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Some classical function systems in separable Orlicz spaces

by

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Abstract. The boundedness of (sub)sequences of partial Fourier and Fourier–Walsh sums in subspaces of separable Orlicz spaces is studied. The boundedness of the shift operator and Paley function with respect to the Haar system is also investigated. These results are applied to get the analogues of the classical theorems on basicness of the trigonometric and Walsh systems in nonreflexive separable Orlicz spaces.

0. Introduction. A fundamental result in the study of orthonormal systems is: the trigonometric and Walsh systems are bases in L^p for $1 < p < \infty$ [14], [15]. Moreover, a necessary and sufficient condition for the trigonometric (and for the Walsh) system to be a basis in a separable Orlicz space is the reflexivity of the space [6], [16]. In this paper we are concerned with any separable Orlicz space. Let us denote by L_N such a space. Of course when L_N is nonreflexive neither system is a basis in the whole space L_N, but what is happening if we restrict ourselves to an Orlicz subspace L_Q of L_N? We prove (Theorem 1.1) that these systems are both simultaneously bases (or not bases) of L_Q (in the norm of L_N; see Definition 1.2). We also get a necessary and sufficient condition on the subspace L_Q for both systems to be bases of L_Q (in the norm of L_N) and we describe the "maximal" subspace with that property: it is the Orlicz space L_{R_N} (see Definition 1.1). To prove these results we study the boundedness of the sequences of partial Fourier and Fourier–Walsh sums. We also investigate subsequences of these sums to get more precise results.

The second part of this article is devoted to the shift operator T, the Paley function P with respect to the Haar system and the majorant S^* of Fourier–Haar partial sums. These operators are bounded in L^p for $1<p<\infty$. It is well known that the norms of P and S^* are equivalent [3], [4]. A necessary and sufficient condition for T to be bounded in an Orlicz space is

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the reflexivity of this space [10]. Then we can develop the same ideas as in the first part. We work in any separable Orlicz space L_N and we study the boundedness of these operators from an Orlicz subspace L_Q (of L_N) into L_N. They are simultaneously bounded (or not) and as before we find that L_{R_N} is the "maximal" subspace L_Q with the property that these operators are bounded from L_Q into L_N.

Let us note that when L_N is reflexive our results coincide with the well known results mentioned above.

1. Preliminaries. Let I be a bounded interval of $\mathbb{R}$. We denote by $L^0 = L^0(I)$ the Lebesgue space of functions that are measurable and finite almost everywhere on I; $m(A)$ is the Lebesgue measure of the set $A \subset I$ and $\mathbf{1}_A$ is the characteristic function of A. The constants appearing in the article will be denoted by C.

Let $L_M = L_M(I)$ be the Orlicz space (see [12]) generated by an $\mathcal{N}$-function M, i.e. M is a convex continuous even function such that $M(0) = 0$ and

$$(1) \qquad \lim_{u\to\infty} \frac{M(u)}{u} = \lim_{u\to 0} \frac{u}{M(u)} = \infty.$$

This space is endowed with the norm

$$(2) \qquad \|f\|_M = \inf\Big\{\kappa > 0 : \int_I M(f(x)/\kappa)\,dx \le 1\Big\}.$$

Let M^* be the Young function complementary to M.

It is well known [12] that

$$(3) \qquad [L_{M_1} \subset L_{M_2}] \Leftrightarrow [\exists C > 0\ \exists u_0 \ge 0\ \forall u \ge u_0\ M_2(u) \le M_1(Cu)] \Rightarrow [\exists C > 0\ \|\cdot\|_{M_2} \le C\|\cdot\|_{M_1}].$$

In what follows N will be an $\mathcal{N}$-function satisfying the Δ_2 condition, that is,

$$(4) \qquad \exists C > 0\ \exists u_0 \ge 0\ \forall u \ge u_0 \quad N(2u) \le CN(u).$$

DEFINITION 1.1. R_N is an $\mathcal{N}$-function generating an Orlicz space L_{R_N} such that

$$(5) \qquad R_N(u) = u\int_1^u \xi^{-2}N(\xi)\,d\xi, \quad u \ge 2.$$

In what follows the definition of R_N for $0 \le u < 2$ does not play any role.

