References

[1] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.
[2] —, *Fractal Geometry*, Wiley, 1990.
[3] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.
[4] —, —, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.
[5] A. B. Gulisashvili, *Traces of differentiable functions on subsets of the euclidean space*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149 (1986), 52–66 (in Russian).
[6] A. Jonsson, *Atomic decomposition of Besov spaces on closed sets*, in: Function Spaces, Differential Operators and Non-Linear Analysis, Teubner-Texte Math. 133, Teubner, Leipzig, 1993, 285–289.
[7] A. Jonsson, *Besov spaces on closed sets by means of atomic decompositions*, preprint, Umeå, 1993.
[8] A. Jonsson and H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Math. Reports 2 Part 1, Harwood Acad. Publ., London, 1984.
[9] —, —, *The dual of Besov spaces on fractals*, Studia Math. 112 (1995), 285–300.
[10] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[11] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.
[12] H. Triebel and H. Winkelvoß, *Intrinsic atomic characterizations of function spaces on domains*, Math. Z. 221 (1996), 647–673.
[13] —, —, *The dimension of a closed subset of* $\mathbb{R}^n$ *and related function spaces*, Acta Math. Hungar. 68 (1995), 117–133.
[14] H. Winkelvoß, *Function spaces related to fractals. Intrinsic atomic characterizations of function spaces on domains*, Thesis, Jena, 1995.

Mathematisches Institut
Friedrich-Schiller-Universität Jena
D-07740 Jena, Germany
E-mail: triebel@minet.uni-jena.de

Received September 26, 1995
Revised version June 10, 1996 (3530)

STUDIA MATHEMATICA 121 (2) (1996)

A quantitative asymptotic theorem for contraction semigroups with countable unitary spectrum

by

CHARLES J. K. BATTY (Oxford), ZDZISŁAW BRZEŹNIAK (Hull) and DAVID A. GREENFIELD (Benfleet)

Abstract. Let T be a semigroup of linear contractions on a Banach space X, and let $X_s(T) = \{x \in X : \lim_{s\to\infty} \|T(s)x\| = 0\}$. Then $X_s(T)$ is the annihilator of the bounded trajectories of T^*. If the unitary spectrum of T is countable, then $X_s(T)$ is the annihilator of the unitary eigenvectors of T^*, and $\lim_s \|T(s)x\| = \inf\{\|x-y\| : y \in X_s(T)\}$ for each x in X.

1. Introduction. Let T be a semigroup of linear contractions on a Banach space X, and suppose that the unitary spectrum of T is countable. Let

$$X_s(T) = \{x \in X : \lim_{s\to\infty} \|T(s)x\| = 0\}.$$

The ABLP Theorem [2], [19], [6], [25] shows that $X_s(T) = X$ if the adjoint semigroup T^* has no unitary eigenvalues.

A variant of the ABLP Theorem [20], [21], [6] shows that $X = X_s(T) \oplus X_b(T)$, where T acts as a group of isometries on $X_b(T)$, provided that T satisfies a suitable ergodic spectral condition (which is automatic if X is reflexive). It follows easily that, for any x in X,

$$(*)\qquad \lim_s \|T(s)x\| = \inf\{\|x-y\| : y \in X_s(T)\}.$$

There are several instances where T is a C_0-semigroup generated by a differential operator A, and results have been obtained which identify the space $X_s(T)$ or which evaluate $\lim_s \|T(s)x\|$, without the conditions above being satisfied [4], [7], [10], [32]. Typically, $X = L^1(\mathbb{R}^n)$ and A^* has finitely many independent unitary eigenvectors in $L^\infty(\mathbb{R}^n)$. These results can usually be obtained by means of some more or less explicit estimates

1991 *Mathematics Subject Classification*: 47D03, 35B40.
Key words and phrases: contraction semigroup, unitary spectrum, unitary eigenvector trajectory, asymptotic stability, trivially asymptotically stable, countable, spectral synthesis.

on kernel functions, or by using the fact that T is an analytic semigroup, so that $\|AT(t)\| \le c/t$ for some constant c. However, they can also be derived by applying the ABLP Theorem on a subspace Y of X (the pre-annihilator of all the unitary eigenvectors of A^*).

In this paper, we will show that (∗) holds for all contraction semigroups with countable unitary spectrum. The main part of the argument (Section 4) concerns the special case when $X_s(T) = \{0\}$, and then the conclusion is that T is both isometric and invertible (Theorem 4.8). The argument there uses some techniques from harmonic analysis which were previously exploited by Esterle, Strouse and Zouakia [12] to give a proof of the ABLP Theorem.

Another variant of the ABLP Theorem (Proposition 5.1) is that $X_s(T)$ is the annihilator of the unitary eigenvectors of T^*, if the unitary spectrum is countable. We shall show (Theorem 3.1) that, even if the unitary spectrum is uncountable, $X_s(T)$ is the annihilator of the bounded trajectories of T^*. A result of this type was established by Vũ Quôc Phóng [36], under some additional assumptions.

We are very grateful to Wolfgang Arendt, E. Brian Davies and Jan van Neerven for helpful discussions and for making preprints available.

2. Preliminaries. The results in this paper are applicable to a contraction T and its powers $\{T^n : n = 0, 1, 2, \ldots\}$ or to a C_0-semigroup of contractions $\{T(t) : t \ge 0\}$. We shall give a unified proof by working in the more general context of representations of abelian semigroups as in [6] and [5], and we shall adopt the terminology and conventions of those papers with only minor changes.

Throughout, G will be a locally compact abelian group with dual Γ, and S will be a measurable subsemigroup of G with non-empty interior S^0 in G, satisfying $S - S = G$. We shall regard S as being ordered by $\preceq$, where $s \preceq t$ if $t - s \in S$; all limits over s will be with respect to this ordering. We shall take G to be equipped with Haar measure, and S with the restriction of that measure, and we shall regard $L^1(S)$ as a subspace of $L^1(G)$. For f in $L^1(G)$ and t in G, we will let f_t be the translate of f: $f_t(\tau) = f(\tau - t)$. The Fourier transform on $L^1(G)$ and $L^1(S)$ is defined by

$$\widehat{f}(\chi) = \int_G f(t)\chi(t)\,dt \quad (\chi \in \Gamma).$$

The most important examples are:

(i) $G = \mathbb{Z}$, $\Gamma = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $S = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$,

$$\widehat{f}(\chi) = \sum_{n\in\mathbb{Z}} f(n)\chi^n \quad (f \in \ell^1(\mathbb{Z}),\ \chi \in \mathbb{T});$$

(ii) $G = \mathbb{R}$, $\Gamma = \mathbb{R}$, $S = \mathbb{R}_+ = [0, \infty)$,

$$\widehat{f}(\chi) = \int_{\mathbb{R}} f(t)e^{it\chi}\,dt \quad (f \in L^1(\mathbb{R}),\ \chi \in \mathbb{R}).$$

For a closed subset E of Γ, we shall let J_E denote the ideal of all functions in $L^1(G)$ which are of spectral synthesis with respect to E, so J_E is the closure in $L^1(E)$ of the set of functions f such that $\widehat{f} = 0$ near E. We shall also let $J_E^+ = J_E \cap L^1(S)$. We say that E is a *set of spectral synthesis* if $J_E = \{f \in L^1(G) : \widehat{f} = 0 \text{ on } E\}$. The following proposition recalls two facts about countable closed subsets of Γ, the first being well known and the second having been proved in [15, Théorème 2] in the case $S = \mathbb{Z}_+$ and being [12, Lemme 3.8] for $S = \mathbb{R}_+$. For completeness, we give the proof of the second result.

