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(H_p, L_p)-type inequalities for the two-dimensional dyadic derivative

by

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Abstract. It is shown that the restricted maximal operator of the two-dimensional dyadic derivative of the dyadic integral is bounded from the two-dimensional dyadic Hardy–Lorentz space $H_{p,q}$ to $L_{p,q}$ ($2/3 < p < \infty$, $0 < q \le \infty$) and is of weak type (L_1, L_1). As a consequence we show that the dyadic integral of a two-dimensional function $f \in L_1$ is dyadically differentiable and its derivative is f a.e.

1. Introduction. It is known that

$$f(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(s)\,ds \quad \text{a.e.}$$

if $f \in L_1[0,1)$. The dyadic analogue of this result can be formulated as follows. Butzer and Wagner [5] introduced the dyadic derivative to be the limit of

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \dotplus 2^{-j-1})) \quad (x \in [0,1))$$

as $n \to \infty$ where $\dotplus$ denotes the dyadic addition (see e.g. Schipp, Wade, Simon and Pál [13]). The dyadic integral $\mathbf{I}f$ is defined by the convolution of f and the function W whose kth Walsh–Fourier coefficient is $1/k$ ($k \ne 0$). The boundedness of $\mathbf{I}^* f = \sup_{n\in\mathbb{N}} |\mathbf{d}_n(\mathbf{I}f)|$ from $L_p[0,1)$ to $L_p[0,1)$ ($1 < p \le \infty$) and the weak type $(L_1[0,1), L_1[0,1))$ inequality

$$\sup_{\gamma>0} \gamma\lambda(\sup_{n\in\mathbb{N}} \mathbf{I}^* f > \gamma) \le C\|f\|_1 \quad (f \in L_1[0,1)) \tag{1}$$

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are due to Schipp [9]. The dyadic analogue of the differentiation theorem follows easily from the last weak type inequality:

$$\lim_{n\to\infty} \mathbf{d}_n(\mathbf{I}f) = f \quad \text{a.e.}$$

if $f \in L_1[0,1)$ is of mean zero (see Schipp [9]).

The weak type inequality was extended by the author [15]. We proved that

$$\|\mathbf{I}^* f\|_{p,q} \le C\|f\|_{H_{p,q}} \quad (1/2 < p < \infty,\ 0 < q \le \infty) \tag{2}$$

where $H_{p,q}$ denotes the one-dimensional dyadic Hardy–Lorentz space. As a special case we obtain (1) from this by choosing $p = 1$ and $q = \infty$.

The two-dimensional differentiation theorem

$$f(x,y) = \lim_{h,k\to 0} \frac{1}{hk} \int_x^{x+h}\int_y^{y+k} f(s,t)\,ds\,dt \quad \text{a.e.} \tag{3}$$

if $f \in L\log L[0,1)^2$ can be found in Zygmund [18]. The dyadic analogue of this result is

$$\lim_{n,m\to\infty} \mathbf{d}_{n,m}(\mathbf{I}f) = f \quad \text{a.e.} \quad (f \in L\log L[0,1)^2)$$

where $\mathbf{I}f$ now denotes the convolution of f and $W \times W$ and, moreover,

$$(\mathbf{d}_{n,m}f)(x,y) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 2^{i+j-2}(f(x,y) - f(x, y \dotplus 2^{-j-1}) - f(x \dotplus 2^{-i-1}, y) + f(x \dotplus 2^{-i-1}, y \dotplus 2^{-j-1})) \tag{4}$$

(see Schipp and Wade [12] and also Weisz [17]). Recently the author [15] generalized this convergence result for $f \in H_1^\sharp \supset L\log L[0,1)^2$ where $H_1^\sharp$ is the two-dimensional dyadic hybrid Hardy space.

In this paper the Hardy–Lorentz spaces $H_{p,q}$ of dyadic martingales on the unit square are introduced with the $L_{p,q}$ Lorentz norms of the maximal function $\sup_{n\in\mathbb{N}} |f_{n,n}|$. Of course, $H_p = H_{p,p}$ are the usual Hardy spaces $(0 < p \le \infty)$.

