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Uniform approximation with linear combinations of reproducing kernels

by

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Abstract. We show several theorems on uniform approximation of functions. Each of them is based on the choice of a special reproducing kernel in an appropriate Hilbert space. The theorems have a common generalization whose proof is founded on the idea of the Kaczmarz projection algorithm.

1. Introduction and results. The purpose of this paper is to give some new applications and a new geometric interpretation of an algorithm $\mathcal{A}$ for uniform approximation of functions. $\mathcal{A}$ was introduced in [3]; it derives from our attempts to explain some learning processes in the brain (see [8]). Of all algorithms for approximating functions $\mathcal{A}$ is probably the simplest. The analysis of $\mathcal{A}$ presented here relies on the idea of the Kaczmarz projection algorithm (see [2, 5]). Other related results are given in [3, 6, 7]. Here we will apply reproducing kernels and the Fourier transform to obtain new applications, and we will show the connection with the algorithm of Kaczmarz (which was not known to us when we wrote the earlier papers). Also we point out five open problems.

The algorithm $\mathcal{A}$ is defined as follows. There is an unknown function $f: X \to \mathbb{C}$, where X is an abstract set, and there is a given function $h: X^2 \to \mathbb{C}$ with $h(x,x) = 1$ for all $x \in X$. We assume that $\mathcal{A}$ had already produced an approximation $f_n : X \to \mathbb{C}$ of f and that it receives a point $(x_n, f(x_n))$. Then $\mathcal{A}$ forms a new approximation

$$(\mathcal{A}_1) \qquad f_{n+1}(x) = f_n(x) + a_n h(x, x_n),$$

where a_n is such that $f_{n+1}(x_n) = f(x_n)$, in other words

$$(\mathcal{A}_2) \qquad a_n = f(x_n) - f_n(x_n),$$

i.e., *a_n is the error committed by $\mathcal{A}$ at time n*. (In the algorithm and in the Theorem below we may replace everywhere $\mathbb{C}$ by $\mathbb{R}$.) Thus given any f_0 and

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a sequence $(x_0, f(x_0)), (x_1, f(x_1)), \ldots$ the algorithm $\mathcal{A}$ creates a sequence $f_1, f_2, \ldots$

We assume without loss of generality that $f_0 \equiv 0$ (otherwise we replace f by $f - f_0$). Hence

$$f_n(x) = \sum_{i<n} a_i h(x, x_i).$$

We will prove that if h satisfies a certain condition to be stated in a while, there exists a Hilbert space H of functions $\varphi : X \to \mathbb{C}$ such that $\mathcal{A}$ yields interesting results for $f \in H$.

The required condition on h is that there exists an $a \in X$ such that $h(a, y) \neq 0$ for all $y \in X$, and the function

$$k(x, y) = \frac{h(y, a)}{\overline{h}(a, y)} h(x, y), \tag{k}$$

where $\overline{h}$ is the complex conjugate of h, be *positive definite* in the sense that for every finite set $X_0 \subseteq X$ and any $g : X_0 \to \mathbb{C}$ we have

$$\sum_{x,y \in X_0} g(x)\overline{g}(y)k(x, y) \geq 0. \tag{p}$$

This condition may be hard to check directly, but it is *equivalent* to the existence of a Hilbert space H_k of functions $\varphi : X \to \mathbb{C}$ such that k is a reproducing kernel for H_k, i.e., $k_y \in H_k$ and

$$\langle \varphi, k_y \rangle = \varphi(y)$$

for all $\varphi \in H_k$ and all $y \in X$, where $k_y(x) = k(x, y)$. We define H to be H_k.

For a proof of this equivalence (and a construction of H_k) see, e.g., S. Saitoh [10], Chapter 2, (see also [11]). We are able to check (p) for various functions h by producing explicitly H_k. Several examples will be given below.

Vice versa, given a reproducing kernel k, and the corresponding Hilbert space H_k normalized such that $\|k_a\| = 1$, we can define

$$h(x, y) = \frac{\langle k_y, k_x \rangle}{\|k_y\|^2} = \frac{k(x, y)}{\|k_y\|^2}. \tag{h}$$

Then $h(x, x) = 1$ and the formula (k) is true. Let us add that $H (= H_k)$ is the closed linear span of the set $\{k_y : y \in X\}$. We shall prove that H is also the closed linear span of the set $\{h_y : y \in X\}$, where $h_y(x) = h(x, y)$. The construction of H in [10] also shows that if $\varphi_n \to \varphi$ in H, then $\varphi_n(x) \to \varphi(x)$ for all $x \in X$. However, the converse of this implication is not true in general and this leads us to some unsolved problems which will be stated below.

