brunel, H. Fong, and L. Sucheston, *An ergodic superproperty of Banach spaces defined by a class of matrices*, Proc. Amer. Math. Soc. 49 (1975), 373–378.

[5] D. van Dulst, *Reflexive and Superreflexive Banach Spaces*, MCT, 1982.

[6] U. Krengel, *Ergodic Theorems*, de Gruyter, 1985.

[7] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948), 167–190.

[8] Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math. Oxford (2) 34 (1983), 77–86.

[9] R. A. Raimi, *Invariant means and invariant matrix methods of summability*, Duke Math. J. 30 (1963), 81–94.

[10] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. 36 (1972), 104–110.

[11] —, *Mappings of positive integers and subspaces of m*, Portugal. Math. 38 (1979), 29–38.

[12] S.-Y. Shaw, *Mean ergodic theorems and linear functional equations*, J. Funct. Anal. 87 (1989), 428–441.

[13] A. M. Sinclair, *The states of a Banach algebra generate the dual*, Proc. Edinburgh Math. Soc. (2) 17 (1971), 193–200.

[14] K. Yosida and S. Kakutani, *Operator-theoretical treatment of Markoff's process and mean ergodic theorem*, Ann. of Math. 42 (1941), 188–228.

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