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Sums of idempotents and a lemma of N. J. Kalton

by

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Abstract. A lemma of Gelfand–Hille type is proved. It is used to give an improved version of a result of Kalton on sums of idempotents.

1. Introduction. In an elegant short paper [9], Kalton gave the following condition for the sum of two idempotent elements of a Banach algebra to be idempotent.

THEOREM 1 (Kalton). *Let p, q be idempotents in a complex Banach algebra A. Then the sum $p+q$ is idempotent if (and only if) the sequence $(\|(p+q)^n\|)_{n\geq 1}$ is bounded.*

Proof. See [9]. (Of course, the "only if" is trivial.)

In this note we shall give a refinement of Kalton's method which will, in particular, show that the boundedness condition in Theorem 1 may be replaced by an $o(n)$-condition.

The first step is exactly as in [9]. Suppose that p, q are idempotents in a complex Banach algebra A, which we may suppose to be unital. Let B be the closed unital subalgebra of A generated by pq, so that B is a commutative Banach algebra. We then form a Banach algebra B_0, by adjoining to B an element ξ that satisfies $\xi^2 = pq$. (The algebra B_0 is normed by setting $\|b + c\xi\| = \|b\| + \theta\|c\|$, for all $b, c \in B$, where θ is a constant, $\theta > \sqrt{\|pq\|}$.) Then we have the following algebraic identity.

LEMMA 1 (Kalton). *With the above notation,*

$$p(p+q)^m q = \tfrac{1}{2}\xi((1+\xi)^{m+1} - (1-\xi)^{m+1}),$$

for all $m \geq 0$.

Proof. See [9], page 449.

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2. A lemma of Gelfand–Hille type. Throughout this section, A is a complex unital Banach algebra. We shall prove a result (Lemma 3) which is related to a classical result of Gelfand and Hille, and also extends part of Kalton's proof of Theorem 1. It may be useful to place the new result in context by briefly outlining the development of results of this kind.

In [7] (1941), Gelfand proved the following result:

THEOREM (Gelfand). *Let $x \in A$ with $\mathrm{Sp}\, x = \{0\}$ and suppose that*

$$\|(1+x)^n\| \le K \quad (n \in \mathbb{Z}),$$

for some constant K. Then $x = 0$.

Remark. The result of Gelfand is usually stated in the equivalent form that, if the element $u \in A$ has $\mathrm{Sp}\, u = \{1\}$ and if $\{\|u^n\| : n \in \mathbb{Z}\}$ is bounded, then $u = 1$. (Just take $x = u - 1$ and apply the theorem as stated.)

In [8] (1944), the result was improved by Hille:

THEOREM (Gelfand–Hille). *Let $x \in A$ with $\mathrm{Sp}\, x = \{0\}$ and suppose that*

$$\|(1+x)^n\| \le K(1+|n|)^r \quad (n \in \mathbb{Z}),$$

for some constant K and non-negative integer r. Then $x^{r+1} = 0$.

Moreover, if $r \ge 1$ and if $\|(1+x)^n\| = o(|n|^r)$ (as $n \to \pm\infty$), then $x^r = 0$.

In [12] (1950), Shilov showed that, in the Gelfand theorem, it is *not* sufficient to assume the boundedness condition merely for $n \in \mathbb{Z}^+$. (See also [1], §5, Example 1. Also, a very simple example is given in [13], page 370: $I + T = (I + V)^{-1}$, where V is the Volterra integration operator on $L^2[0,1]$; this has $\|(I+T)^n\| = 1$ $(n \in \mathbb{Z}^+)$.) Five years later, in [4] (1955), Bohnenblust and Karlin conjectured that, if $x \ne 0$ but $\mathrm{Sp}\, x = \{0\}$, then the ray $R_x \equiv \{1 + \alpha x : \alpha > 0\}$ can not be tangent, at 1, to the unit sphere S of A. Their definition of tangency need not concern us, since they showed ([4], page 219) that R_x is tangent to S at 1 if and only if x is *dissipative*, i.e. if and only if $\|e^{tx}\| \le 1$ for all $t \in \mathbb{R}^+$. The conjecture of Bohnenblust and Karlin was thus equivalent to the statement that every quasi-nilpotent dissipative element of a Banach algebra must be zero. In this form, a counter-example to the conjecture was provided (in 1961) by Lumer and Phillips [11], Theorem 2.2.