Our results will show that under various natural assumptions about X, f, h, and $x_0, x_1, \ldots$ the algorithm $\mathcal{A}$ yields $f_n \to f$ uniformly, in particular $a_n \to 0$ and in some cases $\sum |a_i|^2 < \infty$. A few concrete examples are

the following. (Their derivations from the Theorem stated at the end of this section are given in Section 2.)

EXAMPLE (A). Let X be the open interval $(0, 1)$ and

$$h(x, y) = \begin{cases} \dfrac{x}{y} & \text{for } x \leq y, \\ \dfrac{1-x}{1-y} & \text{for } x \geq y. \end{cases}$$

If f is absolutely continuous and $\lim_{x\uparrow 1} f(x) = \lim_{x\downarrow 0} f(x) = 0$, then $\mathcal{A}$ yields

$$|f(x) - f_n(x)| \leq \left\{ x(1-x) \left(\int_0^1 |f'|^2 - \sum_{i<n} \frac{|a_i|^2}{x_i(1-x_i)} \right) \right\}^{1/2}.$$

Notice that this immediately implies that if $x_0, x_1, \ldots$ are such that

$$\frac{|a_n|^2}{x_n(1-x_n)} \geq \delta \sup_{x \in (0,1)} \frac{|f(x) - f_n(x)|^2}{x(1-x)},$$

where δ is a positive constant, and $\int |f'|^2 < \infty$, then $f_n \to f$ uniformly.

(For related facts see [3], §2 and §6.)

UNSOLVED PROBLEMS. 1. Let $\int |f'|^2 < \infty$ and $x_0, x_1, \ldots$ be everywhere dense in $(0, 1)$. Must $f_n \to f$ uniformly?

2. Let $\int |f'|^2 < \infty$ and let each element of the set $\{x_0, x_1, \ldots\}$ appear infinitely many times in the sequence $x_0, x_1, \ldots$ Must $f_n \to f$ uniformly on the set $\{x_0, x_1, \ldots\}$?

3. Suppose that $f_n \to f$ uniformly where f_n are given by $\mathcal{A}$. Must

$$\sum_{i=0}^{\infty} \frac{|a_i|^2}{x_i(1-x_i)} = \int_0^1 |f'|^2 \ ?$$

(We know only that, if $\int |f' - f_n'|^2 \to 0$, then the above equality holds.)

4. Suppose that for every $\varepsilon > 0$ there exist $y_0, \ldots, y_{n-1}$ and $b_0, \ldots, b_{n-1}$ such that

$$\Big| f(x) - \sum_{i<n} b_i h(x, y_i) \Big| < \varepsilon$$

for all $x \in (0, 1)$. Does the algorithm $\mathcal{A}$ imply that $\lim_{n\to\infty} a_n = 0$? (By the assertion of (A) the answer is yes if $\int |f'|^2 < \infty$.) A related problem was formulated in [6], §5.2.

Similar problems are also open for the examples (B), (C), (D) and (E) which follow.

EXAMPLE (B). Let X be the open ray $(0,\infty)$ and

$$h(x,y)=\begin{cases}\dfrac{x}{y} & \text{for } x\le y,\\ 1 & \text{for } x\ge y.\end{cases}$$

If f is absolutely continuous and $\lim_{x\downarrow 0} f(x)=0$, then $\mathcal{A}$ yields

$$|f(x)-f_n(x)|\le\Big\{x\Big(\int_0^\infty |f'|^2-\sum_{i<n}\frac{|a_i|^2}{x_i}\Big)\Big\}^{1/2}.$$

And hence, if $x_0,x_1,\ldots$ are such that

$$\frac{|a_i|^2}{x_i}\ge\delta\sup_{x\in(0,\infty)}\frac{|f(x)-f_n(x)|^2}{x},$$

where δ is a positive constant, and $\int|f'|^2<\infty$, then $\frac{1}{x}|f(x)-f_m(x)|^2\to 0$ uniformly in $(0,\infty)$.

(For related examples see [3].)