PROPOSITION 2.1. *Let E be a countable closed subset of Γ.*

(1) *E is a set of spectral synthesis.*

(2) *The map $\theta_E : f + J_E^+ \mapsto f + J_E$ is an isometric isomorphism of $L^1(S)/J_E^+$ onto $L^1(G)/J_E$.*

Proof. (1) This is well known; see [31, Theorem 7.2.4], for example.

(2) Let $s_0 \in S^0$. By taking non-negative functions supported in small neighbourhoods of s_0 in S, we can find a net (f_α) in $L^1(S)$ such that $\|f_\alpha\|_1 = 1$ and $\|g * f_\alpha - g_{s_0}\|_1 \to 0$ for all g in $L^1(G)$. Let $f'_\alpha(t) = f_\alpha(-t)$ $(t \in G)$, and let $\psi \in L^\infty(G)$. Then $(\psi * f'_\alpha)$ is weak*-convergent to ψ_{-s_0}, for the duality of $L^\infty(G)$ with $L^1(G)$. Hence

$$\|\psi\|_\infty = \|\psi_{-s_0}\|_\infty \le \liminf_\alpha \|\psi * f'_\alpha\|_\infty.$$

Moreover, $\psi * f'_\alpha$ is bounded and uniformly continuous.

Suppose that $\psi \in J_E^\perp$, the annihilator of J_E in $L^\infty(G)$. Then the spectrum of $\psi * f'_\alpha$ is contained in the countable set E. A result of Loomis [17] shows that $\psi * f'_\alpha$ is almost periodic and it follows that

$$\|\psi * f'_\alpha\|_\infty = \|(\psi * f'_\alpha)|S\|_\infty \le \|(\psi|S) * f'_\alpha\|_\infty.$$

Hence

$$\|\psi\|_\infty \le \liminf_\alpha \|\psi * f'_\alpha\|_\infty \le \liminf_\alpha \|(\psi|S) * f'_\alpha\|_\infty \le \|\psi|S\|_\infty.$$

This shows that the map $\theta_E^* : \psi \in J_E^\perp \mapsto \psi|S \in (J_E^+)^\perp$ is isometric, so its image has weak*-compact unit ball and is therefore weak*-closed. Since θ_E is injective, it follows that θ_E is a surjective isometry. ■

A *representation* of S on a complex Banach space X is a strongly continuous homomorphism T of S into the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators on X, so $T(s + t) = T(s)T(t)$ $(s, t \in S)$. All our representations will be bounded ($\sup_s \|T(s)\| < \infty$), and usually they will be contractive ($\|T(s)\| \le 1$). There is no great loss of generality in the latter assumption, since a bounded representation can always be made contractive by changing to an equivalent norm on X. We shall say that T is *asymptotically stable* if $T(s) \to 0$ in the strong operator topology, or in other words, if $X_s(T) = X$, where $X_s(T)$ is as in the introduction.

If T is a bounded representation and $f \in L^1(S)$, then we define $\widehat{f}(T)$ in $\mathcal{B}(X)$ by

$$\widehat{f}(T)x = \int_S f(s)T(s)x\,ds,$$

and we let $\mathcal{A}_T$ be the norm-closure of $\{\widehat{f}(T) : f \in L^1(S)\}$ in $\mathcal{B}(X)$. The map $f \mapsto \widehat{f}(T)$ is an algebra homomorphism, and $\mathcal{A}(T)$ is a commutative Banach algebra. If $s \in S^0$, there is a net $(\widehat{f}_\alpha(T))$ in $\mathcal{A}_T$ such that $\widehat{f}_\alpha(T)$ converges to $T(s)$ in the strong operator topology, and $\|\widehat{f}_\alpha(T)\widehat{g}(T) - T(s)\widehat{g}(T)\| \to 0$ for all g in $L^1(S)$. This is obtained by taking non-negative functions f_α with $\|f_\alpha\|_1 = 1$ supported by small neighbourhoods of s in S.

The *unitary spectrum* $\mathrm{Sp_u}(T)$ is the set of all χ in Γ such that $|\widehat{f}(\chi)| \le \|\widehat{f}(T)\|$ for all f in $L^1(S)$. Any χ in $\mathrm{Sp_u}(T)$ induces a character ϕ_χ of $\mathcal{A}_T$ by

$$\phi_\chi(\widehat{f}(T)) = \widehat{f}(\chi).$$

If each $T(s)$ is an invertible isometry on X, then T extends to a representation U of G by isometries on X, and $\mathrm{Sp_u}(T)$ coincides with the standard definition of the Arveson spectrum [28] or the finite L-spectrum [18] of U.

A *unitary eigenvalue* of T is a character χ in Γ for which there exists a non-zero *unitary eigenvector* x for T, so $T(s)x = \chi(s)x$ for all s. Similarly, a unitary eigenvalue of the adjoint T^* is a character χ for which there is a non-zero ϕ in X^* such that $T(s)^*\phi = \chi(s)\phi$ for all s. All unitary eigenvalues of T and T^* belong to $\mathrm{Sp_u}(T)$.

Let $\chi \in \Gamma$. Then $\chi \in \mathrm{Sp_u}(T)$ if and only if χ is an approximate eigenvalue of T, that is, there is a net (x_α) in X such that $\|x_\alpha\| = 1$ and $\|T(s)x_\alpha - \chi(s)x_\alpha\| \to 0$ uniformly on compact subsets of S. Moreover, for any B in $\mathcal{A}_T$, $\|Bx_\alpha - \phi_\chi(B)x_\alpha\| \to 0$ [6, Proposition 2.2].

The following result is implicit in [6, Proposition 4.1] and [5, Lemma 4.7].