We verify here the same results for the two-dimensional dyadic derivative as we proved in [14] for Cesàro means of two-dimensional Walsh–Fourier series. We denote the restricted maximal operator $\sup_{|n-m|\le\alpha} |\mathbf{d}_{n,m}(\mathbf{I}f)|$ for any $\alpha \ge 0$ by $\mathbf{I}_\alpha^* f$ and prove inequality (2) for this operator $(2/3 < p < \infty,\ 0 < q \le \infty)$. The two-dimensional version of (1) follows from this with $p = 1$ and $q = \infty$. Note that the unrestricted maximal operator is investigated in Weisz [17].

It is known that if $\alpha^{-1} \le |h/k| \le \alpha$ for any $\alpha > 0$ then (3) holds for all $f \in L_1[0,1)^2$. The dyadic analogue of this follows from the two-dimensional

version of (1):

$$\lim_{\substack{n,m\to\infty\\ |n-m|\le\alpha}} \mathbf{d}_{n,m}(\mathbf{I}f) = f \quad \text{a.e.} \quad (f \in L[0,1)^2).$$

This convergence is also proved by Gát [7] with another method.

2. Martingales and Hardy–Lorentz spaces. In this paper the unit square $[0,1)^2$ and the Lebesgue measure λ are considered. By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \le k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. If I_1 and I_2 are dyadic intervals and $\lambda(I_1) = \lambda(I_2)$ then the set

$$I := I_1 \times I_2$$

is a *dyadic square*. Clearly, the dyadic square of area 2^{-2n} containing $(x,y) \in [0,1)^2$ is given by

$$I_{n,n}(x,y) := I_n(x) \times I_n(y).$$

The σ-algebra generated by the dyadic squares $\{I_{n,n}(x) : x \in [0,1)^2\}$ will be denoted by $\mathcal{F}_{n,n}$ $(n \in \mathbb{N})$, more precisely,

$$\mathcal{F}_{n,n} = \sigma\{[k2^{-n}, (k+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n}) : 0 \le k < 2^n,\ 0 \le l < 2^n\}$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$. We will investigate martingales of the form $f = (f_{n,n}, n \in \mathbb{N})$ with respect to $(\mathcal{F}_{n,n}, n \in \mathbb{N})$. We briefly write L_p instead of the real $L_p([0,1)^2, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{[0,1)^2} |f|^p\,d\lambda)^{1/p}$ $(0 < p \le \infty)$.

The distribution function of a Borel-measurable function f is defined by

$$\lambda(\{|f| > \gamma\}) := \lambda(\{x : |f(x)| > \gamma\}) \quad (\gamma \ge 0).$$

The *weak* L_p space L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\gamma>0} \gamma[\lambda(\{|f| > \gamma\})]^{1/p} < \infty,$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\widetilde{f}(t) := \inf\{\gamma : \lambda(\{|f| > \gamma\}) \le t\}.$$

The Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty$ and $0 < q < \infty$,

$$\|f\|_{p,q} := \left(\int_0^\infty \widetilde{f}(t)^q t^{q/p}\,\frac{dt}{t}\right)^{1/q},$$

while for $0 < p \le \infty$,

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \tilde{f}(t).$$

Let

$$L_{p,q} := L_{p,q}([0,1)^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities:

$$L_{p,p} = L_p, \quad L_{p,\infty} = L_p^* \quad (0 < p \le \infty)$$

(see e.g. Bennett and Sharpley [1] or Bergh and Löfström [2]).

The *maximal function* of a martingale $f = (f_{n,n}, n \in \mathbb{N})$ is defined by

$$f^* := \sup_{n\in\mathbb{N}} |f_{n,n}|.$$

It is easy to see that, in case $f \in L_1$, the maximal function can also be given by

$$f^*(x,y) = \sup_{n\in\mathbb{N}} \frac{1}{\lambda(I_{n,n}(x,y))} \Big| \int_{I_{n,n}(x,y)} f \, d\lambda \Big|.$$

For $0 < p, q \le \infty$ the *martingale Hardy–Lorentz space* $H_{p,q}$ consists of all martingales $f = (f_{n,n}, n \in \mathbb{N})$ for which

$$\|f\|_{H_{p,q}} := \|f^*\|_{p,q} < \infty.$$

Note that in case $p = q$ the usual definition of Hardy space $H_{p,p} = H_p$ is obtained.