EXAMPLE (C). Let X be a locally compact abelian group, and let $\check{}$ denote the inverse Fourier transform which maps the appropriate space $A(X)$ of continuous functions on X *onto* the space $L^1(\widehat{X})$, where $\widehat{X}$ is the group of characters of X. Let $h_1: X\to\mathbb{C}$, $h_1\in A(X)$, be such that

$$h_1(0)=1\quad\text{and}\quad \check{h}_1\ge 0.$$

Then, if $f\in A(X)$ and $h(x,y)=h_1(x-y)$, the algorithm $\mathcal{A}$ yields

$$|f(x)-f_n(x)|\le\Big(\int_{\widehat{X}}\frac{|\check{f}|^2}{\check{h}_1}-\sum_{i<n}|a_i|^2\Big)^{1/2}.$$

And if $x_0,x_1,\ldots$ are such that there exists a $\delta>0$ such that

$$|a_n|\ge\delta\sup_{x\in X}|f(x)-f_n(x)|$$

and $\int(|\check{f}|^2/\check{h}_1)<\infty$, then $f_n\to f$ uniformly in X. (In the above formulas $|\check{f}(s)|^2/\check{h}_1(s)$ is interpreted as 0 whenever $\check{f}(s)=0$, even in the case when $\check{h}_1(s)=0$.)

If X is the circle group $\mathbb{R}/\mathbb{Z}$, then a natural choice of h_1 is the function

$$h_1(x)=\sum_{k\in\mathbb{Z}}e^{-(x-k)^2/a^2}\Big/\sum_{k\in\mathbb{Z}}e^{-k^2/a^2},$$

where $a>0$ is some constant. Notice that, as $a\to 0$, the space of functions f for which $\sum_{k\in\mathbb{Z}}|\check{f}(k)|^2/\check{h}_1(k)<\infty$ becomes larger.

PROBLEM 5. Extend the above result to the case of some classical non-abelian groups, e.g., the group $SO(n,\mathbb{R})$.

Remark 1. The above example (C) is related to Wiener's Tauberian Theorem which can be stated as follows. *If $h_1\in L^1(X)$ and the Fourier transform $\widehat{h}_1$ has no zeros in $\widehat{X}$, then the set of linear combinations*

$$\sum_{i<n}a_ih_1(x-x_i)\qquad(a_i\in\mathbb{C},\ x_i\in X)$$

is dense in $L^1(X)$ (see [9 or 12]). For $h_1\in A(X)$, our assumption $\check{h}_1\ge 0$ is stronger, but it yields the above quantitative estimate of the uniform approximation.

EXAMPLE (D). Let $X=\{z\in\mathbb{C}:|z|<1\}$ and H be the Hilbert space of holomorphic functions $\varphi: X\to\mathbb{C}$ for which the norm

$$\|\varphi\|=\Big(\iint_{\Delta}|\varphi(u+vi)|^2\,du\,dv\Big)^{1/2}$$

is finite, where $\Delta=\{(u,v)\in\mathbb{R}^2: u^2+v^2<1\}$. Let

$$h(x,y)=\Big(\frac{1-|y|^2}{1-x\overline{y}}\Big)^2.$$

Then, for $f\in H$, $\mathcal{A}$ yields

$$|f(x)-f_n(x)|\le\frac{1}{\sqrt{\pi}\,(1-|x|^2)}\Big(\|f\|^2-\pi\sum_{j<n}|a_j|^2(1-|x_j|^2)\Big)^{1/2}.$$

And, if $x_0,x_1,\ldots$ are such that

$$(1-|x_n|^2)|a_n|\ge\delta\sup_{x\in X}(1-|x|^2)|f(x)-f_n(x)|,$$

where δ is a positive constant, then

$$(1-|x|^2)|f(x)-f_n(x)|\to 0\quad\text{uniformly in } X.$$

Remark 2. Here h was obtained from Bergman's reproducing kernel

$$k(x,y)=\frac{1}{\pi(1-x\overline{y})^2}$$

for the unit disk (see [10]) by the formula (h). As we shall see in the main theorem this generalizes to all reproducing kernels, and beyond.