The following properties are equivalent:

(6) N satisfies the Δ_2 condition,

(7) L_N is separable,

$$(8) \qquad \|f\|_N < \infty \Leftrightarrow \int_I N(f) < \infty,$$

$$(9) \qquad \exists r \ge 1\ \exists u_0 \ge 1\ \forall u \ge u_0 \quad u^r \int_u^\infty t^{-r-1}N(t)\,dt \le CN(u).$$

(For (6)⇔(7)⇔(8) see [12], for (6)⇔(9) see [18].)

We note that the Δ_2 condition implies the following properties:

(10) R_N satisfies the Δ_2 condition,

$$(11) \qquad \exists C \ge 1\ \exists u_0 \ge 0\ \forall u \ge u_0 \quad N(u) \le CR_N(u).$$

We remark that (11) is equivalent to (see (3), (5))

$$(12) \qquad L_{R_N} \subset L_N.$$

We use the following Marcinkiewicz's interpolation type theorem (see (5)–(9)).

THEOREM A ([20], Vol. II, p. 118, Th. 4.34; [18]). *Let A be a quasilinear operator which has simultaneously strong type (p,p) for all $1 < p < \infty$ and weak type* (1,1). *Let L_N be a separable Orlicz space. Then A is defined on L_{R_N} and*

(a) $\exists C > 0\ \forall f \in L_{R_N} \quad \int_I N(Af) \le C\Big(1 + \int_I R_N(f)\Big),$

(b) $\|A\|_{L_{R_N}\to L_N} < \infty.$

We remark that (b) follows from (a) immediately (see [20], Vol. I, p. 174, Th. 10.14).

LEMMA A. *Let N satisfy the Δ_2 condition. Then $L_N = L_{R_N}$ if and only if L_N is reflexive.*

Proof. This follows from the chain of equivalences:

$$\begin{aligned}[L_N = L_{R_N}] &\Leftrightarrow [L_N \subset L_{R_N}] \quad \text{(by (12))}\\ &\Leftrightarrow [\exists C > 0\ \exists u_0 \ge 0\ \forall u \ge u_0\ R_N(u) \le N(Cu)] \quad \text{(by (11))}\\ &\Leftrightarrow \Big[\exists l > 1\ \exists u_0 \ge 0\ \forall u \ge u_0\ N(u) \le \frac{1}{2l}N(lu)\Big] \quad \text{(see [1], [2], [7])}\\ &\Leftrightarrow [N^* \text{ satisfies the } \Delta_2 \text{ condition}] \quad \text{(see [12])}\\ &\Leftrightarrow [L_N \text{ is reflexive}] \quad \text{(by (7), (4); see also [12]).}\end{aligned}$$

Let us denote by $\mathcal{T}$, $\mathcal{H}=\{h_n\}_{n=1}^\infty$, $\mathcal{W}=\{w_n\}_{n=0}^\infty$ the trigonometric, Haar, Walsh (in the Paley enumeration) systems respectively, defined on I; here and in what follows we set $I=[0,2\pi]$ for the trigonometric case and $I=[0,1]$ for the other cases (for the definitions of $\mathcal{T}$, $\mathcal{H}$, $\mathcal{W}$ see for example [8]). By $S_n^{\mathcal{T}}$, $S_n^{\mathcal{H}}$, $S_n^{\mathcal{W}}$ we denote as usual the Fourier, Fourier–Haar, Fourier–Walsh partial sum operators defined on $L_1=L_1(I)$.

DEFINITION 1.2 [17]. We say that a sequence $\{x_n\}$ in a Banach space B is a *basis* of a subspace X *in the norm* of B if for every x in X there is a unique series $\sum_n a_n x_n$, $a_n=a_n(x)$, which converges to x in the norm of B.

We consider the following problem: when are trigonometric and Walsh systems both bases of the Orlicz subspace L_Q in the norm of the whole Orlicz space L_N?

The main result is:

THEOREM 1.1. *Let L_N be a separable Orlicz space and L_Q be an Orlicz subspace of L_N. The following assertions are equivalent*:

(a) *the trigonometric system is a basis of L_Q in the norm of L_N*,
(b) *the Walsh system is a basis of L_Q in the norm of L_N*,
(c) $L_Q\subset L_{R_N}$ (see also (5) for the definition of R_N).

Since both systems are dense in L_N (see also (12)) it is sufficient to prove (see for example [20], Vol. I, p. 266, Th. 6.4):

THEOREM 1.2. *Under the same assumptions as in Theorem* 1.1 *the following assertions are equivalent*:

(a) $\sup_n \|S_n^{\mathcal{T}}\|_{L_Q\to L_N}<\infty$,
(b) $\sup_n \|S_n^{\mathcal{W}}\|_{L_Q\to L_N}<\infty$,
(c) $L_Q\subset L_{R_N}$.