PROPOSITION 2.2. *Let T be a bounded representation of S on X and P be an idempotent in $\mathcal{A}_T$. Let $Y = \{Px : x \in X\}$ and $K = \{x \in X : Px = 0\}$. Then*

$$\mathrm{Sp_u}(T|_Y) = \{\chi \in \mathrm{Sp_u}(T) : \phi_\chi(P) = 1\},$$
$$\mathrm{Sp_u}(T|_K) = \{\chi \in \mathrm{Sp_u}(T) : \phi_\chi(P) = 0\}.$$

Proof. Suppose that $\chi \in \mathrm{Sp_u}(T)$ and $\phi_\chi(P) = 1$. There is a net (x_α) in X such that $\|x_\alpha\| = 1$ and $\|T(s)x_\alpha - \chi(s)x_\alpha\| \to 0$ uniformly on compact subsets of S. Then $\|Px_\alpha - x_\alpha\| \to 0$, so $\|Px_\alpha\| \to 1$. If $y_\alpha = \|Px_\alpha\|^{-1}Px_\alpha \in Y$, then $\|y_\alpha\| = 1$ and $\|T|_Y(s)y_\alpha - \chi(s)y_\alpha\| \to 0$ uniformly on compact sets. Thus $\chi \in \mathrm{Sp_u}(T|_Y)$.

Conversely, suppose that $\chi \in \mathrm{Sp_u}(T|_Y)$. There is a net (y_α) in Y such that $\|y_\alpha\| = 1$ and $\|T(s)y_\alpha - \chi(s)y_\alpha\| \to 0$ uniformly on compact sets. Then $\chi \in \mathrm{Sp_u}(T)$ and $\|y_\alpha - \phi_\chi(P)y_\alpha\| \to 0$, so $\phi_\chi(P) = 1$.

The proof for $\mathrm{Sp_u}(T|_K)$ is similar. ∎

Finally, we remark that in the two standard examples of S, $\mathrm{Sp_u}(T)$ coincides with standard notions. A bounded representation of $\mathbb{Z}_+$ is of the form $\{T_1^n : n \in \mathbb{Z}_+\}$, where T_1 is a power-bounded operator on X, and $\mathrm{Sp_u}(T) = \sigma(T_1) \cap \mathbb{T}$. A bounded representation T of $\mathbb{R}_+$ is a bounded C_0-semigroup, and $\mathrm{Sp_u}(T) = \{\chi \in \mathbb{R} : i\chi \in \sigma(A)\}$, where A is the generator of T.

3. Trajectories and asymptotic stability. Let $V : S \to \mathcal{B}(Z)$ be a homomorphism, where Z is some Banach space. We will say that $z : G \to Z$ is a *trajectory* for V if it satisfies

$$V(s)(z(t)) = z(s+t) \quad (s \in S,\ t \in G).$$

When, in addition, there exists an M such that $\|z(t)\| \le M$ for all $t \in G$ then z will be called a *bounded trajectory for V through* $z(0)$.

Note that if V is strongly continuous on S, then any trajectory is norm-continuous on G. Similarly if Z is a dual space and V is point-weak*-continuous, then any trajectory is weak*-continuous on G.

Suppose that T is a bounded representation of S on X which is not asymptotically stable. Under certain supplementary conditions, Vũ Quôc Phóng [36] showed that T^* has non-zero bounded trajectories. Here, we show that this holds in general.

THEOREM 3.1. *Let T be a bounded representation of S on a Banach space X, and let M denote the set of ϕ in X^* through which there exists a bounded trajectory for T^*. The pre-annihilator $M_\perp$ of M in X is $X_s(T)$.*

Proof. If $\phi \in M$, there exists a bounded trajectory ψ of T^*, satisfying $\psi(0) = \phi$. For any $x \in X_s(T)$ and $s \in S$,

$$|\phi(x)| = |\psi(0)x| = |T^*(s)\psi(-s)x| \le \|\psi\|_\infty \|T(s)x\| \to 0 \quad \text{as } s \to \infty,$$

so $x \in M_\perp$.

Let (Y, Q, U) be the limit isometric representation of (X, T), as in [6, Proposition 3.1]. If $x \in X \setminus X_s(T)$, then $Qx \ne 0$. By Douglas's Theorem [11], U may be extended to a representation V of G by isometries on a Banach space Z containing Y. Let $\xi \in Z^*$ be such that $\xi(Qx) = 1$, and let $\psi(t) = Q^*U(t)^*\xi$ $(t \in G)$. Then ψ is a bounded trajectory for T^* and $\psi(0)(x) = 1$, so $x \notin M_\perp$. ∎

COROLLARY 3.2. *Let T be a bounded representation of S on X which is not asymptotically stable. Then T^* has a non-zero, bounded, uniformly norm-continuous trajectory.*

Proof. By Theorem 3.1, $M \neq \{0\}$, so T^* has a non-zero bounded trajectory ϕ. If W is a compact neighbourhood of the identity in G and

$$\psi(t) = \frac{1}{|W|} \int_W \phi(s+t)\, ds,$$

then ψ is a bounded, uniformly norm-continuous trajectory. If W is sufficiently small, then ψ is non-zero. ∎

Remark. It follows from Theorem 3.1 and Corollary 3.2 that the supplementary condition (i) or (ii) can be omitted from various results in [36, Section 2] and from [37, Theorem 6.3].

EXAMPLE 3.3. Let $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$. Consider the multiplier C_0-semigroup on $X = C_0(\mathbb{C}_-)$ given by

$$(T(s)f)(z) = e^{sz} f(z) \quad (s \in \mathbb{R}_+,\ z \in \mathbb{C}_-).$$

The dual of X is the space $M(\mathbb{C}_-)$ of complex, regular measures on $\mathbb{C}_-$. Given μ in $M(\mathbb{C}_-)$, it is easily seen that there exists a bounded trajectory through μ if and only if μ is supported by $i\mathbb{R}$. On the other hand, given $f \in C_0(\mathbb{C}_-)$, $\|T(s)f\|$ converges to zero if and only if $f|_{i\mathbb{R}} = 0$. Theorem 3.1 confirms this fact.

In this example,

$$\lim_s \|T(s)f\| = \|f|_{i\mathbb{R}}\|_\infty = \|f + X_{\mathrm{s}}(T)\|,$$

for all f. We shall see in Examples 4.1 and 4.2 that such a formula is not true in general for contractive representations, but we shall see in Theorem 5.3 that it is true when the unitary spectrum is countable.

4. Trivially asymptotically stable representations.

A bounded representation T of S on X will be called *trivially asymptotically stable* if $X_{\mathrm{s}}(T) = \{0\}$. Any representation by isometries is trivially asymptotically stable, and we now give some examples of non-isometric, trivially asymptotically stable representations.

EXAMPLE 4.1. Let $S = \mathbb{Z}_+$, $X = \ell^p(\mathbb{Z})$ $(1 \le p \le \infty)$ or $X = c_0(\mathbb{Z})$,

$$(Tx)_n = \alpha_n x_{n-1},$$

where $\alpha_n = 1$ if $n \ge 1$, $0 < \alpha_n < 1$ if $n \le 0$ and $\alpha_n \to 0$ as $n \to -\infty$. Then $\|T\| = 1$, $\sigma(T) = \{z \in \mathbb{C} : |z| \le 1\}$, and $\{T^n : n \in \mathbb{Z}_+\}$ is trivially asymptotically stable, but T is neither isometric nor invertible.

