STUDIA MATHEMATICA 121 (1) (1996)

Hankel convolution on distribution spaces with exponential growth

by

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Abstract. We study the Hankel transformation and Hankel convolution on spaces of distributions with exponential growth.

1. Introduction. The Hankel integral transformation is usually defined by

$$h_\mu(\varphi)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\varphi(x)\,dx, \quad x \in (0,\infty).$$

Here J_μ is the Bessel function of the first kind and order μ. Throughout this paper the order μ will always be greater than $-1/2$. Also in the sequel we will denote by I the real interval $(0,\infty)$. A. H. Zemanian [21]–[23] investigated the h_μ transformation on certain spaces of distributions. He introduced in [21] the space $\mathcal{H}_\mu$ that consists of all complex-valued functions $\varphi = \varphi(x)$, $x \in I$, such that

$$\lambda^\mu_{k,m}(\varphi) = \sup_{x\in(0,\infty)} \left| x^k \left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x))\right| < \infty$$

for every $k, m \in \mathbb{N}$. When endowed with the topology generated by the family $\{\lambda^\mu_{k,m}\}_{k,m\in\mathbb{N}}$ of seminorms, $\mathcal{H}_\mu$ becomes a Fréchet space and the Hankel transformation is an isomorphism from $\mathcal{H}_\mu$ onto itself. The dual space of $\mathcal{H}_\mu$ is denoted by $\mathcal{H}'_\mu$.

For every $a \in I$, A. H. Zemanian [22] defined the subspace $\beta_{\mu,a}$ of $\mathcal{H}_\mu$ consisting of $\varphi \in \mathcal{H}_\mu$ such that $\varphi(x) = 0$ for $x \geq a$. The space $\beta_\mu = \bigcup_{a>0} \beta_{\mu,a}$ endowed with the inductive topology is a dense subspace of

1991 *Mathematics Subject Classification*: Primary 46F12.
Key words and phrases: Hankel transform, Hankel convolution, distribution, Bessel functions.
Research partially supported by Consejera de Educación, Gobierno Autónomo de Canarias, Proyecto 967/15-9-95 and by DGICYT Grant PB 95-0591 (Spain).

$\mathcal{H}_\mu$ and the Hankel transform on β_μ is characterized in [22, Theorem 1]. As usual the dual of β_μ is denoted by β'_μ.

F. M. Cholewinski [9], I. I. Hirschman [13] and D. T. Haimo [11] studied a convolution for a variant of the Hankel transformation that, after straightforward manipulations, allows defining a convolution for the h_μ transformation. A measurable function φ on I is said to be in L_μ if, and only if, $x^{\mu+1/2}\varphi$ is absolutely integrable on I. For every $\varphi,\psi\in L_\mu$ we define the *Hankel convolution* $\varphi\#\psi$ of φ and ψ by

$$(\varphi\#\psi)(x)=\int_0^\infty \varphi(y)(\tau_x\psi)(y)\,dy,\qquad x\in I,$$

where $(\tau_x\psi)(y)=\int_0^\infty D_\mu(x,y,z)\psi(z)\,dz$, $x,y\in I$, and

$$D_\mu(x,y,z)=\int_0^\infty t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)\,dt,\qquad x,y,z\in I.$$

The operator τ_x, $x\in I$, is called the *Hankel translation*.

In a series of papers J. J. Betancor and I. Marrero [3]–[8] and J. J. Betancor and B. J. González [2] have investigated the Hankel convolution on the Zemanian spaces $\mathcal{H}'_\mu$ and β'_μ of generalized functions.

The Fourier transform of distributions of exponential growth had been investigated earlier (see [12] and [24], among others). However, the Hankel transformation has not been defined on distributions of exponential growth. In this paper we investigate the Hankel transformation and Hankel convolution on distributions of exponential growth. In Section 2 we introduce two Fréchet function spaces, namely: $\mathcal{X}_\mu$ of all smooth complex-valued functions $\varphi=\varphi(x)$, $x\in I$, such that $e^{kx}\left(\frac{1}{x}D\right)^m(x^{-\mu-1/2}\varphi(x))$ is bounded on I for every $k,m\in\mathbb{N}$; and $\mathcal{Q}_\mu$ of all complex-valued functions Φ such that $x^{-\mu-1/2}\Phi(x)$ is an even entire function rapidly decreasing in any horizontal strip. It is established that h_μ is an isomorphism from $\mathcal{X}_\mu$ onto $\mathcal{Q}_\mu$. The Hankel convolution of distributions in $\mathcal{X}'_\mu$, the dual space of $\mathcal{X}_\mu$, is studied in Section 3.

The following boundedness properties of Bessel functions that can be found in [19] will be very useful in the sequel.

There exists $C>0$ such that

$$|z^{-\mu}J_\mu(z)|\le Ce^{|\operatorname{Im} z|},\qquad z\in\mathbb{C}. \tag{1}$$

If $H_\mu^{(1)}$ denotes the Hankel function of the first kind and order μ then there exists $C>0$ such that

$$|z^{1/2}H_\mu^{(1)}(z)|\le Ce^{-\operatorname{Im} z},\qquad z\in\mathbb{C},\ |z|\ge 1. \tag{2}$$

Throughout this paper C will always denote a suitable positive constant not necessarily the same at each occurrence.

2. The spaces $\mathcal{X}_\mu$ and $\mathcal{Q}_\mu$ and the Hankel transformation. In this section we introduce new function spaces that the Hankel transformation maps isomorphically.

The space $\mathcal{X}_\mu$ is formed by all smooth complex-valued functions $\varphi(x)$, $x\in I$, for which

$$\gamma^\mu_{k,m}(\varphi)=\sup_{x\in I}\left|e^{kx}\left(\frac{1}{x}D\right)^m(x^{-\mu-1/2}\varphi(x))\right|<\infty,$$

for every $k,m\in\mathbb{N}$. When endowed with the topology generated by the family $\{\gamma^\mu_{k,m}\}_{k,m\in\mathbb{N}}$ of seminorms, $\mathcal{X}_\mu$ becomes a Fréchet space. Moreover, according to [17, Proposition 4.1.5], $\mathcal{X}_\mu$ is nuclear. Proceeding as in [5, Lemma 2.2] we can see that the seminorms

$$\eta^\mu_{k,m}(\varphi)=\sup_{x\in I}|e^{kx}x^{-\mu-1/2}S_\mu^m\varphi(x)|,\qquad \varphi\in\mathcal{X}_\mu,\ k,m\in\mathbb{N},$$

where S_μ denotes the Bessel operator $x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$, induce on $\mathcal{X}_\mu$ the same topology as that defined by $\{\gamma^\mu_{k,m}\}_{k,m\in\mathbb{N}}$. Note that $\mathcal{X}_\mu$ is continuously contained in $\mathcal{H}_\mu$.

E. L. Koh and C. K. Li [14] and E. L. Koh and A. H. Zemanian [15] have defined the Hankel transformation on certain spaces of generalized functions. It is easy to see that the Fréchet function spaces introduced in [14] and [15] contain $\mathcal{X}_\mu$.