For that we need some lemmas.

LEMMA 1.1. *Let N and Q be $\mathcal{N}$-functions with N satisfying the Δ_2 condition. Then for every $n\ge n_0=\min\{n: Q(200n)>200n\}$ we have the inequality*

$$\int_0^{2\pi} N(Q^{-1}(200n)S_n^{\mathcal{T}}(\mathbf{1}_{[0,1/(200n)]}))\ge Cn^{-1}R_N(Q^{-1}(200n)).$$

Proof. Let

$$D_n(x)=\frac{\sin{(n+1/2)x}}{2\sin(x/2)}$$

be the Dirichlet kernel and $\xi_{k,n}=2k\pi/(2n+1)$, $k=1,\dots,2n+1$, be the zeros of D_n on $[0,2\pi]$. Let

$$t_{k,n}=\frac12[\xi_{k,n}+\xi_{k-1,n}]=\frac{2k-1}{2n+1}\pi,$$
$$I_{k,n}=\left[t_{k,n}-\frac{1}{200n},t_{k,n}+\frac{1}{200n}\right]=[a_{k,n},b_{k,n}],$$
$$J_{k,n}=\left[a_{k,n}-\frac{1}{200n},b_{k,n}+\frac{1}{200n}\right].$$

It is well known [13] that for $t\in J_{k,n}$, $k=1,\dots,2n+1$,

$$D_n(t)>\frac{n}{5k}.$$

Thus for $t\in J_{k,n}$, $k=1,\dots,2n+1$,

$$D_n(t)>\frac{1}{5t}.$$

If $x\in I_{k,n}$ and $z\in[0,1/(200n)]$ then $x-z\in J_{k,n}$ and it follows for $x\in I_{k,n}$ that

$$|S_n^{\mathcal{T}}(\mathbf{1}_{[0,1/(200n)]}(x))|=\Big|\int_0^{2\pi}\mathbf{1}_{[0,1/(200n)]}(z)D_n(x-z)\,dz\Big|$$
$$=\int_0^{1/(200n)}|D_n(x-z)|\,dz>\frac{1}{1000nx}.$$

As N is an increasing function satisfying the Δ_2 condition, we have

$$\int_0^{2\pi}N(Q^{-1}(200n)S_n^{\mathcal{T}}(\mathbf{1}_{[0,1/(200n)]}))$$
$$\ge C\sum_{k=1}^{2n+1}\int_{I_{k,n}}N\Big(\frac{Q^{-1}(200n)}{200nx}\Big)\,dx$$
$$\ge\frac{C}{100n}\,\frac{2n+1}{\pi}\sum_{k=1}^{n}\int_{a_{k,n}}^{a_{k+1,n}}N\Big(\frac{Q^{-1}(200n)}{200nx}\Big)\,dx$$
$$\ge C\int_{\pi/(2n+1)}^{1}N\Big(\frac{Q^{-1}(200n)}{200nx}\Big)\,dx$$
$$\ge\frac{CQ^{-1}(200n)}{200n}\int_{Q^{-1}(200n)/(200n)}^{(2n+1)Q^{-1}(200n)/(200n\pi)}\frac{N(u)}{u^2}\,du.$$

Since Q is an $\mathcal{N}$-function (see (1)), there exists $n_0=\min\{n:Q(200n)>200n\}$

and for $n \geq n_0$ we have

$$\int_0^{2\pi} N(Q^{-1}(200n)S_n^{\mathcal{T}}(\mathbf{1}_{[0,1/(200n)]})) \geq Cn^{-1}R_N(Q^{-1}(200n)).$$