EXAMPLE 4.2. Let $S = \mathbb{R}_+$, $X = L^1(\mathbb{R})$ equipped with the norm

$$\|f\|_X = \int_{-\infty}^{0} |f(s)|\, ds + \frac{1}{2} \int_{0}^{\infty} |f(s)|\, ds,$$

and let

$$(T(s)f)(t) = f(t-s).$$

Then $\|T(s)\| = 1$, $\sigma(T(s)) = \mathbb{T}$, $\sigma(A) = i\mathbb{R}$ and $\mathrm{Sp_u}(T) = \mathbb{R}$, and T is trivially asymptotically stable and invertible, but not isometric.

Let E be a closed subset of $\mathbb{R}$, and J_E be the ideal of functions in $L^1(\mathbb{R})$ which are of spectral synthesis with respect to E. Then T induces a trivially asymptotically stable C_0-semigroup T_E of invertible contractions on $X_E := X/J_E$ (with the quotient norm arising from $\|\cdot\|_X$). Moreover, T_E is isometric with respect to the quotient norm $\|\cdot\|_1$ arising from the standard norm on $L^1(\mathbb{R})$, $\mathrm{Sp_u}(T_E) = E$ [5, Theorem 2.3], and

$$\|T_E(s)(f + J_E)\|_{X_E} \to \tfrac{1}{2}\|f + J_E\|_1 \quad \text{as } s \to \infty.$$

Hence T_E is isometric on X_E if and only if

$$(**) \qquad \|f + J_E\|_{X_E} = \tfrac{1}{2}\|f + J_E\|_1$$

for all f in $L^1(\mathbb{R})$.

Now, suppose that E is countable, and let $f \in L^1(\mathbb{R})$. By Proposition 2.1, there exists g in $L^1(\mathbb{R}_+)$ such that $f + J_E = g + J_E$ and, for any $\varepsilon > 0$, there exists h in $J_E \cap L^1(\mathbb{R}_+)$ such that

$$\|g - h\|_1 < \|g + J_E\|_1 + \varepsilon = \|f + J_E\|_1 + \varepsilon.$$

Hence

$$\|f + J_E\|_{X_E} = \|g + J_E\|_{X_E} \le \|g - h\|_X = \tfrac{1}{2}\|g - h\|_1 < \tfrac{1}{2}(\|f + J_E\|_1 + \varepsilon).$$

Thus $(**)$ holds if E is countable, so T_E is isometric.

Any isometric representation with countable unitary spectrum is invertible [5, Theorem 5.1]. Our main objective in this section is to show that any trivially asymptotically stable, contractive representation with countable unitary spectrum is isometric and hence invertible (Theorem 4.8). Thus the case of Example 4.2 when E is countable is one instance of a general result. Before starting the proof, we first give a general way of constructing trivially asymptotically stable representations which will be useful in Section 5.

PROPOSITION 4.3. *Let T be a contractive representation of S on X, and $\widetilde{T}$ be the representation of S on $X/X_{\mathrm{s}}(T)$ (in the quotient norm) induced by T. For x in X,*

$$\lim_s \|\widetilde{T}(s)(x + X_{\mathrm{s}}(T))\| = \lim_s \|T(s)x\|.$$

Hence $\widetilde{T}$ is trivially asymptotically stable.

Proof. Let $x \in X$, and $l = \lim_s \|\widetilde{T}(s)(x + X_s(T))\| = \lim_s \|T(s)x + X_s(T)\|$. For $\varepsilon > 0$, there exist y in $X_s(T)$ and t in S such that $\|T(t)x - y\| < l + \varepsilon$. For s in S,

$$\|T(s+t)x\| \le \|T(s+t)x - T(s)y\| + \|T(s)y\| \le l + \varepsilon + \|T(s)y\|.$$

Hence $\lim_s \|T(s)x\| \le l + \varepsilon$ for any $\varepsilon > 0$. On the other hand, $\|\widetilde{T}(s)(x + X_s(T))\| \le \|T(s)x\|$ for all s, so the result follows. ∎

LEMMA 4.4. *Let T be a trivially asymptotically stable, contractive representation of S on X with unitary spectrum E, and let $f \in L^1(S)$. Then $\|\widehat{f}(T)\| \le \|f + J_E^+\|$. If E is countable, then $\|\widehat{f}(T)\| \le \|f + J_E\|$.*

Proof. Let $g \in J_E^+$. By [6, Theorem 4.3] (see also Proposition 5.5), $\|T(s)\widehat{g}(T)x\| \to 0$ for each x in X. Since T is trivially asymptotically stable, $\widehat{g}(T)x = 0$. Thus, $\widehat{g}(T) = 0$. It follows immediately that $\|\widehat{f}(T)\| \le \|f - g\|_1$ for all g in J_E^+, so $\|\widehat{f}(T)\| \le \|f + J_E^+\|$. The last statement follows from Proposition 2.1. ∎

PROPOSITION 4.5. *Let T be a trivially asymptotically stable, contractive representation of S whose unitary spectrum E is a compact set of spectral synthesis, and let $f \in L^1(S)$ with $\widehat{f} = 1$ on E. Then $\widehat{f}(T) = I$.*

Proof. Since $(\widehat{f})^2 = \widehat{f}$ on E and E is a set of spectral synthesis, $f * f - f \in J_E$, so $\widehat{f}(T)^2 = \widehat{f}(T)$, by Lemma 4.4. Thus $\widehat{f}(T)$ is an idempotent in $\mathcal{A}_T$. Let $K = \ker \widehat{f}(T)$. By Proposition 2.2,

$$\mathrm{Sp_u}(T|_K) = \{\chi \in E : \widehat{f}(\chi) = 0\} = \emptyset.$$

By the ABLP Theorem, $\|T(s)x\| \to 0$ for all $x \in K$. Since T is trivially asymptotically stable, this implies that $K = \{0\}$, so $\widehat{f}(T) = I$. ∎

PROPOSITION 4.6. *Let T be a trivially asymptotically stable, contractive representation of S on X whose unitary spectrum E is countable. There is a contractive homomorphism $\pi : L^1(G) \to \mathcal{A}_T$ such that*

(1) *$\pi(f) = \widehat{f}(T)$ for all f in $L^1(S)$,*
(2) *$\pi(g_s) = T(s)\pi(g)$ for all g in $L^1(G)$ and s in S,*
(3) *if $\widehat{g} = 0$ on E, then $\pi(g) = 0$.*

Proof. Let $g \in L^1(G)$. By Proposition 2.1, there exists f in $L^1(S)$ such that $g - f \in J_E$. Define $\pi(g) = \widehat{f}(T)$. By Lemma 4.4, $\pi(g)$ is independent of the choice of f, and $\|\pi(g)\| \le \|f + J_E\| \le \|g\|_1$. It is easily verified that π is a homomorphism, and that (1) holds. If $\widehat{g} = 0$ on E, then $g \in J_E$ since E is a set of spectral synthesis, so $\pi(g) = 0$.