However, it should be noted that the earlier example of Shilov, mentioned above, was *already* a counter-example to the Bohnenblust–Karlin conjecture, since it is elementary (see e.g. Lemma 2 below) that if $\{\|(1+x)^n\| : n \in \mathbb{Z}^+\}$ is bounded, then x is dissipative-equivalent, i.e. $\|e^{tx}\| \le K$ $(t \ge 0)$, for some constant K (which is equivalent to the element x being dissipative for some equivalent norm on A). There is also a formulation in terms of numerical range (see [5], §3, Definition 5 and Theorem 6); but we shall not make explicit use of numerical-range ideas in this paper, preferring to make direct use of elementary complex analysis. It is then a simple remark that x being dissipative-equivalent is equivalent to e^x having bounded non-negative powers. Hence, if $y = e^x - 1$, then x is dissipative-equivalent if and only if the sequence $(\|(1+y)^n\|)_{n\ge1}$ is bounded. If $\mathrm{Sp}\, x = \{0\}$ then $\mathrm{Sp}\, y = \{0\}$ and, conversely, if $\mathrm{Sp}\, y = \{0\}$ then it is possible to *choose* $x \in A$ with $\mathrm{Sp}\, x = \{0\}$ such that $e^x = 1 + y$.

As just mentioned, there is also a slightly less obvious connection between the two viewpoints. The simple proof introduces a method that will appear in a somewhat more elaborate form in the proof of Lemma 3 below.

LEMMA 2. *If $\|(1+x)^n\| \le K$ for all $n \in \mathbb{Z}^+$, then $\|e^{tx}\| \le K$ for all $t \in \mathbb{R}^+$.*

Proof. Let $0 \le s \le 1$; then for every $n \ge 1$,

$$(1+sx)^n = [(1-s) + s(1+x)]^n = \sum_{k=0}^{n} \binom{n}{k} (1-s)^{n-k} s^k (1+x)^k.$$

So $\|(1+sx)^n\| \le K$ for all $n \ge 1$ and all $0 \le s \le 1$.

If now $t \in \mathbb{R}^+$, then for any integer $n \ge t$, we may take $s = t/n$, and deduce that $\|(1+tx/n)^n\| \le K$ for all sufficiently large n. Then, letting $n \to \infty$ (with t fixed, but arbitrary) we have $\|e^{tx}\| \le K$ for all $t \in \mathbb{R}^+$.

As a corollary, we give what is, in effect, a proof of the result of Bohnenblust and Karlin [4] that the identity element is a vertex of the closed unit ball.

COROLLARY 1. *Let $x \in A$ with both $\|(1+x)^n\| \le K$ and $\|(1-x)^n\| \le K$ for all $n \in \mathbb{Z}^+$ and some constant K. Then $x = 0$.*

Proof. First, $\mathrm{Sp}\, x = \{0\}$: for if $\lambda \in \mathrm{Sp}\, x$, then $(1+\lambda)^n \in \mathrm{Sp}(1+x)^n$, so that $|1+\lambda|^n \le K$ $(n \in \mathbb{Z}^+)$ and therefore $|1+\lambda| \le 1$. Similarly, $|1-\lambda| \le 1$, and so $\lambda = 0$. Thus $\mathrm{Sp}\, x = \{0\}$.

By Lemma 2, applied to each of x and $-x$, we deduce that $\|e^{tx}\| \le K$ for all $t \in \mathbb{R}$. We may now apply the above theorem of Gelfand to the element $u = e^x$, which satisfies $\mathrm{Sp}\, u = \{1\}$, to deduce that $e^x = 1$. Since $\mathrm{Sp}\, x = \{0\}$, it then follows that $x = 0$.