It is well known that for a martingale $f = (f_{n,n}, n \in \mathbb{N})$,

$$\sup_{\gamma>0} \gamma\lambda(f^* > \gamma) \le \sup_{n\in\mathbb{N}} \|f_{n,n}\|_1 \tag{5}$$

and

$$\|f^*\|_p \le \frac{p}{p-1} \|f\|_p \quad (1 < p \le \infty),$$

hence $H_p \sim L_p$ whenever $1 < p \le \infty$ (see Neveu [8]), where $\sim$ denotes the equivalence of the norms and spaces. Moreover, it is proved in Weisz [16] that

$$H_{p,q} \sim L_{p,q} \quad (1 < p \le \infty,\ 0 < q \le \infty).$$

A bounded measurable function a is a *p-atom* if $a = 1$ or there exists a dyadic square Q such that

(i) $\int_Q a \, d\lambda = 0$,

(ii) $\|a\|_\infty \le \lambda(Q)^{-1/p}$,

(iii) $\{a \ne 0\} \subset Q$.

Using the atomic decomposition we verified the next theorem in [14].

THEOREM A. *Suppose that the operator T is sublinear and, for each $p_0 \le p \le 1$, there exists a constant $C_p > 0$ such that*

$$\int_{[0,1)^2 \setminus Q} |Ta|^p \, d\lambda \le C_p \tag{6}$$

for every p-atom a where the support of a is contained in Q as in (i)–(iii). *If T is bounded from L_∞ to L_∞ then for every $p_0 \le p \le 1$,*

$$\|Tf\|_p \le C_p \|f\|_{H_p} \quad (f \in H_p \cap L_1).$$

The following interpolation result concerning Hardy–Lorentz spaces will be used in this paper (see Weisz [16]).

THEOREM B. *If a sublinear operator T is bounded from H_{p_0} to L_{p_0} and from L_∞ to L_∞ then it is also bounded from $H_{p,q}$ to $L_{p,q}$ if $p_0 < p < \infty$ and $0 < q \le \infty$.*

3. The two-dimensional dyadic derivative. First we introduce the Walsh system. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \quad 0 \le x_k < 2,\ x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \quad (n \in \mathbb{N})$$

are called *Rademacher functions*. The product system generated by these functions is the *one-dimensional Walsh system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \le n_k < 2$ and $n_k \in \mathbb{N}$.

The Kronecker product $(w_{n,m}; n, m \in \mathbb{N})$ of two Walsh systems is said to be the *two-dimensional Walsh system*. Thus

$$w_{n,m}(x,y) := w_n(x) w_m(y).$$

Recall that the *Walsh–Dirichlet kernels*

$$D_n := \sum_{k=0}^{n-1} w_k$$

satisfy

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0, 2^{-n}), \\ 0 & \text{if } x \in [2^{-n}, 1), \end{cases} \tag{7}$$

for $n \in \mathbb{N}$ (see e.g. Schipp, Wade, Simon and Pál [13]).

For each function f defined on $[0,1)^2$ Butzer and Engels [3] introduced the concept of the two-dimensional dyadic derivative by (4). Then f is said to be *dyadically differentiable* at $x, y \in [0,1)$ if $(\mathbf{d}_{n,m} f)(x,y)$ converges as $n, m \to \infty$. It was verified by Butzer and Wagner [4] that every Walsh function is dyadically differentiable and

$$\lim_{\min(n,m)\to\infty} \mathbf{d}_{n,m}(w_k \times w_l)(x,y) = kl(w_k \times w_l)(x,y)$$

for all $x, y \in [0,1)$ and $k, l \in \mathbb{N}$. Let W be the function whose Walsh–Fourier coefficients satisfy

$$\widehat{W}(k) := \int_0^1 W w_k \, d\lambda := \begin{cases} 1 & \text{if } k = 0, \\ 1/k & \text{if } k \in \mathbb{N},\ k \neq 0. \end{cases}$$

The two-dimensional dyadic integral of $f \in L_1$ is introduced by

$$\mathbf{I} f(x,y) := f * (W \times W)(x,y) := \int_0^1\int_0^1 f(t,u) W(x \dot{+} t) W(y \dot{+} u) \, dt \, du.$$

Notice that $W \in L_2 \subset L_1$, so $\mathbf{I}$ is well defined on L_1.