For s in S we have $g_s - f_s \in J_E$, so

$$\pi(g_s) = \widehat{f}_s(T) = T(s)\widehat{f}(T) = T(s)\pi(g). \ \blacksquare$$

LEMMA 4.7. *Let T be a trivially asymptotically stable, contractive representation of S on X whose unitary spectrum E is countable, and let π be as in Proposition 4.6. Let $\mathcal{K}_E$ be the set of all f in $L^1(G)$ such that $\widehat{f}$ has compact support and $\widehat{f}|_E$ takes only the values 0 and 1. For f in $\mathcal{K}_E$, let Y_f be the image of the idempotent $\pi(f)$, and let $Y = \bigcup_{f \in \mathcal{K}_E} Y_f$. Then Y is a linear subspace of X and, for each s in S^0, $T(s)$ maps X into $\overline{Y}$ (the closure of Y in X).*

Proof. Suppose that $x_1, x_2 \in X$ and $f_1, f_2 \in \mathcal{K}_E$. Let $f = f_1 + f_2 - f_1 * f_2$. Then $f \in \mathcal{K}_E$ and

$$\begin{aligned}\pi(f)(\pi(f_1)x_1 + \pi(f_2)x_2) &= \pi(f * f_1)x_1 + \pi(f * f_2)x_2 \\ &= \pi(f_1)x_1 + \pi(f_2)x_2,\end{aligned}$$

since π is a homomorphism and $\widehat{f}.\widehat{f}_j = \widehat{f}_j$ on E. Thus Y is a linear subspace.

Let $s \in S^0$ and $x \in X$. For any $\varepsilon > 0$, there exists k in $L^1(S)$ such that $\|\widehat{k}(T)x - T(s)x\| < \varepsilon/2$, and there exists g in $L^1(G)$ such that $\widehat{g}$ has compact support and $\|k - k * g\|_1 < \varepsilon/(2\|x\|)$ [31, Theorem 2.6.6]. Since E is countable, $\mathrm{supp}\,\widehat{g} \cap E$ is contained in a compact, relatively open subset of E, so there is an open subset V of Γ containing $\mathrm{supp}\,\widehat{g} \cap E$ such that $E \cap V$ is compact. There exists f in $L^1(G)$ such that $\widehat{f}$ has compact support, $\widehat{f} = 1$ on $E \cap V$ and $\widehat{f} = 0$ on $E \setminus V$. Since $\widehat{g} = \widehat{f}.\widehat{g}$ on E,

$$\pi(g)\widehat{k}(T)x = \pi(f)\pi(g)\widehat{k}(T)x \in Y.$$

Moreover,

$$\begin{aligned}\|\pi(g)\widehat{k}(T)x - T(s)x\| &= \|\pi(g)\pi(k)x - T(s)x\| \\ &\le \|\pi(g * k - k)x\| + \|\pi(k)x - T(s)x\| \\ &\le \|g * k - k\|_1\|x\| + \|\widehat{k}(T)x - T(s)x\| < \varepsilon. \ \blacksquare\end{aligned}$$

THEOREM 4.8. *Let T be a trivially asymptotically stable, contractive representation of S on X whose unitary spectrum E is countable. For each s in S, $T(s)$ is an invertible isometry.*

Proof. First, suppose that E is compact. There exists k in $L^1(G)$ such that $\widehat{k} = 1$ on E. By Proposition 2.1 (or by [5]), there exists h in $L^1(S)$ such that $h - k \in J_E$, so $\widehat{h} = 1$ on E. Define

$$U(t) = \pi(h_t) \quad (t \in G),$$

where $h \in L^1(S)$, $\widehat{h} = 1$ on E and π is as in Proposition 4.6. If $h' \in L^1(S)$ and $(h')\widehat{\ } = 1$ on E, then $h_t - h'_t \in J_E$, so $U(t)$ is independent of the choice of h.

For s, t in G we have $h_{t+s} - h_t * h_s \in J_E$, so

$$U(t+s) = \pi(h_{t+s}) = \pi(h_t * h_s) = \pi(h_t)\pi(h_s) = U(t)U(s),$$

since π is a homomorphism. By Proposition 4.5, $\widehat{h}(T) = I$, so $U(0) = I$. For s in S,

$$U(s) = \widehat{h}_s(T) = T(s)\widehat{h}(T) = T(s).$$

Given $\varepsilon > 0$, there exists k in $L^1(G)$ such that $\widehat{k} = 1$ on E and $\|k\|_1 < 1+\varepsilon$ [31, Theorem 2.6.8]. Then we may take h in $L^1(S)$ such that $h-k \in J_E$. For t in G,

$$\begin{aligned}\|U(t)\| &= \|\pi(h_t)\| \le \|h_t + J_E\| \\ &= \|h + J_E\| = \|k + J_E\| \le \|k\|_1 < 1+\varepsilon.\end{aligned}$$

This shows that U is a representation of G by isometries on X, extending T. This establishes the result in the case when E is compact.

Now, consider the general case. We will use the notation and results of Lemma 4.7. Let $f \in \mathcal{K}_E$. By Proposition 2.2, $\mathrm{Sp_u}(T|_{Y_f}) = \{\chi \in E : \widehat{f}(\chi) = 1\}$, which is compact and countable. By the case previously considered, $T(s)|_{Y_f}$ is an invertible isometry of Y_f, for each s in S. It follows that $T(s)|_{\overline{Y}}$ is an invertible isometry of $\overline{Y}$.

Let $x \in X$, $s_0 \in S^0$, so $T(s_0)x \in \overline{Y}$, by Lemma 4.7. By the previous paragraph, there exists x' in $\overline{Y}$ such that $T(s_0)x = T(s_0)x'$. Since T is trivially asymptotically stable, $T(s_0)$ is injective, so $x = x' \in \overline{Y}$. Thus, $\overline{Y} = X$, and each $T(s)$ is an invertible isometry of X. ∎

5. Some quantitative asymptotic results. In this section, we combine the ideas of the previous two sections to obtain a quantitative refinement of the ABLP Theorem (Theorem 5.3). The first result is a version of Theorem 3.1 for representations with countable unitary spectrum, in which arbitrary bounded trajectories of T^* are replaced by unitary eigenvectors. It is both a refinement and a corollary of the ABLP Theorem. We are grateful to Wolfgang Arendt for supplying a short proof.

PROPOSITION 5.1. *Let T be a bounded representation of S on X with countable unitary spectrum, and let N be the linear span of the unitary eigenvectors of T^*. The pre-annihilator $N_\perp$ of N in X is $X_s(T)$.*

Proof. It is clear that if $\|T(s)x\| \to 0$ and $T(s)^*\phi = \chi(s)\phi$ for some $\chi \in \Gamma$, then $\phi(x) = 0$.