In view of its application later in this paper, it will be useful to give an alternative to the use of Gelfand's theorem in the last proof. (In effect, we shall just be giving one of the proofs of the Gelfand result.) We define $F(\lambda) = e^{\lambda x}$ for $\lambda \in \mathbb{C}$. Then, since $\mathrm{Sp}\, x = \{0\}$, it is elementary that F is an entire A-valued function of minimal exponential type. But F is bounded on $\mathbb{R}$ and so, by a well-known result (e.g. [3], (6.2.13)), it follows that F

is constant. In particular, $F(1) = F(0)$, i.e. $e^x = 1$ and, as before, the quasinilpotence of x then implies that $x = 0$.

In 1981, Esterle proved ([6], Theorem 9.1):

THEOREM (Esterle). *Let $x \in A$ with $\operatorname{Sp} x = \{0\}$ and suppose that*

$$\|(1+x)^n\| \le K \quad (n \in \mathbb{Z}^+),$$

for some constant K. Then $x(1+x)^n \to 0$ as $n \to \infty$.

In [10] (1986), Katznelson and Tzafriri strengthened this last result by weakening the condition on the spectrum to $\operatorname{Sp}(1+x) \cap \mathbb{T} \subseteq \{1\}$, where $\mathbb{T}$ is the unit circle. (Note that the condition that $\|(1+x)^n\| \le K$ for all $n \in \mathbb{Z}^+$ already implies that $\operatorname{Sp}(1+x)$ is a subset of the closed unit disc.)

In [2] (1989), it was shown that this result of Katznelson and Tzafriri could be deduced rather simply from the 1941 result of Gelfand above. It was thus natural, in view of Hille's improvement of the Gelfand result, to ask whether the result of Esterle, or even that of Katznelson and Tzafriri, might still hold assuming only that $\|(1+x)^n\| = o(n)$ as $n \to \infty$ (of course, in addition to the appropriate spectral condition). In fact it was shown in [2], Theorem 4.2, that at least the result of Katznelson and Tzafriri could not be generalized in this way. An example of Atzmon (unpublished) shows that not even the result of Esterle may be so extended.

We now give the main result of this note. It will be seen that it resembles the Gelfand–Hille result above; it also refines part of Kalton's proof in [9].

LEMMA 3. *Let $x \in A$ and suppose that, for integers $k \ge 0$, $r \ge 0$,*

$$\|x^k((1+x)^n - (1-x)^n)\| \le Cn^r \quad (n \in \mathbb{Z}^+),$$

for some constant C. Then $x^{k+r+2} = 0$ if r is odd, while $x^{k+r+1} = 0$ if r is even.

Moreover, if r is odd and if $\|x^k((1+x)^n - (1-x)^n)\| = o(n^r)$ as $n \to \infty$, then $x^{k+r} = 0$.

Remark. If r is even, then an $o(n^r)$-condition gives no more information than the $O(n^r)$-condition.

Proof of Lemma 3. We first show that $\operatorname{Sp} x = \{0\}$. Analogously to the proof of Corollary 1, if $\lambda \in \operatorname{Sp} x$ then $\lambda^k((1+\lambda)^n - (1-\lambda)^n) \in \operatorname{Sp}[x^k((1+x)^n - (1-x)^n)]$, so that

$$|\lambda^k((1+\lambda)^n - (1-\lambda)^n)| \le Cn^r \quad (n \ge 1).$$

But it is an elementary exercise to see that, if $\lambda \ne 0$, then $|\lambda^k((1+\lambda)^n - (1-\lambda)^n)|$ (or at least a subsequence of it) grows exponentially as a function of n, as $n \to \infty$. Hence $\lambda = 0$, i.e. $\operatorname{Sp} x = \{0\}$.

We now consider again the growth condition

$$\|x^k((1+x)^n - (1-x)^n)\| \le Cn^r \quad (n \in \mathbb{Z}^+).$$

Again as in the proof of Lemma 1, we have, for all $n \in \mathbb{Z}^+$ and all $0 \le s \le 1$,

$$\begin{aligned}\|x^k((1+sx)^n - (1-sx)^n)\| &= \Big\|x^k \sum_{p=0}^{n} \binom{n}{p} (1-s)^{n-p} s^p ((1+x)^p - (1-x)^p)\Big\| \\ &\le C \sum_{p=0}^{n} \binom{n}{p} (1-s)^{n-p} s^p p^r = CS(n,s),\end{aligned}$$