We denote by $\theta_{\mathcal{X}}$ the space of multipliers of $\mathcal{X}_\mu$, that is, a function f is in $\theta_{\mathcal{X}}$ whenever $f\varphi$ is in $\mathcal{X}_\mu$ for every $\varphi\in\mathcal{X}_\mu$. A procedure similar to the one used in [3] and [20] allows one to prove that $f\in\theta_{\mathcal{X}}$ if, and only if,

(i) f is smooth on I, and
(ii) for every $m\in\mathbb{N}$ there exist $k\in\mathbb{N}$ and $C>0$ such that

$$\left|\left(\frac{1}{x}D\right)^m f(x)\right|\le Ce^{kx},\qquad x\in I.$$

The dual space of $\mathcal{X}_\mu$ will be denoted as usual by $\mathcal{X}'_\mu$. $\theta_{\mathcal{X}}$ is also the space of multipliers in $\mathcal{X}'_\mu$. By using standard techniques (see, for example, [1], [5] and [18]) it is not hard to establish that a functional T on $\mathcal{X}_\mu$ is in $\mathcal{X}'_\mu$ if, and only if, there exist $r\in\mathbb{N}$ and essentially bounded functions f_k on I, $0\le k\le r$, such that

$$T=\sum_{k=0}^r S_\mu^k(e^{rx}x^{-\mu-1/2}f_k).$$

The space $\mathcal{Q}_\mu$ consists of all complex-valued functions Φ such that

(i) $z^{-\mu-1/2}\Phi(z)$ is an even entire function, and
(ii) for every $k, m \in \mathbb{N}$,

$$w^{\mu}_{k,m}(\Phi) = \sup_{|\mathrm{Im}\, z|\le k} (1+|z|^2)^m |z^{-\mu-1/2}\Phi(z)| < \infty.$$

When endowed with the topology generated by the family $\{w^{\mu}_{k,m}\}_{k,m\in\mathbb{N}}$ of norms, $\mathcal{Q}_\mu$ is a nuclear Fréchet space. The dual of $\mathcal{Q}_\mu$ is denoted by $\mathcal{Q}'_\mu$.

We also introduce the space $\theta_{\mathcal{Q}}$ of all complex-valued, even and entire functions F such that for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ for which

$$\sup_{|\mathrm{Im}\, z|\le k} (1+|z|^2)^{-m}|F(z)| < \infty.$$

It is clear that F is a multiplier of $\mathcal{Q}_\mu$ whenever $F \in \theta_{\mathcal{Q}}$.

We now establish that the Hankel transformation maps $\mathcal{X}_\mu$ onto $\mathcal{Q}_\mu$ homeomorphically.

THEOREM 2.1. *The Hankel transformation h_μ is an isomorphism from $\mathcal{X}_\mu$ onto $\mathcal{Q}_\mu$. Moreover, the inverse of h_μ is also h_μ.*

Proof. Let φ be in $\mathcal{X}_\mu$. Define $\Phi = h_\mu(\varphi)$. By (1), for every $k \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty |(xz)^{-\mu}J_\mu(xz)|x^{\mu+1/2}|\varphi(x)|\,dx &\le C\int_0^\infty e^{x|\mathrm{Im}\, z|}x^{\mu+1/2}|\varphi(x)|\,dx\\ &\le C\sup_{x\in I}|e^{(k+1)x}x^{-\mu-1/2}\varphi(x)| \quad \text{if } |\mathrm{Im}\, z|\le k.\end{aligned}$$

Hence $z^{-\mu-1/2}\Phi(z)$ is an even entire function. Also, since $\mathcal{H}_\mu$ contains $\mathcal{X}_\mu$, by [23, Lemma 5.4-1] we can write for every $k, m \in \mathbb{N}$,

$$\sup_{|\mathrm{Im}\, z|\le k}(1+|z|^2)^m|z^{-\mu-1/2}\Phi(z)| \le C\{\eta^{\mu}_{k+1,0}(\varphi)+\eta^{\mu}_{k+1,m}(\varphi)\}.$$

Therefore h_μ is a continuous mapping from $\mathcal{X}_\mu$ into $\mathcal{Q}_\mu$.

Let now $\Phi \in \mathcal{Q}_\mu$. Since Φ is absolutely integrable in $(0,\infty)$ we can define

$$\varphi(x) = \int_0^\infty (xy)^{1/2}J_\mu(xy)\Phi(y)\,dy, \quad x \in I,$$

and the integral is absolutely convergent for every $x \in I$. Also according to well-known properties of Bessel functions [23, 5.1(7)], for every $m \in \mathbb{N}$ and $x \in I$ one has

$$\text{(3)}\quad \left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x)) = (-1)^m\int_0^\infty (xy)^{-\mu-m}J_{\mu+m}(xy)y^{2m+\mu+1/2}\Phi(y)\,dy.$$

The integral in (3) is again absolutely convergent for each $x \in I$. Proceeding as in [10, proof of Lemma 6.1] we can conclude from (3) that

$$\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x)) = \frac{(-1)^m}{2}\int_{-\infty}^{\infty}(x(\xi+i\eta))^{-\mu-m}H^{(1)}_{\mu+m}(x(\xi+i\eta)) \times (\xi+i\eta)^{2m+\mu+1/2}\Phi(\xi+i\eta)\,d\xi$$

for every $x > 1$, $\eta > 0$ and $m \in \mathbb{N}$. Here $H^{(1)}_\mu$ denotes the Hankel function of the first kind and order μ. Since $|x(\xi+i\eta)| > \eta$ for $x > 1$, (2) yields

$$\text{(4)}\quad \left|\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x))\right| \le Ce^{-\eta x}\int_{-\infty}^{\infty}|\xi+i\eta|^m|\Phi(\xi+i\eta)|\,d\xi$$

for every $x > 1$, $\eta > 0$ and $m \in \mathbb{N}$, where the positive constant C depends on η.

Now we choose $l \in \mathbb{N}$ such that $l > \mu + 3/2$. Then from (4) we deduce that for every $k, m \in \mathbb{N}$,

$$\begin{aligned}\text{(5)}\quad &\left|e^{kx}\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x))\right|\\ &\le Ce^{-x}\int_{-\infty}^{\infty}|\xi+i(k+1)|^m|\Phi(\xi+i(k+1))|\,d\xi\\ &\le C\int_{-\infty}^{\infty}|\xi+i(k+1)|^{\mu+1/2-l}|\xi+i(k+1)|^{m+l}\\ &\quad\times|(\xi+i(k+1))^{-\mu-1/2}\Phi(\xi+i(k+1))|\,d\xi\\ &\le Cw^{\mu}_{k+1,m+l}(\Phi) \quad \text{for every } x > 1.\end{aligned}$$

Moreover, for every $x \in (0,1)$ and $k, m \in \mathbb{N}$ we have

$$\begin{aligned}\text{(6)}\quad &\left|e^{kx}\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\varphi(x))\right|\\ &\le e^k\int_0^\infty |(xy)^{-\mu-m}J_{\mu+m}(xy)|\cdot|y^{2m+\mu+1/2}\Phi(y)|\,dy \le Cw^{\mu}_{1,m+n}(\Phi),\end{aligned}$$

where $n \in \mathbb{N}$ and $n > \mu+1$. By combining (5) and (6) we conclude that h_μ is a continuous mapping from $\mathcal{Q}_\mu$ into $\mathcal{X}_\mu$.