EXAMPLE (E). Let H be the linear space of entire functions $\varphi:\mathbb{C}\to\mathbb{C}$ such that $\|\varphi\|<\infty$, where

$$\|\varphi\|=\Big(\iint_{\mathbb{R}^2}|\varphi(x+iy)|^2p(x,y)\,dx\,dy\Big)^{1/2},$$

where p is a continuous function with $p(x,y)>0$ for all $(x,y)\in\mathbb{R}^2$. It is easy to check that H with this norm is a Hilbert space (see [1], or the proof of Proposition 3 in Chapter I, §9 of [13]). Of course p must converge to 0 fast enough as $x^2+y^2\to\infty$, if H is to contain more than the function 0. Now, it is easy to check that for every $z\in\mathbb{C}$ the evaluation functional $\varphi\mapsto\varphi(z)$ is bounded. Hence H has a reproducing kernel.

For example, if $p(x,y)=e^{-x^2-y^2}$, then H is the space of entire functions called the Fock space (see [4]), with an orthonormal basis

$$\varphi_n(z)=z^n/\sqrt{n!\pi}\qquad(n=0,1,\ldots).$$

Hence, in this case the reproducing kernel is

$$k(z,w)=\sum_{n=0}^{\infty}\varphi_n(z)\overline{\varphi_n(w)}=\frac{1}{\pi}e^{z\bar{w}},$$

and

$$h(z,w)=\frac{k(z,w)}{k(w,w)}=e^{(z-w)\bar{w}}.$$

And the algorithm $\mathcal{A}$ yields representations of entire functions $f\in H$ in the form

$$f(z)=\sum a_je^{(z-w_j)\bar{w}_j},$$

where

$$\pi\sum|a_j|^2e^{-|w_j|^2}=\|f\|^2.$$

For other examples of reproducing kernels see [10] and references therein.

Now we will state our general theorem which yields the above examples. (It improves Theorem 1 of §1 of [3].) We have two ways (a) and (b) of introducing the underlying concepts.

(a) H is complex (or real) Hilbert space, X an abstract set, and a function $k:X\to H\backslash\{0\}$ is given. We define

$$\text{(h)}\qquad h(x,y)=\frac{\langle k(y),k(x)\rangle}{\|k(y)\|^2}.$$

Hence $h(x,x)=1$ as required in the algorithm $\mathcal{A}$. [In this case k is not a function of two variables. However, we can define $k(y)(x)=k(x,y)$ by the formula (k) (at the beginning of this paper), and we can check that this function turns out to be positive definite, i.e., (p) is satisfied.]

(b) $h:X^2\to\mathbb{C}$ is given and $h(x,x)=1$ for all $x\in X$. We define $k:X^2\to\mathbb{C}$ by the formula (k) and assume that h is such that k is positive definite, i.e., (p) holds. Then we form the space $H=H_k$ with its inner product described in [10] and normalized such that $\|k_a\|=1$. [In this case it is easy to check that the formula (h) follows from the formula (k).]

Given $f:X\to\mathbb{C}$ (or $\mathbb{R}$) and $x_i\in X$ $(i=0,1,\ldots)$ we define

$$|f|_{xn}=\inf_{\varphi\in H}\{\|\varphi\|:\langle\varphi,\ k(x_i)\rangle=f(x_i)\text{ for }i<n\text{ and }\langle\varphi,k(x)\rangle=f(x)\},$$

with the understanding that $\inf\emptyset=\infty$.

We assume as above that the sequence $a_0,a_1,\ldots$ is defined by the algorithm $\mathcal{A}$.

THEOREM. *Under the assumptions $f_0=0$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and* (a) *or* (b), *for all functions $f:X\to\mathbb{C}$ we have*

$$|f(x)-f_n(x)|\le\|k(x)\|\Big(|f|_{xn}^2-\sum_{i<n}|a_i|^2\|k(x_i)\|^{-2}\Big)^{1/2}.$$

The proof will be given in Section 4.

This theorem implies immediately the following corollary.

COROLLARY. *If $x_0,x_1,\ldots$ are such that*

$$\|k(x_n)\|^{-1}|a_n|\ge\delta\sup_{x\in X}\|k(x)\|^{-1}|f(x)-f_n(x)|,$$

where δ is a positive constant, and if the set $\{|f|_{xn}:x\in X,\ n=1,2,\ldots\}$ is bounded, then $\|k(x)\|^{-1}|f(x)-f_n(x)|\to0$ uniformly on X.