For the converse, we shall apply the ABLP Theorem to the invariant subspace $N_\perp$. Any approximate eigenvalue of $T|_{N_\perp}$ is an approximate eigenvalue of T, so $\mathrm{Sp_u}(T|_{N_\perp})$ is countable. Suppose that $\phi \in (N_\perp)^*$ and $(T(s)|_{N_\perp})^*\phi = \chi(s)\phi$ for all s in S, for some χ in Γ. Let ξ be a translation-invariant bounded linear functional on $L^\infty(S)$ such that $\xi(1) = 1$, and define ψ by

$$\psi(x) = \xi(\overline{\chi(\cdot)}\phi(T(\cdot)x)) \quad (x \in X).$$

Then $\psi \in X^*$ and $T(s)^*\psi = \chi(s)\psi$, so $\psi \in N$. But $\phi = \psi|_{N_\perp} = 0$. This shows that $(T|_{N_\perp})^*$ has no unitary eigenvalues. By the ABLP Theorem, $\lim_s \|T(s)x\| = 0$ for all x in $N_\perp$. ∎

COROLLARY 5.2. *Let T be a bounded representation of S on X with countable unitary spectrum. The almost periodic trajectories of T^* are point-weak*-dense in the space of all bounded trajectories of T^*.*

Proof. By Theorem 3.1 and Proposition 5.1, $M_\perp = N_\perp$, so N is weak*-dense in M.

Let $\phi : G \to X^*$ be a bounded trajectory for T^*, and let $t_1, \dots, t_n \in G$ and W be a weak*-neighbourhood of 0 in X^*. There exist $s_1, \dots, s_n$ in S such that $t_j + s_j \in S$ $(j = 1, \dots, n)$. Let $s = s_1 + \dots + s_n \in S$, and $W' = \bigcap_{j=1}^n (T(t_j+s)^*)^{-1}(W)$, a weak*-neighbourhood of 0. Now $t \mapsto \phi(t-s)$ is a bounded trajectory for T^*, so $\phi(-s) \in M$. Hence there exists $\psi_0 \in N$ such that $\phi(-s) - \psi_0 \in W'$. Since ψ_0 is a finite linear combination of unitary eigenvectors, there is an almost periodic trajectory ψ such that $\psi(-s) = \psi_0$. Now

$$\phi(t_j) - \psi(t_j) = T(t_j + s)^*(\phi(-s) - \psi_0) \in W. \ ∎$$

THEOREM 5.3. *Let T be a contractive representation of S on X with countable unitary spectrum, and let $\overline{N}$ be the weak*-closure of the linear span of the unitary eigenvectors of T^*. For each x in X,*

$$\lim_s \|T(s)x\| = \inf\{\|x - y\| : y \in X_s(T)\} = \sup\{|\phi(x)| : \phi \in \overline{N},\ \|\phi\| \le 1\}.$$

Proof. By Proposition 4.3, T induces a trivially asymptotically stable, contractive representation $\widetilde{T}$ on the quotient space $X/X_s(T)$, and

$$\lim_s \|\widetilde{T}(s)(x + X_s(T))\| = \lim_s \|T(s)x\|.$$

Since $\|\widehat{f}(\widetilde{T})\| \le \|\widehat{f}(T)\|$, it follows that $\mathrm{Sp_u}(\widetilde{T}) \subseteq \mathrm{Sp_u}(T)$, so $\mathrm{Sp_u}(\widetilde{T})$ is countable. By Theorem 4.8, $\widetilde{T}$ is isometric, so

$$\lim_s \|T(s)x\| = \|x + X_s(T)\|.$$

The second equality now follows from Proposition 5.1 and the Hahn–Banach Theorem. ∎

COROLLARY 5.4. *Let T be a bounded representation of S on X with countable unitary spectrum, and let $\overline{N}$ be the weak*-closure of the linear span of the unitary eigenvectors of T^*. For each x in X,*

$$\begin{aligned}\sup\{|\phi(x)| : \phi \in X^*,\ \|\phi\| \le 1\} &\le \limsup_s \|T(s)x\| \\ &\le M \sup\{|\phi(x)| : \phi \in X^*,\ \|\phi\| \le 1\},\end{aligned}$$

where $M = \sup_s \|T(s)\|$.

Proof. Let $\|x\|' = \sup_s \|T(s)x\|$. Since T is contractive with respect to $\|\cdot\|'$, Theorem 5.3 gives

$$\limsup_s \|T(s)x\| = \sup\{|\phi(x)| : \phi \in X^*,\ \|\phi\|' \le 1\}.$$

Since $\|x\| \le \|x\|' \le M\|x\|$ for all x, $M^{-1}\|\phi\| \le \|\phi\|' \le \|\phi\|$ for all ϕ, and the result follows. ■

Examples 4.1 and 4.2 show that Theorem 5.3 may fail if $\mathrm{Sp_u}(T)$ is not countable.

It was shown in [6, Theorem 4.3] (special cases had previously been given in [16], [12] and [35]) that $\lim_s \|T(s)\widehat{f}(T)\| = 0$ when S is a contractive representation with unitary spectrum E and $f \in J_E$. It follows that $\lim_s \|T(s)\widehat{f}(T)\| \le \|f + J_E^{+}\|$ for all f in $L^1(S)$. The following result is a sharper version of this.

PROPOSITION 5.5. *Let T be a contractive representation of S on X with unitary spectrum E, and let $f \in L^1(S)$. Then*

(1) $\lim_s \|T(s)\widehat{f}(T)x\| \le \|f + J_E\| \lim_s \|T(s)x\|$ *for all x in X*,
(2) $\lim_s \|T(s)\widehat{f}(T)\| \le \|f + J_E\|$.

Proof. Let (Y, Q, U) be the limit isometric representation of (X, T), as in [6, Proposition 3.2]. By Douglas's Theorem [11], [5, Proposition 2.1], (Y, U) may be extended to an isometric representation V of G on a Banach space Z containing Y, with $\mathrm{Sp}(V) = \mathrm{Sp_u}(U) \subseteq E$. For g in J_E we have $\widehat{g}(V) = 0$, so

$$\|\widehat{f}(U)\| \le \|\widehat{f}(V)\| = \|\widehat{f}(V) - \widehat{g}(V)\| \le \|f - g\|_1.$$

It follows that

$$\|\widehat{f}(U)\| \le \|f + J_E\|.$$

This means that, for each x in X,

$$\begin{aligned}\lim_s \|T(s)\widehat{f}(T)x\| &= \|Q\widehat{f}(T)x\| = \lim_s \|\widehat{f}(U)Qx\| \\ &\le \|f + J_E\|\|Qx\| = \|f + J_E\| \lim_s \|T(s)x\|.\end{aligned}$$

The proof of the second statement is modelled on [35, Theorem 3.2] and [6, Theorem 4.3]. Consider the representation $\widetilde{T}$ of S on $\mathcal{A}_T$ defined by $\widetilde{T}(s)(B) = T(s)B$. This is strongly continuous and $\mathrm{Sp_u}(\widetilde{T}) \subseteq \mathrm{Sp_u}(T) = E$ (see [6]). By applying (1) to $\widetilde{T}$,

$$\lim_s \|T(s)\widehat{f}(T)\widehat{g}(T)\| \le \|f + J_E\| \lim_s \|T(s)\widehat{g}(T)\| \le \|f + J_E\|\, \|g\|_1.$$