Finally, since $\mathcal{X}_\mu$ is contained in $\mathcal{H}_\mu$ it follows that $h_\mu = h_\mu^{-1}$. ∎

An immediate consequence of [23, Theorem 5.4-1] and Theorem 2.1 is

COROLLARY 2.1. *The space $\mathcal{Q}_\mu$ is continuously contained in $\mathcal{H}_\mu$.* ∎

We define the generalized Hankel transformation between $\mathcal{X}'_\mu$ and $\mathcal{Q}'_\mu$ to be the transpose of the h_μ transformation, that is, the Hankel transform $h'_\mu(T)$ of $T \in \mathcal{X}'_\mu$ (resp. $\mathcal{Q}'_\mu$) is the element of $\mathcal{Q}'_\mu$ (resp. $\mathcal{X}'_\mu$) defined by

$$\langle h'_\mu(T), \varphi\rangle = \langle T, h_\mu(\varphi)\rangle, \quad \varphi \in \mathcal{Q}_\mu \text{ (resp. } \mathcal{X}_\mu).$$

The behaviour of h'_μ on $\mathcal{X}'_\mu$ and $\mathcal{Q}'_\mu$ is deduced from Theorem 2.1.

THEOREM 2.2. *The generalized Hankel transformation is an isomorphism from $\mathcal{X}'_\mu$ (resp. $\mathcal{Q}'_\mu$) onto $\mathcal{Q}'_\mu$ (resp. $\mathcal{X}'_\mu$) when we consider on $\mathcal{X}'_\mu$ and $\mathcal{Q}'_\mu$ the weak $*$ or strong topology.* ∎

3. Hankel convolution on $\mathcal{X}_\mu$ and $\mathcal{X}'_\mu$. We first investigate the behaviour of the Hankel translation and Hankel convolution in $\mathcal{X}_\mu$.

PROPOSITION 3.1. *For every $x \in I$ the Hankel translation τ_x defines a continuous linear mapping from $\mathcal{X}_\mu$ into itself.*

Proof. Let $x \in I$. By [16, (2.1)], since $\mathcal{X}_\mu$ is contained in $\mathcal{H}_\mu$ we have for each $\varphi \in \mathcal{X}_\mu$,

$$(\tau_x\varphi)(y) = h_\mu[t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)h_\mu(\varphi)(t)](y), \quad y \in I. \tag{7}$$

By Theorem 2.1 to see that τ_x is continuous from $\mathcal{X}_\mu$ into itself it is sufficient to prove that the function $\Phi_x(t) = (xt)^{-\mu}J_\mu(xt)$, $t \in \mathbb{C}$, is a multiplier in $\mathcal{Q}_\mu$. Note first that Φ_x is an even entire function. Moreover, from (1) it follows that

$$|\Phi_x(t)| \le Ce^{x|\operatorname{Im} t|}, \quad t \in \mathbb{C}.$$

Thus $\Phi_x \in \theta_{\mathcal{Q}}$ and the proof is finished. ∎

PROPOSITION 3.2. *The Hankel convolution defines a continuous linear mapping from $\mathcal{X}_\mu \times \mathcal{X}_\mu$ into $\mathcal{X}_\mu$.*

Proof. It is easy to infer from [13, Theorem 2.d] that for every $\varphi, \psi \in \mathcal{X}_\mu$ the interchange formula

$$h_\mu(\varphi \# \psi) = y^{-\mu-1/2}h_\mu(\varphi)h_\mu(\psi) \tag{8}$$

holds. Moreover, the mapping $(\Phi, \Psi) \mapsto y^{-\mu-1/2}\Phi\Psi$ is continuous from $\mathcal{Q}_\mu \times \mathcal{Q}_\mu$ into $\mathcal{Q}_\mu$. Then Theorem 2.1 yields the desired result. ∎

Proposition 3.1 allows us to define the Hankel convolution $T \# \varphi$ of $T \in \mathcal{X}'_\mu$ and $\varphi \in \mathcal{X}_\mu$ as follows:

$$(T \# \varphi)(x) = \langle T, \tau_x\varphi\rangle, \quad x \in I.$$

Note that we cannot insure that $T \# \varphi \in \mathcal{X}_\mu$. In fact, define

$$\langle T, \varphi\rangle = \int_0^\infty x^{\mu+1/2}\varphi(x)\,dx, \quad \varphi \in \mathcal{X}_\mu.$$

It is clear that $T \in \mathcal{X}'_\mu$. Let $\varphi \in \mathcal{X}_\mu$. By invoking [13, §2, (2)], we have

$$\begin{aligned}\langle T, \tau_x\varphi\rangle &= \int_0^\infty z^{\mu+1/2}\int_0^\infty D_\mu(x,y,z)\varphi(y)\,dy\,dz \\ &= \int_0^\infty \varphi(y)\int_0^\infty D_\mu(x,y,z)z^{\mu+1/2}\,dz\,dy \\ &= c_\mu^{-1}x^{\mu+1/2}\int_0^\infty y^{\mu+1/2}\varphi(y)\,dy, \quad x \in I,\end{aligned}$$

where $c_\mu = 2^\mu\Gamma(\mu+1)$. Hence $x^{-\mu-1/2}\langle T, \tau_x\varphi\rangle = c_\mu^{-1}\int_0^\infty y^{\mu+1/2}\varphi(y)\,dy$, $x \in I$, and

$$e^x x^{-\mu-1/2}\langle T, \tau_x\varphi\rangle \to \infty \quad \text{as } x \to \infty.$$

Thus $T \# \varphi \notin \mathcal{X}_\mu$.

The next proposition states that for every $T \in \mathcal{X}'_\mu$ and $\varphi \in \mathcal{X}_\mu$, $x^{-\mu-1/2}T \# \varphi$ is a multiplier of $\mathcal{X}_\mu$.

PROPOSITION 3.3. *If $T \in \mathcal{X}'_\mu$ and $\varphi \in \mathcal{X}_\mu$ then $x^{-\mu-1/2}T \# \varphi \in \theta_{\mathcal{X}}$.*

Proof. As established in Section 2 there exist $r \in \mathbb{N}$ and essentially bounded functions f_k on I, $0 \le k \le r$, such that

$$T = \sum_{k=0}^r S_\mu^k(e^{rx}x^{-\mu-1/2}f_k).$$

Hence it is sufficient to prove the result for

$$T = S_\mu^k(e^{rx}x^{-\mu-1/2}f),$$

where f is an essentially bounded function on I and $r, k \in \mathbb{N}$.