Set

$$W_K := \sum_{n=2^K}^{\infty} \frac{w_n}{n}$$

and let us estimate $\mathbf{d}_n W$ and $\mathbf{d}_n W_K$. The following theorem can be proved with the help of the ideas in Schipp, Wade, Simon and Pál [13] (pp. 272–275) and in Weisz [15].

THEOREM 1. *For all $n, K \in \mathbb{N}$ we have*

$$|\mathbf{d}_n W(x) + 1| \le C \sum_{i=1}^{4} F_{0,n}^i(x) \quad \text{and} \quad |\mathbf{d}_n W_K(x)| \le C \sum_{i=1}^{5} F_{K,n}^i(x)$$

where

$$F_{K,n}^1(x) := \frac{1}{2^{K-n} \vee 1} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i) 2^{j-n} D_{2^i}(x \dot{+} 2^{-j-1}),$$

$$F_{K,n}^2(x) := \frac{1}{2^{K-n} \vee 1} \sum_{i=0}^{n-1} (n-i) 2^{i-n} D_{2^i}(x),$$

$$F_{K,n}^3(x) := \sum_{j=0}^{n-1} 2^j \sum_{i=n}^{\infty} 2^{-i} \frac{D_{2^i}(x \dot{+} 2^{-j-1})}{2^{K-i} \vee 1},$$

$$F_{K,n}^4(x) := \sum_{k=0}^{\infty} 2^{-k} \frac{D_{2^{n+k}}(x)}{2^{K-n-k} \vee 1}$$

and

$$F_{K,n}^5(x) := D_{2^K}(x) 1_{\{n>K\}}.$$

4. Inequalities concerning the two-dimensional dyadic derivative. Before considering the operator

$$\mathbf{I}_\alpha^* f := \sup_{|n-m|\le\alpha} |\mathbf{d}_{n,m}(\mathbf{I} f)| \quad (f \in L_1)$$

for any $\alpha \ge 0$ let us modify slightly the dyadic derivative. Set

$$\delta_{n,m} f(x,y) := \int_0^1\int_0^1 f(t,u)[\mathbf{d}_n W(x \dot{+} t) + 1][\mathbf{d}_m W(y \dot{+} u) + 1] \, dt \, du$$

and

$$\mathbf{J}_\alpha^* f := \sup_{|n-m|\le\alpha} |\delta_{n,m} f| \quad (f \in L_1).$$

First we can prove that $\mathbf{J}_\alpha^*$ is bounded from H_p to L_p.

THEOREM 2. *There exist constants C_p depending only on p and α such that for each $2/3 < p \le 1$,*

$$\|\mathbf{J}_\alpha^* f\|_p \le C_p \|f\|_{H_p} \quad (f \in H_p)$$

where $\mathbf{J}_\alpha^* f$ will be defined for $f \in H_p \setminus L_1$ in the proof.

Proof. First assume that $f \in H_p \cap L_1$. By Theorem A the proof of Theorem 2 will be complete if we show that the operator $\mathbf{J}_\alpha^*$ satisfies (6) and is bounded from L_∞ to L_∞.

Since $\|D_{2^n}\|_1 = 1$, we can show that

$$\|F_{0,n}^i\|_1 \le C \quad (i = 1, \ldots, 4;\ n \in \mathbb{N}). \tag{8}$$

From this it follows that $\|\mathbf{d}_n W + 1\|_1 \le C$ for all $n \in \mathbb{N}$, which verifies that $\mathbf{J}_\alpha^*$ is bounded on L_∞.