say, where C is independent of n and s. If $r = 0$, then of course $S(n,s) = 1$. Otherwise, to estimate $S(n,s)$, we split the sum into two parts. Let $n > 2r$; then, say

$$S_1(n,s) = \sum_{p=0}^{r-1} \binom{n}{p} (1-s)^{n-p} s^p p^r \le C',$$

where C' is independent of n and s, and

$$\begin{aligned}S_2(n,s) &= \sum_{p=r}^{n} \binom{n}{p} (1-s)^{n-p} s^p p^r \\ &= \sum_{p=r}^{n} \frac{n(n-1)\dots(n-r+1)}{p(p-1)\dots(p-r+1)} \binom{n-r}{p-r} (1-s)^{n-p} s^p p^r \\ &\le C''(ns)^r \sum_{p=r}^{n} \binom{n-r}{p-r} (1-s)^{n-p} s^{p-r} = C''(ns)^r.\end{aligned}$$

So, with some D independent of n and s, we have

$$\|x^k((1+sx)^n - (1-sx)^n)\| \le D(1+ns)^r,$$

for all $n > 2r$ and all $0 \le s \le 1$ (and this holds for every $r \ge 0$).

Then, for any given $t \in \mathbb{R}^+$, we take $s = t/n$ for $n > \max\{t, 2r\}$, and deduce that

$$\|x^k((1+tx/n)^n - (1-tx/n)^n)\| \le D(1+t)^r.$$

Then let $n \to \infty$ and we have $\|x^k(e^{tx} - e^{-tx})\| \le D(1+t)^r$ for all $t \ge 0$.

In a similar way to the alternative proof of Lemma 1 given above, set $F(\lambda) = x^k(e^{\lambda x} - e^{-\lambda x})$ for $\lambda \in \mathbb{C}$. As before, since $\operatorname{Sp} x = \{0\}$, F is an entire A-valued function of minimal exponential type. It has just been shown that, on the real axis, it satisfies $\|F(t)\| \le D(1+t)^r$ $(t \ge 0)$. But since F is an odd function, also $\|F(t)\| \le D(1+|t|)^r$ for all $t \in \mathbb{R}$. By a further part of the classical theorem on entire functions used before (e.g. [3], (6.2.13)), it follows that F is a polynomial in λ with degree $\deg F \le r$. But $F(\lambda) = 2\sum_{n \text{ odd}} x^{n+k}(\lambda^n/n!)$, so that $x^{n+k} = 0$ for all odd $n > r$. Hence, if r is odd, then $x^{k+r+2} = 0$, while, if r is even, then $x^{k+r+1} = 0$.

Finally, suppose that $\|x^k((1+x)^n - (1-x)^n)\| = o(n^r)$, where r is odd. Then certainly $x^{k+r+2} = 0$. But then, for $n > 2r$,

$$x^k((1+x)^n - (1-x)^n) = 2x^k \sum_{\substack{p \text{ odd} \\ 1 \le p \le r}} \binom{n}{p} x^p,$$

and the $o(n^r)$-condition implies that $x^{k+r} = 0$.

We may now deduce a strengthening of Kalton's result.

THEOREM 2. *Let p, q be idempotents in A and suppose that $\|(p+q)^n\| = o(n)$ as $n \to \infty$. Then $p+q$ is idempotent.*

Proof. By Lemma 1, in the Banach algebra B_0 we have

$$p(p+q)^n q = \tfrac{1}{2}\xi((1+\xi)^{n+1} - (1-\xi)^{n+1}),$$

where $\xi^2 = pq$.

Thus, $\|\xi((1+\xi)^{n+1} - (1-\xi)^{n+1})\| = o(n)$ as $n \to \infty$. By Lemma 3, with $k = 1$, $r = 1$, we have $\xi^2 = 0$, i.e. $pq = 0$. An exactly symmetrical argument gives $qp = 0$, and so $(p+q)^2 = p+q$. The proof is complete.

Remark. In this last result, the $o(n)$-condition may not be weakened to an $O(n)$-condition. This may be seen by considering the 2×2 matrices $p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then p, q are idempotent and $(p+q)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, so that $\|(p+q)^n\| = O(n)$, but $p+q$ is not idempotent.