Let $\varphi \in \mathcal{X}_\mu$. By [16, Proposition 2.1(ii)] and (7) we have

$$\begin{aligned}(T \# \varphi)(x) = \langle T, \tau_x\varphi\rangle &= \int_0^\infty f(y)e^{ry}y^{-\mu-1/2}\tau_x(S_\mu^k\varphi)(y)\,dy \\ &= (-1)^k x^{\mu+1/2}\int_0^\infty f(y)e^{ry}y^{-\mu-1/2} \\ &\quad \times h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\varphi)(t)t^{2k}](y)\,dy, \quad x \in I.\end{aligned}$$

Let now $n \in \mathbb{N}$. By using [21, 5.1(7)] it follows that

$$\left(\frac{1}{x}D\right)^n (x^{-\mu-1/2}(T \# \varphi)(x))$$
$$= (-1)^{n+k} \int_0^\infty f(y) e^{ry} y^{-\mu-1/2}$$
$$\times h_\mu[t^{2(n+k)}(xt)^{-\mu-n} J_{\mu+n}(xt) h_\mu(\varphi)(t)](y)\, dy, \quad x \in I.$$

Then for every $x \in I$,

$$\left|\left(\frac{1}{x}D\right)^n (x^{-\mu-1/2}(T \# \varphi)(x))\right|$$
$$\le C \sup_{y \in I} |e^{(r+1)y} y^{-\mu-1/2} h_\mu[t^{2(n+k)}(xt)^{-\mu-n} J_{\mu+n}(xt) h_\mu(\varphi)(t)](y)|.$$

By Theorem 2.1 and since $z^{-\mu} J_\mu(z)$ belongs to $\theta_{\mathcal{Q}}$, there exist $l, m \in \mathbb{N}$ such that

$$(9) \quad \left|\left(\frac{1}{x}D\right)^n (x^{-\mu-1/2}(T \# \varphi)(x))\right|$$
$$\le C w_{l,m}^\mu (t^{2(k+n)}(xt)^{-\mu-n} J_{\mu+n}(xt) h_\mu(\varphi)(t)), \quad x \in I.$$

Moreover, from (1) one infers that

$$(10) \quad w_{l,m}^\mu ((xt)^{-\mu-n} J_{\mu+n}(xt) t^{2(k+n)} h_\mu(\varphi)(t))$$
$$\le C w_{l,m+k+n}^\mu (h_\mu(\varphi)) e^{xl}, \quad x \in I,$$

where C is independent of $x \in I$.

Finally, according to Theorem 2.1 again and by (9) and (10) we conclude that

$$\left|\left(\frac{1}{x}D\right)^n (x^{-\mu-1/2}(T \# \varphi)(x))\right| \le C e^{xl}, \quad x \in I.$$

Hence $x^{-\mu-1/2}(T \# \varphi)(x) \in \theta_{\mathcal{X}}$ and the proof is complete. ∎

According to Proposition 3.3, if $T \in \mathcal{X}'_\mu$ and $\varphi \in \mathcal{X}_\mu$ then $T \# \varphi$ defines an element of $\mathcal{X}'_\mu$ by

$$\langle T \# \varphi, \psi \rangle = \int_0^\infty (T \# \varphi)(x) \psi(x)\, dx, \quad \psi \in \mathcal{X}_\mu.$$

PROPOSITION 3.4. *If $T \in \mathcal{X}'_\mu$ and $\varphi \in \mathcal{X}_\mu$ then*

$$(11) \quad \langle T \# \varphi, \psi \rangle = \langle T, \varphi \# \psi \rangle, \quad \psi \in \mathcal{X}_\mu,$$

and the interchange formula

$$(12) \quad h'_\mu(T \# \varphi) = x^{-\mu-1/2} h'_\mu(T) h_\mu(\varphi)$$

holds. Moreover, for every $T \in \mathcal{X}'_\mu$ the mapping $\varphi \mapsto T \# \varphi$ is continuous from $\mathcal{X}_\mu$ into $\mathcal{X}'_\mu$ when we consider on $\mathcal{X}'_\mu$ the strong topology.

Proof. Let $\psi \in \mathcal{X}_\mu$. We have

$$\langle T \# \varphi, \psi \rangle = \int_0^\infty (T \# \varphi)(x) \psi(x)\, dx = \int_0^\infty \langle T, \tau_x \varphi \rangle \psi(x)\, dx.$$

Hence, since $(\tau_x \varphi)(y) = (\tau_y \varphi)(x)$, $x, y \in I$, the proof of (11) is finished when we prove that

$$(13) \quad \int_0^\infty \langle T, \tau_x \varphi \rangle \psi(x)\, dx = \left\langle T(y), \int_0^\infty (\tau_x \varphi)(y) \psi(x)\, dx \right\rangle.$$

CLAIM 1.

$$(14) \quad \lim_{a \to \infty} \int_a^\infty (\tau_x \varphi)(y) \psi(x)\, dx = 0$$

and

$$(15) \quad \lim_{a \to 0} \int_0^a (\tau_x \varphi)(y) \psi(x)\, dx = 0,$$

in the sense of convergence in $\mathcal{X}_\mu$.

Proof. First we establish (14). Let $a > 0$. It is clear that

$$\int_a^\infty (\tau_x \varphi)(y) \psi(x)\, dx = (\psi_a \# \varphi)(y), \quad y \in I,$$

where

$$\psi_a(x) = \begin{cases} \psi(x), & x > a, \\ 0, & x \le a. \end{cases}$$

Moreover, by [13, Theorem 2.d],

$$h_\mu(\psi_a \# \varphi) = x^{-\mu-1/2} h_\mu(\psi_a) h_\mu(\varphi).$$

Thus, according to Theorem 2.1, $\psi_a \# \varphi \to 0$ in $\mathcal{X}_\mu$ as $a \to \infty$ if, and only if,

$$x^{-\mu-1/2} h_\mu(\psi_a) h_\mu(\varphi) \to 0 \quad \text{in } \mathcal{Q}_\mu \text{ as } a \to \infty.$$

By (1), $x^{-\mu-1/2} h_\mu(\psi_a)(x)$ is even and entire, and also for every $k \in \mathbb{N}$ one has

$$|x^{-\mu-1/2} h_\mu(\psi_a)(x)| \le \int_a^\infty |(xy)^{-\mu} J_\mu(xy) y^{\mu+1/2} \psi(y)|\, dy$$
$$\le C \int_a^\infty e^{yk} |y^{\mu+1/2} \psi(y)|\, dy, \quad |\mathrm{Im}\, x| \le k.$$

Hence

$$\lim_{a \to \infty} x^{-\mu-1/2} h_\mu(\psi_a) h_\mu(\varphi) = 0$$

in the sense of convergence in $\mathcal{Q}_\mu$. Thus (14) is proved.