Remark 3. Every function $k:X\to H$ can be interpreted as a reproducing kernel for the orthogonal complement $H_0^\perp$ of the space

$$H_0=\{\varphi\in H:\langle\varphi,k(x)\rangle=0\text{ for all }x\in X\}.$$

(H_0 is a closed linear subspace of H, hence $H_0^\perp$ is well defined.) Namely for all $\varphi\in H_0^\perp$ and $x\in X$ we define $\varphi(x)$ by the formula

$$\varphi(x)=\langle\varphi,k(x)\rangle.$$

Thus k is a reproducing kernel by definition, and it is clear that if $f\in H_0^\perp$, then $|f|_{xn}\le\|f\|$.

2. Derivation of (A), (B), (C), (D) and (E) from the Theorem

Proof of (A). Let H be the Hilbert space of absolutely continuous functions $\varphi:(0,1)\to\mathbb{C}$ satisfying the conditions $\lim_{x\uparrow1}\varphi(x)=\lim_{x\downarrow0}\varphi(x)=0$ and

$$\|\varphi\|=\Big(\int_0^1|\varphi'|^2\Big)^{1/2}<\infty.$$

(To check that H is a Hilbert space it suffices to observe that the map

$$\psi\mapsto\Big(\int_0^x\psi(t)\,dt\Big)_{x\in(0,1)}$$

is an isomorphism of the Hilbert space $\{\psi\in L^2(0,1):\int_0^1\psi=0\}$ onto H.)

Then let $(h(y))(x)=h(x,y)$ where $h(x,y)$ is defined as in (A). We put $k(y)=y(1-y)h(y)$. Then an easy calculation shows that k is a reproducing kernel, i.e., for all $\varphi\in H$, $\langle\varphi,k(y)\rangle=\varphi(y)$. And of course $|f|_{xn}\le\|f\|$, hence all assertions of (A) follow from the Theorem and the Corollary.

Proof of (B). The argument is similar to the above. (Here $\|\varphi\| = (\int_0^\infty |\varphi'|^2)^{1/2}$ and $k(y) = yh(y)$.)

Proof of (C). Again the argument is similar. (Here

$$\|\varphi\| = \Big(\int_{\hat{X}} (|\check{\varphi}|^2/\check{h}_1)\Big)^{1/2},$$

and $(k(y))(x) = (h(y))(x) = h_1(x-y)$. Hence $(h(y))\check{}(s) = \check{h}_1(s)s(y)$ for all $y \in X$, $s \in \widehat{X}$. And an easy calculation shows that k is a reproducing kernel and $\|k(y)\| = 1$ for all y.)

Proof of (D) and (E). Examples (D) and (E) are obvious specializations of the Theorem and Corollary. (Compare Remark 3.)

3. The algorithm of Kaczmarz. Let H be a Hilbert space and $\mathbf{b}_i \in H$, for $i = 0, 1, \ldots$ Consider the infinite system of linear equations

$$\langle \mathbf{x}, \mathbf{b}_i \rangle = c_i \quad (i = 0, 1, \ldots) \tag{1}$$

where $c_i \in \mathbb{C}$ and $c_i = 0$ whenever $\mathbf{b}_i = 0$. The algorithm of Kaczmarz [5] seeks an approximate solution of (1) by vectors $\mathbf{x}_n$ which are defined as follows.

$$(\mathcal{K}) \qquad \begin{cases} \mathbf{x}_0 = 0, \\ \mathbf{x}_{n+1} = (\text{the orthogonal projection of } \mathbf{x}_n \\ \qquad\qquad \text{into the hyperplane } \{\mathbf{x} \in H : \langle \mathbf{x}, \mathbf{b}_n \rangle = c_n\}) \\ \qquad\quad = \mathbf{x}_n + (c_n - \langle \mathbf{x}_n, \mathbf{b}_n \rangle)\mathbf{b}_n \|\mathbf{b}_n\|^{-2}, \end{cases}$$

where $\mathbf{b}_n \|\mathbf{b}_n\|^{-2}$ is interpreted as 0 if $\mathbf{b}_n = 0$. (This algorithm has many applications, see the survey [2].)

We need the following lemmas about the recursion $(\mathcal{K})$.