Let us write n in base 2:

$$n = \sum_{i=0}^{\infty} \varepsilon_i 2^i, \quad \varepsilon_i \in \{0,1\}.$$

Put

$$s(n) = \sum_{i=0}^{\infty} |\varepsilon_i - \varepsilon_{i+1}|.$$

We need some facts from [11] collected in the next

LEMMA 1.2 [11]. *For every even integer n there exist subsets E_i, F_i $(i = 1, \ldots, s(n)/2)$ of $[0,1]$ such that*

$$[0,1] \supset F_1 \supset E_1 \supset F_2 \supset E_2 \supset \ldots \supset F_{s(n)/2} \supset E_{s(n)/2},$$
$$\frac{1}{2}m(F_i) = m(E_i) = 4^{-i},$$

and if

$$g_n(x) = 2^{s(n)}\mathbf{1}_{E_{s(n)/2}}(x), \quad x \in [0,1],$$

then

$$w_n(x)S_n^{\mathcal{W}}(w_n g_n, x) = \begin{cases} \frac{1}{4}4^i, & x \in E_i \setminus F_{i+1},\ i = 1, \ldots, s(n)/2 - 1, \\ -\frac{1}{6}4^i, & x \in F_i \setminus E_i,\ i = 1, \ldots, s(n)/2. \end{cases}$$

LEMMA 1.3. *Let N and Q be defined as in Lemma* 1.1. *Then there exists $C > 0$ such that for every even integer n satisfying the condition*

$$4Q^{-1}(2^{s(n)}) > 2^{s(n)} \tag{13}$$

the following inequality holds:

$$\int_0^1 N(Q^{-1}(2^{s(n)})S_n^{\mathcal{W}}(w_n(x)\mathbf{1}_{E_{s(n)/2}}(x)))\,dx \geq C2^{-s(n)}R_N(Q^{-1}(2^{s(n)})).$$

Proof. Let E_i, F_i be the sets defined as in Lemma 1.2. Then for all $i = 1, \ldots, s(n)/2$ and $x \in F_i \setminus F_{i+1}$,

$$|S_n^{\mathcal{W}}(w_n(x)\mathbf{1}_{E_{2^{s(n)/2}}}(x))| \geq \frac{4^i}{6 \cdot 2^{s(n)}}.$$

As N is an increasing function satisfying the Δ_2 condition (see (13)) we have

$$\int_0^1 N(Q^{-1}(2^{s(n)})S_n^{\mathcal{W}}(w_n(x)\mathbf{1}_{E_{2^{s(n)/2}}}(x)))\,dx$$
$$\geq \sum_{i=1}^{s(n)/2-1} \int_{F_i \setminus F_{i+1}} N\left(\frac{4^i Q^{-1}(2^{s(n)})}{6 \cdot 2^{s(n)}}\right)$$
$$\geq C \sum_{i=1}^{s(n)/2-1} N\left(\frac{4^i Q^{-1}(2^{s(n)})}{2^{s(n)}}\right)4^{-i}$$
$$\geq C \sum_{i=1}^{s(n)/2-1} \int_{4^i}^{4^{i+1}} 4^{-2i} N\left(\frac{4^i Q^{-1}(2^{s(n)})}{2^{s(n)}}\right)$$
$$\geq \frac{CQ^{-1}(2^{s(n)})}{2^{s(n)}} \int_{4Q^{-1}(2^{s(n)})/2^{s(n)}}^{Q^{-1}(2^{s(n)})} \frac{N(u)}{u^2}\,du$$
$$\geq \frac{C}{2^{s(n)}} R_N(Q^{-1}(2^{s(n)})).$$

Proof of Theorem 1.2. The implications (c)⇒(a) and (c)⇒(b) follow from Theorem A, since the operators $S_n^{\mathcal{T}}$ and $S_n^{\mathcal{W}}$ have strong type (p,p) for all $1 < p < \infty$ and weak type $(1,1)$ (see [20], Vol. I, [9], [14], [15], [19]). (We only remark that the corresponding estimates do not depend on n.)

(a)⇒(c). In particular, for

$$f_n(x) = 200n\mathbf{1}_{[0,1/(200n)]}(x), \quad x \in [0, 2\pi],\ n \geq 1,$$

we have

$$\exists C > 0\ \forall n \geq 1 \quad \|S_n^{\mathcal{T}} f_n\|_N \leq C\|f_n\|_Q.$$

Since

$$f_n/\|f_n\|_Q = Q^{-1}(200n)\mathbf{1}_{[0,1/(200n)]} \quad \text{(see (2))}$$

and

$$\left\|S_n^{\mathcal{T}}\left(\frac{f_n}{C\|f_n\|_Q}\right)\right\|_N \leq 1,$$

it follows that (see (2))

$$\int_0^{2\pi} N\left(S_n^{\mathcal{T}}\left(\frac{Q^{-1}(200n)}{C}\mathbf{1}_{[0,1/(200n)]}(x)\right)\right) dx \leq 1.$$

Thus using the Δ_2 condition for N (see Lemma 1.1), we have

$$\exists C \geq 0\ \forall n \geq n_0 \quad R_N(Q^{-1}(200n)) \leq Cn,$$

where n_0 is defined as in Lemma 1.1.