In a similar way (15) can be established.

CLAIM 2. *Let* $0 < a < b < \infty$. *Then*

$$\int_a^b \langle T, \tau_x\varphi\rangle \psi(x)\,dx = \Big\langle T(y), \int_a^b (\tau_x\varphi)(y)\psi(x)\,dx\Big\rangle. \tag{16}$$

Proof. We use the Riemann sums technique. Let $m \in \mathbb{N} - \{0\}$. Define $x_n = a + n(b-a)/m$, $n = 0, 1, \ldots, m$. Linearity of T leads to

$$\int_a^b \langle T, \tau_x\varphi\rangle \psi(x)\,dx = \lim_{m\to\infty} \Big\langle T(y), \frac{b-a}{m}\sum_{n=1}^m (\tau_y\varphi)(x_n)\psi(x_n)\Big\rangle.$$

Hence to establish our claim it is sufficient to see that

$$\lim_{m\to\infty} \frac{b-a}{m}\sum_{n=1}^m (\tau_y\varphi)(x_n)\psi(x_n) = \int_a^b (\tau_y\varphi)(x)\psi(x)\,dx \tag{17}$$

in the sense of convergence in $\mathcal{X}_\mu$. Moreover, according to (7), [13, Theorem 2.d] and Theorem 2.1, (17) is equivalent to

$$\lim_{m\to\infty} \frac{b-a}{m}\sum_{n=1}^m t^{-\mu-1/2} h_\mu(\varphi)(t)(tx_n)^{1/2} J_\mu(tx_n)\psi(x_n) = t^{-\mu-1/2} h_\mu(\varphi)(t) h_\mu(\psi_{a,b})(t) \tag{18}$$

in the sense of convergence in $\mathcal{Q}_\mu$, where

$$\psi_{a,b}(t) = \begin{cases} \psi(t), & t \in (a,b),\\ 0, & t \notin (a,b).\end{cases}$$

We now prove (18). Let $l, k \in \mathbb{N}$. From (1) we deduce

$$\begin{aligned} I_m(t) = (1+|t^2|)^l \Big| t^{-\mu-1/2} h_\mu(\varphi)(t) \Big(&\frac{b-a}{m}\sum_{n=1}^m (tx_n)^{-\mu} J_\mu(tx_n) x_n^{\mu+1/2}\psi(x_n) \\ &- \int_a^b (tx)^{-\mu} J_\mu(tx) x^{\mu+1/2}\psi(x)\,dx\Big)\Big| \\ \le C(1+|t|^2)^l |h_\mu(\varphi)(t) t^{-\mu-1/2}|,& \quad |\mathrm{Im}\, t| \le k, \end{aligned}$$

where C is independent of $m \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $R > 0$ such that if $|\mathrm{Re}\, t| > R$ and $|\mathrm{Im}\, t| \le k$ then

$$I_m(t) < \varepsilon \quad \text{for every } m \in \mathbb{N}. \tag{19}$$

Moreover, there exists $m_0 \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ with $m \ge m_0$,

$$\Big| \frac{b-a}{m}\sum_{n=1}^m (tx_n)^{-\mu} J_\mu(tx_n) x_n^{\mu+1/2}\psi(x_n) - \int_a^b (tx)^{-\mu} J_\mu(tx) x^{\mu+1/2}\psi(x)\,dx\Big| < \varepsilon \tag{20}$$

for $|\mathrm{Re}\, t| \le R$ and $|\mathrm{Im}\, t| \le k$. Combining (19) and (20) gives (18). Thus our claim is proved.

We now show (13). For every $0 < a < b < \infty$, by Claims 1 and 2,

$$\begin{aligned} \int_0^\infty &\langle T, \tau_x\varphi\rangle\psi(x)\,dx - \Big\langle T(y), \int_0^\infty (\tau_x\varphi)(y)\psi(x)\,dx\Big\rangle \\ &= \int_0^a \langle T, \tau_x\varphi\rangle\psi(x)\,dx + \int_a^b \langle T, \tau_x\varphi\rangle\psi(x)\,dx + \int_b^\infty \langle T, \tau_x\varphi\rangle\psi(x)\,dx \\ &\quad - \Big\langle T(y), \int_0^a (\tau_x\varphi)(y)\psi(x)\,dx\Big\rangle - \Big\langle T(y), \int_a^b (\tau_x\varphi)(y)\psi(x)\,dx\Big\rangle \\ &\quad - \Big\langle T(y), \int_b^\infty (\tau_x\varphi)(y)\psi(x)\,dx\Big\rangle \to 0 \end{aligned}$$

as $a \to 0$ and as $b \to \infty$. Hence (13) is established.

Let now $\psi \in \mathcal{Q}_\mu$. By using (8) and (11) we obtain

$$\begin{aligned} \langle h'_\mu(T \# \varphi), \psi\rangle &= \langle T \# \varphi, h_\mu(\psi)\rangle = \langle T, \varphi \# h_\mu(\psi)\rangle \\ &= \langle h'_\mu(T), h_\mu(\varphi \# h_\mu(\psi))\rangle = \langle x^{-\mu-1/2} h'_\mu(T) h_\mu(\varphi), \psi\rangle. \end{aligned}$$

Thus (12) is proved.

Finally, by invoking Proposition 3.2 we can conclude that for each $T \in \mathcal{X}'_\mu$ the mapping $\varphi \mapsto T \# \varphi$ is continuous from $\mathcal{X}_\mu$ into $\mathcal{X}'_\mu$ when on $\mathcal{X}'_\mu$ we consider the strong topology. ■

Our next objective is to introduce a subspace, $\mathcal{X}'_{\mu,\#}$, of $\mathcal{X}'_\mu$ such that $S \# \varphi \in \mathcal{X}_\mu$ for every $S \in \mathcal{X}'_{\mu,\#}$ and $\varphi \in \mathcal{X}_\mu$. The new space $\mathcal{X}'_{\mu,\#}$ contains $\mathcal{X}_\mu$ and ε'_μ ([5]). Also we will define the Hankel convolution on $\mathcal{X}'_\mu \times \mathcal{X}'_{\mu,\#}$.

Let $m \in \mathbb{Z}$. The space $X_{\mu,m,\#}$ consists of all smooth complex-valued functions $\varphi = \varphi(x)$, $x \in I$, such that

$$\alpha^k_{m,\mu}(\varphi) = \sup_{x\in I} |e^{mx} x^{-\mu-1/2} S^k_\mu \varphi(x)| < \infty$$

for every $k \in \mathbb{N}$. When endowed with the topology generated by the system $\{\alpha^k_{m,\mu}\}_{k\in\mathbb{N}}$ of seminorms, $X_{\mu,m,\#}$ is a Fréchet space. It is clear that $\mathcal{X}_\mu$ is contained in $X_{\mu,m,\#}$. We define $\mathcal{X}_{\mu,m,\#}$ as the closure of $\mathcal{X}_\mu$ in $X_{\mu,m,\#}$. It is obvious that $\mathcal{X}_{\mu,m,\#}$ is a Fréchet space. Moreover, $\mathcal{X}_{\mu,m+1,\#}$ is continuously contained in $\mathcal{X}_{\mu,m,\#}$.