However, following a suggestion of Zemánek, there is a different kind of improvement, in terms of arithmetic means.

For any element x of A, let

$$M_n(x) = \frac{1}{n+1}(1 + x + \ldots + x^n).$$

We remark that there is a discussion of the Gelfand–Hille theorems in connection with growth conditions on arithmetic means in [13]. It is clear that if, say, $\|x^n\| = O(n^r)$ then also $\|M_n(x)\| = O(n^r)$. In the reverse direction, noting that $x^n = (n+1)M_n - nM_{n-1}$ there is at least the simple remark that if $\|M_n(x)\| = O(n^r)$ (or $o(n^r)$), then $\|x^n\| = O(n^{r+1})$ (respectively $o(n^{r+1})$).

To see that, in general, we may have $\|M_n(x)\|$ growing more slowly than $\|x^n\|$, consider the example of the Banach algebra $A = C^1[0,1]$, with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ $(f \in A)$. Let $a(t) = -t$ $(0 \le t \le 1)$. Then $\|a^n\| = 1 + n$, but an elementary calculation shows that the sequence $\|M_n(a)\|$ is bounded.

However, for a sum of idempotents we have the following result.

THEOREM 3. *Let p, q be idempotents in A and let $\|M_n(p+q)\| = o(n)$ as $n \to \infty$. Then $p+q$ is idempotent.*

Proof. By the above comments, we have at least $\|(p+q)^n\| = o(n^2)$. Then by Lemma 3 (case $k = 1$, $r = 2$) and Lemma 1, it follows that $\xi^4 = 0$, i.e. $(pq)^2 = 0$, and, similarly, $(qp)^2 = 0$. Thus, for $n \ge 3$,

$$(p+q)^n = (p+q) + (n-1)(pq+qp) + \tfrac{1}{2}(n-1)(n-2)(pqp+qpq).$$

But $\|(p+q)^n\| = o(n^2)$, so that $pqp + qpq = 0$.

Then $(p+q)^n = (p+q) + (n-1)(pq+qp)$, which holds even for $n \ge 1$, so that

$$(n+1)M_n(p+q) = 1 + n(p+q) + \tfrac{1}{2}(n-1)n(pq+qp).$$

But it is assumed that $\|M_n(p+q)\| = o(n)$, so that $pq + qp = 0$ and $p+q$ is idempotent.

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Some classical function systems in separable Orlicz spaces

by

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Abstract. The boundedness of (sub)sequences of partial Fourier and Fourier–Walsh sums in subspaces of separable Orlicz spaces is studied. The boundedness of the shift operator and Paley function with respect to the Haar system is also investigated. These results are applied to get the analogues of the classical theorems on basicness of the trigonometric and Walsh systems in nonreflexive separable Orlicz spaces.

0. Introduction. A fundamental result in the study of orthonormal systems is: the trigonometric and Walsh systems are bases in L^p for $1 < p < \infty$ [14], [15]. Moreover, a necessary and sufficient condition for the trigonometric (and for the Walsh) system to be a basis in a separable Orlicz space is the reflexivity of the space [6], [16]. In this paper we are concerned with any separable Orlicz space. Let us denote by L_N such a space. Of course when L_N is nonreflexive neither system is a basis in the whole space L_N, but what is happening if we restrict ourselves to an Orlicz subspace L_Q of L_N? We prove (Theorem 1.1) that these systems are both simultaneously bases (or not bases) of L_Q (in the norm of L_N; see Definition 1.2). We also get a necessary and sufficient condition on the subspace L_Q for both systems to be bases of L_Q (in the norm of L_N) and we describe the "maximal" subspace with that property: it is the Orlicz space L_{R_N} (see Definition 1.1). To prove these results we study the boundedness of the sequences of partial Fourier and Fourier–Walsh sums. We also investigate subsequences of these sums to get more precise results.

The second part of this article is devoted to the shift operator T, the Paley function P with respect to the Haar system and the majorant S^* of Fourier–Haar partial sums. These operators are bounded in L^p for $1<p<\infty$. It is well known that the norms of P and S^* are equivalent [3], [4]. A necessary and sufficient condition for T to be bounded in an Orlicz space is

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