LEMMA 1. *If* $\langle \mathbf{x}^*, \mathbf{b}_i \rangle = c_i$ *for* $i \le n$, *then*

$$(\alpha) \qquad \|\mathbf{x}^* - \mathbf{x}_{i+1}\|^2 + \|\mathbf{x}_{i+1} - \mathbf{x}_i\|^2 = \|\mathbf{x}^* - \mathbf{x}_i\|^2 \quad \text{for } i = 0, \ldots, n,$$

$$(\beta) \qquad \|\mathbf{x}^*\|^2 - \|\mathbf{x}^* - \mathbf{x}_{n+1}\|^2 = \sum_{i=0}^{n} \|\mathbf{x}_{i+1} - \mathbf{x}_i\|^2 = \sum_{i=0}^{n} |c_i - \langle \mathbf{x}_i, \mathbf{b}_i \rangle|^2 \|\mathbf{b}_i\|^{-2}.$$

Proof. (α) is the Pythagorean theorem in H and (β) follows from (α). ∎

LEMMA 2. *If* $\langle \mathbf{x}^*, \mathbf{b}_i \rangle = c_i$ *for* $i \le n$, *then*

$$|c_n - \langle \mathbf{x}_n, \mathbf{b}_n \rangle| \le \|\mathbf{b}_n\| \Big(\|\mathbf{x}^*\|^2 - \sum_{i=0}^{n-1} |c_i - \langle \mathbf{x}_i, \mathbf{b}_i \rangle|^2 \|\mathbf{b}_i\|^{-2} \Big)^{1/2}.$$

Proof. By omitting in Lemma 1(β) the term $-\|\mathbf{x}^* - \mathbf{x}_{n+1}\|^2$ and rearranging the other terms.

Remark. In the above lemmas, the term $|c_i - \langle \mathbf{x}_i, \mathbf{b}_i \rangle|^2 \|\mathbf{b}_i\|^{-2}$ is interpreted as 0 whenever $\mathbf{b}_i = 0$.

4. Proof of the Theorem. We substitute in Lemma 2:

(1) $f(x_i)$ for c_i for $i < n$ and $f(x)$ for c_n.
(2) f_i for $\mathbf{x}_i$ for $i \le n$.
(3) $k(x_i)$ for $\mathbf{b}_i$ for $i < n$ and $k(x)$ for $\mathbf{b}_n$.
(4) Any $\varphi \in H$ such that $\langle \varphi, k(x_i) \rangle = f(x_i)$ for $i < n$, and $\langle \varphi, h(x) \rangle = f(x)$, for $\mathbf{x}^*$. (If no such φ exists, then the Theorem is trivially true.)

Then, by the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ of the Theorem, the assumption $(\mathcal{K})$ of Lemma 2 is true. By $(\mathcal{A}_2)$, $c_i - \langle \mathbf{x}_i, \mathbf{b}_i \rangle = a_i$. And, since φ was arbitrary such that (4) holds, Lemma 2 yields the Theorem.

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Trace and determinant in Banach algebras

by

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Abstract. We show that the trace and the determinant on a semisimple Banach algebra can be defined in a purely spectral and analytic way and then we obtain many consequences from these new definitions.

1. Introduction. Determinants of infinite matrices were for the first time investigated by the astronomer G. W. Hill in his studies on lunar theory and his ideas were put into a rigorous form by H. Poincaré in 1886. Ten years later H. von Koch refined and generalized Poincaré's results. In 1903, I. Fredholm developed a determinant theory for integral operators. Unlike von Koch, I. Fredholm studied eigenvalues and looked at the analyticity of $\det(I+\lambda M)$. Fredholm's determinant theory is certainly one of the first milestones in the history of functional analysis. In the early fifties A. F. Ruston, T. Leżański and A. Grothendieck almost simultaneously defined determinants for nuclear or integral operators on a Banach space. In the seventies, A. Pietsch developed an axiomatic approach to the determinant of elements of certain operator ideals. In 1978, J. Puhl [16] studied the trace on the socle and nuclear elements of a semisimple Banach algebra, basing his difficult arguments on the standard trace defined for finite-rank linear operators. For more historical information and references on this matter look at [11], Chapters 4 and 5, and [15], 7.5 and 7.6.

The aim of this paper is to show that the trace and determinant on the socle of a Banach algebra can be developed in a purely spectral and analytic way, that is to say internally, without using operators on the algebra. Then we use the analytic properties of the spectrum to prove that the trace and determinant are entire functions and to deduce the basic properties of the trace and determinant in a purely analytic way. The essential ingredient in all these arguments is the fact that the spectrum is an analytic multifunction. So this point of view gives us the possibility of extending almost all the

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