We now give a representation for elements $T \in \mathcal{X}'_{\mu,m,\#}$ on $\mathcal{X}_\mu$.

PROPOSITION 3.5. *Let $m \in \mathbb{Z}$. If $T \in \mathcal{X}'_{\mu,m,\#}$ then there exist $r \in \mathbb{N}$ and essentially bounded functions f_k on I, $k = 0, \ldots, r$, such that*

$$T = \sum_{k=0}^{r} S_\mu^k [x^{-\mu-1/2} e^{(m+2)x} f_k] \quad \text{on } \mathcal{X}_\mu.$$

Proof. Let $T \in \mathcal{X}'_{\mu,m,\#}$. There exist $n \in \mathbb{N}$ and $C > 0$ such that

$$|\langle T, \varphi\rangle| \le C \max_{0\le k\le n} \alpha^k_{m,\mu}(\varphi), \quad \varphi \in \mathcal{X}_{\mu,m,\#}. \tag{21}$$

Let $\varphi \in \mathcal{X}_\mu$ and $k \in \mathbb{N}$. Assume first that $m \in \mathbb{N}$. For every $x \in I$ one has

$$x^{-\mu-1/2} S_\mu^k \varphi(x) = \int_\infty^x D_t[t^{-\mu-1/2} S^k_{\mu,t}\varphi(t)]\, dt.$$

Then

$$\begin{aligned}
|e^{mx} x^{-\mu-1/2} S_\mu^k \varphi(x)| &\le e^{mx} \int_x^\infty |D_t[t^{-\mu-1/2} S^k_{\mu,t}\varphi(t)]|\, dt \\
&\le \int_0^\infty e^{mt} |D_t[t^{-\mu-1/2} S^k_{\mu,t}\varphi(t)]|\, dt \\
&= \int_0^1 e^{mt} t^{-2\mu-1} \Big|\int_0^t u^{\mu+1/2} S_\mu^{k+1}\varphi(u)\, du\Big|\, dt \\
&\quad + \int_1^\infty e^{mt} t^{-2\mu-1} \Big|\int_t^\infty u^{\mu+1/2} S_\mu^{k+1}\varphi(u)\, du\Big|\, dt \\
&\le \int_0^1 e^{mt} \int_0^t u^{-\mu-1/2} |S_\mu^{k+1}\varphi(u)|\, du\, dt \\
&\quad + \int_1^\infty e^{-t} t^{-2\mu-1} \int_t^\infty e^{u(m+1)} u^{\mu+1/2} |S_\mu^{k+1}\varphi(u)|\, du\, dt \\
&\le C\Big(\int_0^\infty u^{-\mu-1/2} |S_\mu^{k+1}\varphi(u)|\, du \\
&\quad + \int_0^\infty e^{(m+2)u} u^{-\mu-1/2} |S_\mu^{k+1}\varphi(u)|\, du\Big), \quad x \in I.
\end{aligned}$$

Hence

$$\alpha^k_{m,\mu}(\varphi) \le C \int_0^\infty e^{(m+2)u} u^{-\mu-1/2} |S_\mu^{k+1}\varphi(u)|\, du. \tag{22}$$

Assume now $m \in \mathbb{Z}$, $m \le -1$. For each $x \in I$ we can write

$$\begin{aligned}
|e^{mx} x^{-\mu-1/2} S_\mu^k \varphi(x)| &\le \int_0^\infty |D_t(e^{mt} t^{-\mu-1/2} S_\mu^k \varphi(t))|\, dt \\
&\le |m| \int_0^\infty e^{mt} t^{-\mu-1/2} |S_\mu^k\varphi(t)|\, dt \\
&\quad + \int_0^\infty e^{mt} |D_t[t^{-\mu-1/2} S_\mu^k\varphi(t)]|\, dt \\
&\le |m| \int_0^\infty e^{mt} t^{-\mu-1/2} |S_\mu^k\varphi(t)|\, dt \\
&\quad + \int_0^\infty e^{mt} t^{-2\mu-1} \int_0^t u^{\mu+1/2} |S_\mu^{k+1}\varphi(u)|\, du\, dt \\
&\le C\Big(\int_0^\infty e^{mt} t^{-\mu-1/2} |S_\mu^k\varphi(t)|\, dt \\
&\quad + \int_0^\infty e^{(m+1)t} t^{-\mu-1/2} |S_\mu^{k+1}\varphi(t)|\, dt\Big), \quad x \in I.
\end{aligned}$$

Then

$$\begin{aligned}
\alpha^k_{m,\mu}(\varphi) &\le C\Big(\int_0^\infty e^{mt} t^{-\mu-1/2} |S_\mu^k\varphi(t)|\, dt \\
&\quad + \int_0^\infty e^{(m+1)t} t^{-\mu-1/2} |S_\mu^{k+1}\varphi(t)|\, dt\Big).
\end{aligned} \tag{23}$$

By combining (21)–(23) we conclude that

$$|\langle T, \varphi\rangle| \le C \max_{0\le k\le r} \int_0^\infty e^{(m+2)t} t^{-\mu-1/2} |S_\mu^k\varphi(t)|\, dt, \quad \varphi \in \mathcal{X}_\mu, \tag{24}$$

for some $r \in \mathbb{N}$. The desired result can be deduced from (24) by using a standard procedure (see e.g. [1] and [18]). ∎

We denote by $\mathcal{X}_{\mu,\#}$ the space $\bigcup_{m\in\mathbb{Z}} \mathcal{X}_{\mu,m,\#}$ endowed with the inductive topology. We now characterize the elements of $\mathcal{X}'_\mu$ that belong to $\mathcal{X}'_{\mu,\#}$.

THEOREM 3.1. *Let $T \in \mathcal{X}'_\mu$. The following assertions are equivalent:*

(i) $T \in \mathcal{X}'_{\mu,\#}$.

(ii) $x^{-\mu-1/2}h'_\mu(T) \in \theta_Q$.

(iii) *For every $m \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and continuous functions f_k on I, $k = 0, \ldots, r$, such that*

$$T = \sum_{k=0}^{r} S_\mu^k f_k \tag{25}$$

and $e^{mx} f_k$ is bounded on I for every $k = 0, \ldots, r$.