Since R_N and Q^{-1} are increasing and Q is convex, we obtain

$$\exists C>0\ \exists u_0\ge 0\ \forall u\ge u_0\qquad R_N(u)\le CQ(u)\le Q(Cu).$$

This implies (c) (see (3)).

(b)⇒(c). We proceed as in the proof of the previous implication using the function

$$f_{m_k}(x)=w_{m_k}(x)g_{m_k}(x),\qquad x\in I,\ k\ge 1,$$

where g_n is defined as in Lemma 1.2 and $m_k=\sum_{i=1}^{k-1}2^{2i-1}$. We note that

$$(14)\qquad s(m_k)=2k,\quad m_k<m_{k+1}<8m_k,\quad 2^{s(m_k)-1}<m_k<2^{s(m_k)+1}.$$

It follows from Lemma 1.3 that

$$\exists C>0\ \exists k_0\ge 1\ \forall k\ge k_0\qquad R_N(Q^{-1}(2^{s(m_k)}))\le C2^{s(m_k)}$$

and also (see (14))

$$\exists C>0\ \exists u_0\ge 0\ \forall u\ge u_0\qquad R_N(Q^{-1}(u))\le Cu.$$

The result follows as in the first part.

In fact, the method applied in the proof of Theorem 1.2 gives more information about subsequences.

THEOREM 1.3. *Let L_N be a separable Orlicz space and Q be a Young function such that*

$$Q(u)=o(R_N(u))\qquad \text{as } u\to\infty.$$

Then for every subsequence $\{n_j\}$ such that

(a) $\lim_{j\to\infty}n_j=\infty$, *we have*

$$\sup_j\|S_{n_j}^{\mathcal{T}}\|_{L_Q\to L_N}=\infty;$$

(b) $\sup_j s(n_j)=\infty$, *we have*

$$\sup_j\|S_{n_j}^{\mathcal{W}}\|_{L_Q\to L_N}=\infty.$$

The proof is similar to the one of Theorem 1.2. We only remark that for (b) we consider the sequence $\{n_j'\}$ defined by

$$n_j'=\begin{cases} n_j & \text{if } n_j \text{ is even},\\ n_j-1 & \text{if } n_j \text{ is odd}.\end{cases}$$

Clearly, one has $\sup_j s(n_j')=\infty$ and since

$$\|S_{n_j'}^{\mathcal{W}}f\|_N\le\|S_{n_j}^{\mathcal{W}}f\|_N+\|f\|_{L^1}$$

it is sufficient to prove (b) for $\{n_j'\}$ (see part (b)⇒(c) in the proof of Theorem 1.2).

2. Shift operator, Paley function and majorant of Fourier partial sums with respect to the Haar system. Let us recall the definition of these operators (see [8], pp. 68, 76, [5]). For $f\in L^1$, the *shift operator* is

$$T(f,x)=\sum_{n=1}^{\infty}(f,h_n)h_{n+1}(x);$$

the *Paley function* is

$$P(f,x)=\Big[\sum_{n=1}^{\infty}(f,h_n)^2h_n^2(x)\Big]^{1/2};$$

the *majorant of Fourier partial sums* is

$$S^\star(f,x)=\sup_n|S_n^{\mathcal{H}}(f,x)|.$$

Our main result of this part is

THEOREM 2.1. *Let L_N be a separable Orlicz space. Then the following assertions are equivalent*:

(a) $L_Q\subset L_{R_N}$,
(b) $\|T\|_{L_Q\to L_N}<\infty$,
(c) $\|P\|_{L_Q\to L_N}<\infty$,
(d) $\|S^\star\|_{L_Q\to L_N}<\infty$.