If $a = 1$ then the left hand side of (6) is zero. Let $a \neq 1$ be an arbitrary p-atom with support $Q = I \times J$ and $\lambda(I) = \lambda(J) = 2^{-K}$ $(K \in \mathbb{N})$. Without loss of generality we can suppose that $I = J = [0, 2^{-K})$. If $k < 2^K$ and $l < 2^K$ then $w_{k,l}$ is constant on Q and so

$$\int_0^1\int_0^1 a(t,u) w_k(x \dot{+} t) w_l(y \dot{+} u) \, dt \, du = 0.$$

Since

$$\mathbf{d}_n(w_{i2^n+k}) = k w_{i2^n+k} \quad (0 \le k < 2^n;\ i, n \in \mathbb{N})$$

(see Schipp, Wade, Simon and Pál [13], p. 272) it is not hard to see that

$$\delta_{n,m}a(x,y)=\int_0^1\int_0^1 a(t,u)[\mathbf{d}_n W_K(x\dot{+}t)(\mathbf{d}_m W(y\dot{+}u)+1)$$
$$+(\mathbf{d}_n W(x\dot{+}t)+1)\mathbf{d}_m W_K(y\dot{+}u)$$
$$-\mathbf{d}_n W_K(x\dot{+}t)\mathbf{d}_m W_K(y\dot{+}u)]\,dt\,du.$$

By the fact that $F^i_{K,n}\le F^i_{0,n}$ $(i=1,\ldots,4;n,K\in\mathbb{N})$ and by Theorem 1 we obtain

$$\begin{aligned}\mathbf{J}^*_\alpha a\le{}&\sup_{|n-m|\le\alpha}\sup_{\substack{i=1,\ldots,5\\ j=1,\ldots,4}}|a|*F^i_{K,n}\times F^j_{0,m}\\ &+\sup_{|n-m|\le\alpha}\sup_{\substack{i=1,\ldots,4\\ j=1,\ldots,5}}|a|*F^i_{0,n}\times F^j_{K,m}\\ &+\sup_{|n-m|\le\alpha}\sup_{\substack{i=1,\ldots,5\\ j=1,\ldots,5}}|a|*F^i_{K,n}\times F^j_{K,m}\\ \le{}&2\sup_{|n-m|\le\alpha}\sup_{\substack{i=1,\ldots,5\\ j=1,\ldots,4}}|a|*F^i_{K,n}\times F^j_{0,m}\\ &+2\sup_{|n-m|\le\alpha}\sup_{\substack{i=1,\ldots,4\\ j=1,\ldots,5}}|a|*F^i_{0,n}\times F^j_{K,m}+\sup_{|n-m|\le\alpha}|a|*F^5_{K,n}\times F^5_{K,m}.\end{aligned}$$

Now we investigate the first term, the integral of $[\sup_{|n-m|\le\alpha}|a|*F^i_{K,n}\times F^j_{0,m}]^p$ over $[0,1)^2\setminus Q$ for all $i=1,\ldots,5$ and $j=1,\ldots,4$.

Step 1: *Integrating over* $([0,1)\setminus I)\times J$. We proved in [15] that for all $n,K\in\mathbb{N}$ and $i=1,\ldots,5$,

$$\int_{I^c}\Big(\sup_{n\in\mathbb{N}}\int_I F^i_{K,n}(x\dot{+}t)\,dt\Big)^p\,dx\le C_p 2^{-K}\tag{9}$$

where $I^c:=[0,1)\setminus I$. Taking into account (8) and the definition of the p-atom, we can establish that, for all $i=1,\ldots,5$ and $j=1,\ldots,4$,

$$\begin{aligned}\int_{I^c}\int_J\Big(\sup_{n,m\in\mathbb{N}}\int_I\int_J|a(t,u)|F^i_{K,n}(x\dot{+}t)F^j_{0,m}(y\dot{+}u)\,dt\,du\Big)^p\,dx\,dy\\ \le C_p 2^{2K}\int_{I^c}\int_J\Big(\sup_{n\in\mathbb{N}}\int_I F^i_{K,n}(x\dot{+}t)\,dt\Big)^p\,dx\,dy\le C_p.\end{aligned}\tag{10}$$