(iv) *For every $m \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and bounded continuous functions f_k on I, $k = 0, \ldots, r$, such that (25) holds and $e^{mx} f_k \to 0$ as $x \to \infty$ for every $k = 0, \ldots, r$.*

(v) *For every $m \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and bounded continuous functions f_k on I, $k = 0, \ldots, r$, such that (25) holds and $e^{mx} f_k$ is absolutely integrable on I for each $k = 0, \ldots, r$.*

Proof. (i)⇒(ii). Let $T \in \mathcal{X}'_{\mu,\#}$. Then $T \in \mathcal{X}'_{\mu,m,\#}$ for every $m \in \mathbb{Z}$. Assume now $m < -2$. By Proposition 3.5 there exist $r \in \mathbb{N}$ and essentially bounded functions f_n on I, $n = 0, \ldots, r$, such that

$$T = \sum_{n=0}^{r} S_\mu^n [e^{(m+2)x} x^{-\mu-1/2} f_n] \quad \text{on } \mathcal{X}_\mu.$$

Then by setting $g_n(x) = e^{(m+2)x} x^{-\mu-1/2} f_n$, $n = 0, \ldots, r$, the Fubini theorem leads to

$$\begin{aligned}\langle h'_\mu(T), \Phi\rangle = \langle T, h_\mu(\Phi)\rangle &= \sum_{n=0}^{r} (-1)^n \int_0^\infty g_n(x) h_\mu[y^{2n}\Phi(y)](x)\, dx \\ &= \sum_{n=0}^{r} (-1)^n \int_0^\infty y^{\mu+1/2+2n}\Phi(y) \\ &\quad \times \int_0^\infty g_n(x) x^{\mu+1/2}(xy)^{-\mu} J_\mu(xy)\, dx\, dy, \quad \Phi \in \mathcal{Q}_\mu.\end{aligned}$$

Hence

$$y^{-\mu-1/2} h'_\mu(T)(y) = \sum_{n=0}^{r} (-1)^n y^{2n} \int_0^\infty g_n(x) x^{\mu+1/2}(xy)^{-\mu} J_\mu(xy)\, dx. \tag{26}$$

For every $k \in \mathbb{N}$ by choosing the representation (26) associated with $m = -k-3$ and using (1) we can write

$$|y^{-\mu-1/2} h'_\mu(T)(y)| \le C \sum_{n=0}^{r} |y|^{2n}, \quad |\mathrm{Im}\, y| \le k.$$

Therefore $y^{-\mu-1/2} h'_\mu(T)(y)$ is in θ_Q.

(ii)⇒(iii). Let $m \in \mathbb{N}$. We define $\theta = h'_\mu T$. Then for every $k \in \mathbb{N}$ there exist $C_k > 0$ and $n_k \in \mathbb{N}$ such that

$$|y^{-\mu-1/2}\theta(y)| \le C_k (1 + |y|^2)^{n_k}, \quad |\mathrm{Im}\, y| \le k.$$

Set $v(y) = (M^2 + y^2)^{-l}\theta(y)$, $|\mathrm{Im}\, y| \le m+1$. Here $M \in \mathbb{N}$ is such that $M > m+1$, and $l \in \mathbb{N}$ satisfies $l > n_{m+1} + \mu/2 + 3/4$. Thus v is absolutely integrable on I and $h'_\mu(v) = h_\mu(v)$. Hence by [23, Lemma 5.4-1], we can write

$$\begin{aligned} T = h'_\mu(\theta) &= h'_\mu((M^2 + y^2)^l v(y)) \\ &= \sum_{j=0}^{l} \binom{l}{j} (-1)^j M^{2(l-j)} S_\mu^j h_\mu(v) = \sum_{j=0}^{l} S_\mu^j f_j, \end{aligned}$$

where $f_j = \binom{l}{j}(-1)^j M^{2(l-j)} h_\mu(v)$, $j = 0, \ldots, l$. It is clear that, for every $j = 0, \ldots, l$, f_j is a continuous function on I.

To establish (iii) we have to show that $e^{mx} h_\mu(v)$ is a bounded function on I. Since v is absolutely integrable on I, $e^{mx} h_\mu(v)(x)$ is bounded on $(0,1)$. On the other hand, by using a procedure similar to the one employed in [10, Lemma 6.1] we obtain

$$\begin{aligned} h_\mu(v)(x) \\ = \frac{1}{2} x^{\mu+1/2} \int_{-\infty}^{\infty} (x(\xi + i\eta))^{-\mu} H_\mu^{(1)}(x(\xi + i\eta))(\xi + i\eta)^{\mu+1/2} v(\xi + i\eta)\, d\xi \end{aligned} \tag{27}$$

for every $x > 1$ and $0 < \eta < M$. By (2), (27) leads to

$$\begin{aligned} |e^{mx} h_\mu(v)(x)| \le C e^{-x} \int_{-\infty}^{\infty} \frac{|\xi + i(m+1)|^{\mu+1/2}}{(1 + |\xi + i(m+1)|^2)^{l-n_{m+1}}}\, d\xi \\ \times \sup_{|\mathrm{Im}\, z| \le m+1} |(1 + |z|^2)^{l-n_{m+1}} z^{-\mu-1/2} v(z)|, \quad x > 1. \end{aligned}$$

Since $l > n_{m+1} + \mu/2 + 3/4$, it follows that $e^{mx} h_\mu(v)(x)$ is bounded on $(1, \infty)$.

(iii)⇒(iv) and (iv)⇒(v) are clear.

(v)⇒(i). We have to show that $T \in \mathcal{X}'_{\mu,m,\#}$ for each $m \in \mathbb{Z}$. Choose $r \in \mathbb{N}$ such that $r \ge -m+1$. There exist $k \in \mathbb{N}$ and bounded continuous functions f_i on I, $i = 0, \ldots, r$, for which

$$T = \sum_{i=0}^{k} S_\mu^i f_i$$

and $e^{rx} f_i$ is absolutely integrable on I for every $i = 0, \ldots, k$. Then

$$\langle T, \varphi \rangle = \sum_{i=0}^{k} \int_{0}^{\infty} f_i(x) S_\mu^i \varphi(x)\, dx, \qquad \varphi \in \mathcal{X}_\mu.$$

Therefore

$$|\langle T, \varphi \rangle| \le \sum_{i=0}^{k} \int_{0}^{\infty} |e^{rx} f_i(x)| x^{\mu+1/2} e^{(-m-r)x} e^{mx} x^{-\mu-1/2} |S_\mu^i \varphi(x)|\, dx$$
$$\le C \sum_{i=0}^{k} \alpha_{m,\mu}^i(\varphi), \qquad \varphi \in \mathcal{X}_\mu.$$

Since $\mathcal{X}_\mu$ is a dense subset of $\mathcal{X}_{\mu,m,\#}$, it follows that T can be extended to an element of $\mathcal{X}'_{\mu,m,\#}$ defined by the same formula, and the proof is complete. ■

As an immediate consequence of Theorems 2.1 and 3.1 we infer that $\mathcal{X}_\mu$ is a subspace of $\mathcal{X}'_{\mu,\#}$.