Taking s_0 in S^0 and a net (g_α) in $L^1(S)$ such that $\|g_\alpha\|_1 = 1$ and $\|(\widehat{g}_\alpha(T) - T(s_0))\widehat{f}(T)\| \to 0$, we obtain $\lim_s \|T(s + s_0)\widehat{f}(T)\| \le \|f + J_E\|$, and the result follows. ■

6. Differential operators on $L^1(\mathbb{R}^n)$. Typical examples where our results may be applied are C_0-semigroups on $L^1(\mathbb{R}^n)$ generated by suitable L^1-versions A_1 of elliptic operators $A = \sum_{1\le|\alpha|\le m} a_\alpha(x)D^\alpha$ with bounded, measurable, real coefficients, and Schrödinger operators $A = \frac{1}{2}\Delta - V$. However, it is not always easy to verify the conditions of our results, and we will now discuss this question.

Firstly, one has to establish that A_1 generates a bounded C_0-semigroup on $L^1(\mathbb{R}^n)$. In the case of a second-order uniformly elliptic operator, this can usually be established by means of the Beurling–Deny criteria [8, Theorem 1.3.5], [22, Section II.2], [26]. Under mild regularity conditions, the semigroup is given by

$$T(t)f(x) = \mathbf{E}^x[f(X_t)],$$

where X_t is the solution of a stochastic differential equation associated with A. Such a semigroup is contractive in L^∞-norm, so duality theory may provide a contraction semigroup on $L^1(\mathbb{R})$. For Schrödinger semigroups with absorbing potential $V \ge 0$, L^1-contractivity follows by comparison. However, the situation is more complicated for elliptic operators of higher order (see [10]).

Secondly, one has to show that $\sigma(A_1) \cap i\mathbb{R}$ is countable. For an elliptic operator with constant coefficients, $\sigma(A_1) = \{\sum_{|\alpha|\le m} a_\alpha i^{|\alpha|}\xi^\alpha : \xi \in \mathbb{R}^n\}$ [3], [33, p. 69], so it is possible to verify whether the spectral condition holds. For some operators with variable coefficients, it is known (see [1], [9]) that the spectrum of A_1 coincides with the spectrum of the L^2-realisation A_2 of the operator, which is given by a quadratic form. If the quadratic form is semidefinite, it follows that $\sigma(A_1) \subseteq (-\infty, 0]$. Second-order elliptic operators with variable coefficients generate bounded positive holomorphic semigroups on $L^1(\mathbb{R}^n)$ [30], [27], so $\sigma(A_1) \cap i\mathbb{R} \subseteq \{0\}$ by a general result in the theory of positive semigroups [24, Corollary 2.13, p. 304].

Thirdly, one has to identify the unitary eigenvectors of A_1^* in $L^\infty(\mathbb{R}^n)$. This is a problem in differential equations, and some relevant results may be found in [29], [14] and other references given in the examples below.

Each of the following examples has already been examined in the literature, usually by more explicit methods than ours. We show how they fit within the scope of our abstract results.

EXAMPLE 6.1 [7, Theorem 3.1]. Consider the following symmetric, purely second-order, uniformly elliptic operator on $\mathbb{R}^n$ with bounded measurable coefficients:

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

Then A_1 generates a C_0-semigroup of contractions on $L^1(\mathbb{R}^n)$ [8] and $\sigma(A_1)$

$\subseteq (-\infty, 0]$ (cf. [1]). Moreover, the only solutions in $L^\infty(\mathbb{R}^n)$ of $A_1^* g = 0$ are constants [23], [13, Appendix, Theorem 3]. Thus Theorem 5.3 shows that

$$\lim_{t\to\infty} \|T(t)f\|_1 = \Big| \int_{\mathbb{R}^n} f(x)\,dx \Big|$$

for all f in $L^1(\mathbb{R}^n)$.

EXAMPLE 6.2 [4, Proposition 3.2]. Consider the following Schrödinger operator on $\mathbb{R}^n$:

$$A = \tfrac{1}{2}\Delta - V,$$

where $V \geq 0$ is measurable. Then A generates a C_0-semigroup of contractions on $L^1(\mathbb{R}^n)$ [34] and $\sigma(A_1) = \sigma(A_2) \subseteq (-\infty, 0]$ [27]. It was shown in [4, Proposition 3.2] that the only solutions in $L^\infty(\mathbb{R}^n)$ of $A_1^* g = 0$ are scalar multiples of

$$(\dagger) \qquad g(x) := \lim_{t\to\infty} (T(t)^* 1)(x) = \mathbf{E}^x \Big[\exp\Big(-\int_0^\infty V(B(s))\,ds \Big) \Big],$$

where $B(s)$ is Brownian motion and $\mathbf{E}$ is expectation with respect to Wiener measure. It is easily seen that either $g = 0$ or $\|g\|_\infty = 1$, so Theorem 5.3 shows that

$$\lim_{t\to\infty} \|T(t)f\|_1 = \Big| \int_{\mathbb{R}^n} f(x)g(x)\,dx \Big|$$

for all f in $L^1(\mathbb{R}^n)$.

In this example, $\frac{1}{2}\Delta$ can be replaced by other symmetric, purely second-order, elliptic operators, with the probabilistic formula in $(\dagger)$ interpreted appropriately.

EXAMPLE 6.3 [10, Section 6]. Let $n = 1$, and

$$A = -\Big(\Big(\frac{d}{dx} \Big)^2 + I \Big)^2.$$

Then A_1 generates a bounded (non-contractive) C_0-semigroup on $L^1(\mathbb{R})$ [10, Lemma 4], and $\sigma(A_1) = \{-(\xi^2 - 1)^2 : \xi \in \mathbb{R}\} = (-\infty, 0]$. The only independent solutions of $A_1^* g = 0$ in $L^\infty(\mathbb{R})$ are $g(x) = e^{\pm ix}$. Since

$$\|\alpha e^{ix} + \beta e^{-ix}\|_\infty = |\alpha| + |\beta|,$$

Corollary 5.4 shows that

$$\max(|\widehat{f}(1)|, |\widehat{f}(-1)|) \leq \limsup_{t\to\infty} \|T(t)f\|_1 \leq M \max(|\widehat{f}(1)|, |\widehat{f}(-1)|)$$

for all f in $L^1(\mathbb{R})$, where $M = \sup_t \|T(t)\| \doteq 1.373$ (cf. [10]).