Step 2: *Integrating over* $I\times([0,1)\setminus J)$. If $j<K$ and $x\in I$ then $x\dot{+}2^{-j-1}\notin I$. Hence, it follows from (7) that

$$\int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt=0$$

whenever $x\in I$ and $i>j$. Using this and (7) we can calculate the integrals $\int_I F^i_{K,n}(x\dot{+}t)\,dt$ if $x\in I$ and $i=1,\ldots,5$:

$$\begin{aligned}\int_I F^1_{K,n}(x\dot{+}t)\,dt&=\frac{1}{2^{K-n}\vee 1}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}(n-i)2^{j-n}\int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\\ &\le\begin{cases}0 & \text{if } n\le K,\\ C & \text{if } n>K,\end{cases}\\ \int_I F^2_{K,n}(x\dot{+}t)\,dt&=\frac{1}{2^{K-n}\vee 1}\sum_{i=0}^{n-1}(n-i)2^{i-n}\int_I D_{2^i}(x\dot{+}t)\,dt\\ &\le\begin{cases}C2^{n-K} & \text{if } n\le K,\\ C & \text{if } n>K,\end{cases}\\ \int_I F^3_{K,n}(x\dot{+}t)\,dt&=\sum_{j=0}^{n-1}2^j\sum_{i=n}^{\infty}2^{-i}\frac{1}{2^{K-i}\vee 1}\int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\\ &\le\begin{cases}0 & \text{if } n\le K,\\ C & \text{if } n>K,\end{cases}\\ \int_I F^4_{K,n}(x\dot{+}t)\,dt&=\sum_{k=0}^{\infty}2^{-k}\frac{1}{2^{K-n-k}\vee 1}\int_I D_{2^{n+k}}(x\dot{+}t)\,dt\\ &\le\sum_{k=0}^{K-n-1}2^{n-K}2^{n+k-K}+\sum_{k=(K-n)\vee 0}^{\infty}2^{-k}\\ &\le\begin{cases}C2^{n-K} & \text{if } n\le K,\\ C & \text{if } n>K,\end{cases}\\ \int_I F^5_{K,n}(x\dot{+}t)\,dt&=1_{\{n>K\}}\int_I D_{2^K}(x\dot{+}t)\,dt=\begin{cases}0 & \text{if } n\le K,\\ 1 & \text{if } n>K.\end{cases}\end{aligned}$$

Let $r\in\mathbb{N}$ satisfy $r-1<\alpha\le r$ and observe that

$$\begin{aligned}\sup_{|n-m|\le\alpha}|a|*F^i_{K,n}\times F^j_{0,m}\le{}&\sup_{\substack{|n-m|\le r\\ n,m\le K}}|a|*F^i_{K,n}\times F^j_{0,m}\\ &+\sup_{n,m\ge K-r}|a|*F^i_{K,n}\times F^j_{0,m}\\ ={}&:(A_{i,j})+(B_{i,j}).\end{aligned}$$

for all $i=1,\ldots,5$ and $j=1,\ldots,4$. Of course, $(A_{i,j})(x,y)=0$ if $i=1,3,5$ and $x\in I$. So suppose that $i=2,4$ and $j=1,\ldots,4$. It is easy to see that

$$2^{m-K}F^j_{0,m}\le F^j_{K,m}\quad(m\le K;\ j=1,\ldots,4).\tag{11}$$

Consequently,

$$\begin{aligned}(A_{i,j})(x,y) &= \sup_{\substack{|n-m|\le r\\ n,m\le K}} \int_I\int_J |a(t,u)| F^i_{K,n}(x\dotplus t) F^j_{0,m}(y\dotplus u)\,dt\,du\\ &\le C_p 2^{2K/p} \sup_{\substack{|n-m|\le r\\ n,m\le K}} 2^{n-K}\int_J F^j_{0,m}(y\dotplus u)\,du\\ &\le C_p 2^{2K/p}2^r \sup_{m\le K} 2^{m-K}\int_J F^j_{0,m}(y\dotplus u)\,du\\ &\le C_p 2^{2K/p} \sup_{m\le K}\int_J F^j_{K,m}(y\dotplus u)\,du.\end{aligned}$$

Then the inequality

$$\int_I\int_{J^c} (A_{i,j})^p\,d\lambda \le C_p \tag{12}$$

can be proved as in (10) where $i=1,\dots,5$ and $j=1,\dots,4$.