In [4] there was introduced the space ε_μ consisting of all smooth complex-valued functions $\varphi(x)$, $x \in I$, such that the limit

$$\lim_{x \to 0^+} \left(\frac{1}{x} D\right)^k (x^{-\mu-1/2} \varphi(x))$$

exists for every $k \in \mathbb{N}$. This space is equipped with the topology generated by the family $\{\beta_{m,k}^\mu\}_{m \in \mathbb{N}-\{0\}, k \in \mathbb{N}}$, where for each $m \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$,

$$\beta_{m,k}^\mu(\varphi) = \sup_{x \in (0,m)} \left| \left(\frac{1}{x} D\right)^k (x^{-\mu-1/2} \varphi(x)) \right|, \qquad \varphi \in \varepsilon_\mu.$$

The dual space of ε_μ is denoted by ε'_μ and it was characterized in [4, Proposition 4.4]. Moreover, by [4, Propositions 4.5 and 4.6], ε'_μ is contained in $\mathcal{X}'_{\mu,\#}$.

We now prove that elements of $\mathcal{X}'_{\mu,\#}$ define convolution operators in $\mathcal{X}_\mu$.

PROPOSITION 3.6. *Let $S \in \mathcal{X}'_{\mu,\#}$. Then the mapping $\varphi \mapsto S \# \varphi$ is continuous from $\mathcal{X}_\mu$ into itself.*

Proof. By Proposition 3.4 for every $\varphi \in \mathcal{X}_\mu$ we have

$$h'_\mu(S \# \varphi) = x^{-\mu-1/2} h'_\mu(S) h_\mu(\varphi).$$

Hence by Theorems 2.1 and 3.1, $h'_\mu(S \# \varphi) \in \mathcal{Q}_\mu$ for each $\varphi \in \mathcal{X}_\mu$. Moreover, the mapping $\varphi \mapsto h'_\mu(S \# \varphi)$ is continuous from $\mathcal{X}_\mu$ into $\mathcal{Q}_\mu$. Finally, since h'_μ reduces to h_μ on $\mathcal{Q}_\mu$ we infer from Theorem 2.1 that $\varphi \mapsto S \# \varphi$ defines a continuous mapping from $\mathcal{X}_\mu$ into itself. ■

Proposition 3.6 leads to the following definition. For $T \in \mathcal{X}'_\mu$ and $S \in \mathcal{X}'_{\mu,\#}$, the *Hankel convolution* $T \# S$ is the element of $\mathcal{X}'_\mu$ defined by

$$\langle T \# S, \varphi \rangle = \langle T, S \# \varphi \rangle, \qquad \varphi \in \mathcal{X}_\mu.$$

Note that by Proposition 3.4 the definition of Hankel convolution on $\mathcal{X}'_\mu \times \mathcal{X}'_{\mu,\#}$ is a generalization of the above definition of Hankel convolution on $\mathcal{X}'_\mu \times \mathcal{X}_\mu$.

We now establish the interchange formula for the generalized Hankel convolution.

PROPOSITION 3.7. *Let $T \in \mathcal{X}'_\mu$ and $S \in \mathcal{X}'_{\mu,\#}$. Then*

$$h'_\mu(T \# S) = x^{-\mu-1/2} h'_\mu(T) h'_\mu(S).$$

Proof. According to Propositions 3.4 and 3.6 we can write

$$\langle h'_\mu(T \# S), \Phi \rangle = \langle T \# S, h_\mu(\Phi) \rangle = \langle T, S \# h_\mu(\Phi) \rangle$$
$$= \langle h'_\mu(T), h_\mu(S \# h_\mu(\Phi)) \rangle = \langle h'_\mu(T), x^{-\mu-1/2} h'_\mu(S) \Phi \rangle$$
$$= \langle x^{-\mu-1/2} h'_\mu(T) h'_\mu(S), \Phi \rangle, \qquad \Phi \in \mathcal{Q}_\mu. \ ■$$

As a consequence of Theorem 3.1 and Proposition 3.7 we obtain

COROLLARY 3.1. *If $R, S \in \mathcal{X}'_{\mu,\#}$ then $R \# S \in \mathcal{X}'_{\mu,\#}$.* ■

Next we show some algebraic properties of the generalized Hankel convolution.

PROPOSITION 3.8. *Let $T \in \mathcal{X}'_\mu$ and $R, S \in \mathcal{X}'_{\mu,\#}$. Then*

(i) $(T \# R) \# S = T \# (R \# S)$.
(ii) $R \# S = S \# R$.
(iii) $S_\mu(T \# R) = (S_\mu T) \# R = T \# (S_\mu R)$.
(iv) *If δ_μ denotes the functional on $\mathcal{X}_\mu$ defined by*

$$\langle \delta_\mu, \varphi \rangle = 2^\mu \Gamma(\mu+1) \lim_{x \to 0^+} x^{-\mu-1/2} \varphi(x), \qquad \varphi \in \mathcal{X}_\mu,$$

then $\delta_\mu \in \mathcal{X}'_{\mu,\#}$ and $R \# \delta_\mu = R$.

Proof. (i)–(iii) follow immediately from Proposition 3.7. To see (iv) it is sufficient to note that $y^{-\mu-1/2} h'_\mu(\delta_\mu) = 1$. ■

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Received November 20, 1995
Revised version May 6, 1996 (3573)

STUDIA MATHEMATICA 121 (1) (1996)

Invariance properties of homomorphisms on algebras of holomorphic functions with the Hadamard product

by

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Abstract. Let $H(G_1)$ be the set of all holomorphic functions on the domain G_1. Two domains G_1 and G_2 are called *Hadamard-isomorphic* if $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the Hadamard product. Our main result states that two admissible domains are Hadamard-isomorphic if and only if they are equal.

Introduction. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series with positive radii of convergence. Then the *Hadamard product* of f and g is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Let G be an open domain of $\mathbb{C}$ containing 0 and let $H(G)$ be the set of all holomorphic functions on G. We call a domain G *admissible* if for all $f, g \in H(G)$ the Hadamard product $f*g$ extends to a (unique) function of $H(G)$, i.e. $H(G)$ is a commutative algebra. Examples of admissible domains are the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, or more generally $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ for $r \ge 1$, $\mathbb{C} \setminus \{1\}$ and so-called α-starlike regions like $\mathbb{C}_- := \{z \in \mathbb{C} : z \notin [1, \infty)\}$ (see [3] for details). By the famous Hadamard multiplication theorem a domain G is admissible if and only if the complement G^c of G is a multiplicative semigroup (cf. e.g. [5]).

The aim of this paper is to study homomorphisms on $H(G)$. Let us call two domains G_1 and G_2 *Hadamard-isomorphic* if G_1 and G_2 are admissible and $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the Hadamard product. Our main result states that two admissible domains G_1, G_2 with $1 \in G_1^c$ are Hadamard-isomorphic if and only if they are equal. This stands in sharp contrast to the following classical result: $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the *pointwise multiplication* if and only if G_1 is biholomorphically equivalent to G_2. Indeed, we give a complete description of all isomorphisms for admissible domains G with $1 \in G^c$. Roughly speaking, if G is an admissible domain differ-

1991 *Mathematics Subject Classification*: Primary 46J05; Secondary 30A99.
Key words and phrases: Hadamard product, B_0-algebras, homomorphisms.