To prove Theorem 2.1, we need the following

THEOREM 2.2. *The operator T maps L^1 into L^0 and has weak type* $(1,1)$, *more precisely*

$$\forall y>0\ \forall f\in L^1\qquad m\{x\in I:|T(f,x)|>y\}\le\frac{4}{y}\|f\|_{L^1}.$$

Proof. Let us recall that a dyadic interval Δ_j (on $[0,1]$), $j=2^l+i$, $l=0,1,\ldots$, $i=1,\ldots,2^l$, is defined by

$$\Delta_j=\Big(\frac{i-1}{2^l},\frac{i}{2^l}\Big),\qquad \bar{\Delta}_j=\Big[\frac{i-1}{2^l},\frac{i}{2^l}\Big],$$

and

$$\operatorname{supp}h_j\subset\bar{\Delta}_j\qquad\text{for } j\ge 2.$$

Let

$$g(x)=S_n^{\mathcal{H}}(g,x)$$

be a polynomial with respect to the Haar system and $y>\|g\|_{L^1}$. Then the following Calderón–Zygmund decomposition is well known ([8], p. 73):

There exist measurable everywhere finite functions f_i, $i=1,2$, and a set O_y such that $g=f_1+f_2$,

$$(15)\qquad |f_1|<2y\quad\text{a.e. on } I,$$

(16) $$\|f_1\|_{L^1} \le \|g\|_{L^1},$$

and

$$\operatorname{supp} f_2 \subset O_y.$$

If $O_y \ne \emptyset$, then

(17) $$m(O_y) \le \frac{1}{y}\|g\|_{L^1},$$

(18) $$O_y = \bigcup_k \Delta_{m_k},$$

where Δ_{m_k}, $k = 1, 2, \ldots$, are disjoint dyadic intervals. Moreover,

(19) $$\Delta_j \not\subset O_y \Rightarrow (f_2, h_j) = 0.$$

Using (15), (16), we have

(20) $$m\{x \in I : |T(S_n^{\mathcal H} f_1)| > y\} \le \frac{\|T(S_n^{\mathcal H} f_1)\|_{L^2}^2}{y^2} = \frac{\|S_n^{\mathcal H} f_1\|_{L^2}^2}{y^2} \le \frac{\|f_1\|_{L^2}^2}{y^2} \le \frac{2}{y}\|f_1\|_{L^1} \le \frac{2}{y}\|g\|_{L^1}.$$

Further we have (see (18), (19))

(21) $$\begin{aligned} m(\operatorname{supp} T(S_n^{\mathcal W} f_1)) &= m\Big\{x \in I : \sum_{j=1}^n h_{j+1}(x)(f_2, h_j) \ne 0\Big\} \\ &\le m\Big\{\bigcup_{j:\Delta_j \subset O_y} \bar\Delta_{j+1}\Big\} = m\Big\{\bigcup_k \bigcup_{j:\Delta_j \subset \Delta_{m_k}} \bar\Delta_{j+1}\Big\} \\ &\le m\Big\{\bigcup_k (\bar\Delta_{m_k} \cup \bar\Delta_{m_k+1})\Big\}. \end{aligned}$$

(For the last inequality we use the fact that $\Delta_j \subset \Delta_{m_k}$ implies $\Delta_{j+1} \subset \Delta_{m_k} \cup \Delta_{m_k+1}$.) Since $m(\Delta_{m_k+1}) \le m(\Delta_{m_k})$ (see definition of Δ_j) we have (see also (18))

(22) $$m(\operatorname{supp} T(S_n^{\mathcal H}(f_2))) \le 2m(O_y).$$

From (17), (20) and (22), we obtain

$$\begin{aligned} m\{x \in I &: |T(g,x)| > y\} \\ &= m\{x \in I : |T(S_n^{\mathcal H}(f_1,x)) + T(S_n^{\mathcal H}(f_2,x))| > y\} \\ &\le m\{\operatorname{supp} T(S_n^{\mathcal H} f_2)\} + m\{x \in I : |T(S_n^{\mathcal H}(f_1,x))| > y\} \\ &\le \frac{2}{y}\|g\|_{L^1} + \frac{2}{y}\|g\|_{L^1}. \end{aligned}$$

Thus for $y > 0$ for every polynomial g with respect to the Haar system we have

$$m\{x \in I : |T(g,x)| > y\} \le \frac{4}{y}\|g\|_{L^1}.$$

(We proved this inequality for $y > \|g\|_{L^1}$, but for $0 < y \le \|g\|_{L^1}$ it is evident.) Since the Haar system is a basis in L^1 it follows from the last estimate that for $f \in L^1$ the sequence $T(S_n^{\mathcal H} f)$ converges in measure to Tf, and,

$$\begin{aligned} m\{x \in I : |T(f,x)| > y\} &\le \varlimsup_{n\to\infty} m\{x \in I : |T(S_n^{\mathcal H} f, x)| > y\} \\ &\le \frac{4}{y} \varlimsup_{n\to\infty} \|S_n^{\mathcal H} f\|_{L^1} \le \frac{4}{y}\|f\|_{L^1}. \end{aligned}$$