EXAMPLE 6.4 [32]. Let A be a differential operator on $\mathbb{R}$ of the form

$$Au = \frac{d}{dx}\Big(a(x) \frac{du}{dx} \Big) - \frac{d}{dx}(b(x)u(x)),$$

where $a(x)$, $b(x)$ are bounded, smooth and real, and $\inf_x a(x) > 0$. Then A_1 generates a bounded holomorphic semigroup on $L^1(\mathbb{R})$ [30, Theorem 2.7, p. 410], so $\sigma(A_1) \cap i\mathbb{R} = \{0\}$. The solutions of the formal equation

$$0 = A^* g = \frac{d}{dx}\Big(a(x) \frac{dg}{dx} \Big) + b(x) \frac{dg}{dx}$$

are

$$g(x) = \alpha + \beta h(x),$$

where α, β are arbitrary constants and

$$h(x) = \int_0^x \frac{1}{a(s)} \exp\Big(-\int_0^s \frac{b(r)}{a(r)}\,dr \Big)\,ds.$$

If

$$\int_{-\infty}^\infty \exp\Big(-\int_0^s \frac{b(r)}{a(r)}\,dr \Big)\,ds = \infty,$$

then the only solutions in $L^\infty(\mathbb{R})$ are constants, and Theorem 5.3 shows that

$$\lim_{t\to\infty} \|T(t)f\|_1 = \Big| \int_{-\infty}^\infty f(x)\,dx \Big|.$$

If

$$\int_{-\infty}^\infty \exp\Big(-\int_0^s \frac{b(r)}{a(r)}\,dr \Big)\,ds < \infty,$$

then $h(x)$ is bounded, and Theorem 5.3 gives a new formula

$$\lim_{t\to\infty} \|T(t)f\|_1 = \sup\Big\{ \Big| \int_{-\infty}^\infty f(x)(\alpha + \beta h(x))\,dx \Big| : \|\alpha + \beta h\|_\infty \leq 1 \Big\}$$
$$= \max\Big(\Big| \int_{-\infty}^\infty f(x)\,dx \Big|, \Big| \frac{2\int_{-\infty}^\infty f(x)h(x)\,dx - h(\infty) - h(-\infty)}{h(\infty) - h(-\infty)} \Big| \Big).$$

References

[1] W. Arendt, *Gaussian estimates and interpolation of the spectrum in* L^p, Differential Integral Equations 7 (1994), 1153–1168.

[2] W. Arendt and C. J. K. Batty, *Tauberian theorms and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. 306 (1988), 837–852.

[3] E. Balslev, *The essential spectrum of elliptic differential operators in* $L^p(\mathbb{R}_n)$, ibid. 116 (1965), 193–217.

[4] C. J. K. Batty, *Asymptotic stability of Schrödinger semigroups: path integral methods*, Math. Ann. 292 (1992), 457–492.
[5] C. J. K. Batty and D. A. Greenfield, *On the invertibility of isometric semigroup representations*, Studia Math. 110 (1994), 235–250.
[6] C. J. K. Batty and Vũ Quôc Phóng, *Stability of strongly continuous representations of abelian semigroups*, Math. Z. 209 (1992), 75–88.
[7] Z. Brzeźniak and B. Szafirski, *Asymptotic behaviour of L^1 norm of solutions to parabolic equations*, Bull. Polish Acad. Sci. Math. 39 (1991), 1–10.
[8] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.
[9] —, *Uniformly elliptic operators with measurable coefficients*, J. Funct. Anal. 132 (1995), 141–169.
[10] —, *Long time asymptotics of fourth order parabolic equations*, J. Anal. Math., to appear.
[11] R. G. Douglas, *On extending commutative semigroups of operators*, Bull. London Math. Soc. 1 (1969), 157–159.
[12] J. Esterle, E. Strouse et F. Zouakia, *Stabilité asymptotique de certains semi-groupes d'opérateurs et idéaux primaires de $L^1(\mathbb{R}^+)$*, J. Operator Theory 28 (1992), 203–227.
[13] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
[14] D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations*, J. Anal. Math. 4 (1955–56), 309–340.
[15] J.-P. Kahane et Y. Katznelson, *Sur les algèbres de restrictions des séries de Taylor absolument convergentes à un fermé du cercle*, ibid. 23 (1970), 185–197.
[16] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. 68 (1986), 313–328.
[17] L. H. Loomis, *The spectral characterization of a class of almost periodic functions*, Ann. of Math. 72 (1960), 362–368.
[18] Yu. I. Lyubich, *Introduction to the Theory of Banach Representations of Groups*, Birkhäuser, Basel, 1988.
[19] Yu. I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. 88 (1988), 37–42.
[20] Yu. I. Lyubich and Vũ Quôc Phóng, *A spectral criterion for the almost periodicity of one-parameter semigroups*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 47 (1987), 36–41 (in Russian).
[21] —, —, *A spectral criterion for asymptotic almost periodicity of uniformly continuous representations of abelian semigroups*, ibid. 50 (1988), 38–43 (in Russian); English transl.: J. Soviet Math. 49 (1990), 1263–1266.
[22] Z. M. Ma and M. Röckner, *An Introduction to the Theory of Non-Symmetric Dirichlet Forms*, Springer, Berlin, 1992.
[23] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 (1961), 577–591.
[24] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Math. 1184, Springer, Berlin, 1986.
[25] J. van Neerven, *The Asymptotic Behaviour of a Semigroup of Linear Operators*, Birkhäuser, Basel, 1996.
[26] E. M. Ouhabaz, *L^∞-contractivity of semigroups generated by sectorial forms*, J. London Math. Soc. 46 (1992), 529–542.
[27] R. Hempel and J. Voigt, *The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p-independent*, Comm. Math. Phys. 104 (1986), 243–250.
[28] G. K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.
[29] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer, Berlin, 1984.
[30] D. W. Robinson, *Elliptic Operators and Lie Groups*, Oxford Univ. Press, Oxford, 1991.
[31] W. Rudin, *Fourier Analysis on Groups*, Wiley, New York, 1992.
[32] R. Rudnicki, *Asymptotic stability in L^1 of parabolic equations*, J. Differential Equations 102 (1993), 391–401.
[33] M. Schechter, *Spectra of Partial Differential Operators*, North-Holland, Amsterdam, 1971.
[34] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. 7 (1982), 447–526.
[35] Vũ Quôc Phóng, *Theorems of Katznelson–Tzafriri type for semigroups of operators*, J. Funct. Anal. 103 (1992), 74–84.
[36] —, *On the spectrum, complete trajectories, and asymptotic stability of linear semi-dynamical systems*, J. Differential Equations 105 (1993), 30–45.
[37] —, *Stability and almost periodicity of trajectories of periodic processes*, ibid. 115 (1995), 402–415.

St. John's College
Oxford OX1 3JP
England
E-mail: charles.batty@sjc.ox.ac.uk

Department of Mathematics
University of Hull
Cottingham Road
Hull HU6 7RX, England
E-mail: z.brzezniak@maths.hull.ac.uk

87 Eversley Road
Benfleet
Essex SS7 4JT, England

Received January 15, 1996
Revised version July 15, 1996 (3596)