LEMMA 2.1. *Let N and Q be defined as in Lemma* 1.1. *Then there exists $C > 0$ such that for every $n > n_0 = \min\{n : Q(2^{n+1}) > 2^n\}$ we have the inequality*

$$\int_0^1 N(Q^{-1}(2^n) T(\mathbf{1}_{[0,1/2^n]})) \ge C 2^{-n} R_N(Q^{-1}(2^n)).$$

The same inequality holds for P and S^*.

Proof. We first remark that the existence of n_0 follows from (1). We put

$$f_n(x) = 2^n \mathbf{1}_{[0,1/2^n]}(x), \quad x \in [0,1],\ n = 1, 2, \ldots$$

Then

$$f_n(x) = h_1(x) + \sum_{k=0}^{n-1} 2^{k/2} h_{2^k+1}(x).$$

And for $x \in (2^{-j}, 2^{-j+1})$, $2 \le j \le n-1$, we have

$$\begin{aligned} |Tf_n(x)| &= \Big|h_2(x) + h_3(x) + \sum_{k=1}^{n-1} h_{2^k+2}(x) 2^{k/2}\Big| \\ &\ge \Big|\sum_{k=1}^{n-1} 2^{k/2} h_{2^k+2}(x)\Big| - (1+\sqrt{2}) \ge \frac{2^j}{4}. \end{aligned}$$

Thus for $x \in (2^{-n+1}, 2^{-1})$ we obtain

(23) $$|T(f_n,x)| \ge \frac{1}{8x}.$$

And also

(24) $$|P(f_n,x)| \ge \frac{1}{x}, \quad |S^*(f_n,x)| \ge \frac{1}{x}.$$

Using (4), (10) and (23), we have, for $n \geq n_0$,

$$\int_0^1 N(Q^{-1}(2^n)T(\mathbf{1}_{[0,1/2^n]})) \geq \int_{2^{-n+1}}^{2^{-1}} N\left(\frac{Q^{-1}(2^n)}{2^n} \cdot \frac{1}{8x}\right) dx$$

$$\geq C \frac{Q^{-1}(2^n)}{2^n} \int_{2^{-n-1}Q^{-1}(2^n)}^{2^{-1}Q^{-1}(2^n)} \frac{N(u)}{u^2}\, du$$

$$\geq C2^{-n} R_N(Q^{-1}(2^n)).$$

And the same (see (24)) for Q and $S^{\star}$.

Proof of Theorem 2.1. (a)⇒(b), (c), (d). It is well known [14], [20], [5] that the operators T, Q and $S^{\star}$ have strong type (p,p) for all $1 < p < \infty$ and $P, S^{\star}$ have weak type $(1,1)$ [19], [20]. Using these facts and Theorem 2.2 we apply Marcinkiewcz's interpolation theorem (Theorem A).

(b)⇒(a), (c)⇒(a) and (d)⇒(a) can be proved using Lemma 2.1 and the corresponding part of the proof of Theorem 1.2.

Remarks. 3.1. By Lemma A, Theorems 1.2 and 2.1 give the well known results mentioned above for reflexive Orlicz spaces.

3.2. Particular cases of Theorem 1.2 for the trigonometric system were studied in [13].

3.3. If $\sup_j s(n_j) < \infty$ then Theorem 1.3 is false. This follows from the representation of the Dirichlet–Walsh kernel by the sum of a bounded number of Dirichlet–Walsh (Dirichlet–Haar) kernels $D_{2^n}^{\mathcal{W}} = D_{2^n}^{\mathcal{H}}$ and the fact that the Haar system is a basis in any separable Orlicz space.

3.4. One can prove by a similar method (see Theorem 2.2) that $T^{(k)}f = \sum_{j=1}^{\infty} h_{j+k}(x)(f, h_j)$ has weak type $(1,1)$.

3.5. One can get some interesting applications of the previous results in the multidimensional cases.

3.6. Theorems 1.2, 1.3, 2.1 may be expressed in terms of integral inequalities (see Theorem A) for an arbitrary (non-$\mathcal{N}$-) function N, satisfying (4).

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