Since $F^j_{0,m}=F^j_{K,m}$ for $m>K$, (11) yields that

$$F^j_{0,m}\le 2^r F^j_{K,m}\qquad (m\ge K-r;\ j=1,\dots,4). \tag{13}$$

Then, for each $i=1,\dots,5$ and $j=1,\dots,4$,

$$\int_I\int_{J^c}(B_{i,j})^p\,d\lambda \le C_p 2^{2K}\int_I\int_{J^c}\Big(\sup_{m\ge K-r}\int_J 2^r F^j_{K,m}(y\dotplus u)\,du\Big)^p dx\,dy\le C_p \tag{14}$$

as we have seen in (10).

Step 3: *Integrating over* $([0,1)\setminus I)\times([0,1)\setminus J)$. By (7) it is easy to verify that, for $x\notin I$,

$$\int_I D_{2^i}(x\dotplus t\dotplus 2^{-j-1})\,dt = 2^{i-K}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x) \tag{15}$$

if $j<i\le K-1$,

$$\int_I D_{2^i}(x\dotplus t)\,dt = 2^{i-K}1_{[2^{-K},2^{-i})}(x) \tag{16}$$

if $i\in\mathbb{N}$ and

$$\int_I D_{2^i}(x\dotplus t\dotplus 2^{-j-1})\,dt = 1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-K})}(x) \tag{17}$$

if $i\ge K$.

Now we modify slightly the kernel functions $F^i_{K,n}$ $(i=1,\dots,4)$ and calculate their integrals like (9). By (15),

$$\begin{aligned}&\int_{I^c}\Big(\sup_{n\le K}\int_I 2^{n/2}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}(n-i)2^{j-n}D_{2^i}(x\dotplus t\dotplus 2^{-j-1})\,dt\Big)^p dx\\ &=\int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}(n-i)2^{j-n/2}2^{i-K}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x)\Big)^p dx\\ &= C_p 2^{-Kp}\int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}\frac{n-i}{2}2^{(i-n)/2}2^{i/2+j}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x)\Big)^p dx.\end{aligned}$$

Since the function $f(n):=(n/2)2^{-n/2}$ is decreasing for $n\ge 3$, we obtain

$$\begin{aligned}&\int_{I^c}\Big(\sup_{n\le K}\int_I 2^{n/2}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}(n-i)2^{j-n}D_{2^i}(x\dotplus t\dotplus 2^{-j-1})\,dt\Big)^p dx\\ &\le C_p 2^{-Kp}\int_{I^c}\sum_{j=0}^{K-1}\sum_{i=j+1}^{K-1}2^{ip/2+jp}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x)\,dx\\ &\le C_p 2^{-Kp}\sum_{j=0}^{K-1}\sum_{i=j+1}^{K-1}2^{jp}2^{i(p/2-1)}\\ &\le C_p 2^{-Kp}\sum_{j=0}^{K-1}2^{j(3p/2-1)}\le C_p 2^{Kp/2-K}\end{aligned} \tag{18}$$

provided that $2/3<p\le 1$.

Using (16) we get

$$\begin{aligned}&\int_{I^c}\Big(\sup_{n\le K}\int_I 2^{n/2}\sum_{i=0}^{n-1}(n-i)2^{i-n}D_{2^i}(x\dotplus t)\,dt\Big)^p dx\\ &\le C_p\int_{I^c}\Big(\sup_{n\le K}\sum_{i=0}^{n-1}\frac{n-i}{2}2^{(i-n)/2}2^{i/2}2^{i-K}1_{[0,2^{-i})}(x)\Big)^p dx\\ &\le C_p 2^{-Kp}\sum_{i=0}^{K-1}2^{i(3p/2-1)}\le C_p 2^{Kp/2-K}.\end{aligned} \tag{19}$$

It follows from (15) and (17) that

$$\begin{aligned}&\int_{I^c}\Big(\sup_{n\le K}\int_I\sum_{j=0}^{n-1}2^j\sum_{i=n}^{\infty}2^{-i/2}D_{2^i}(x\dotplus t\dotplus 2^{-j-1})\,dt\Big)^p dx\\ &\le\int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}2^j\sum_{i=j+1}^{K-1}2^{-i/2}2^{i-K}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x)\Big)^p dx\\ &\quad+\int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}2^j\sum_{i=K}^{\infty}2^{-i/2}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-K})}(x)\Big)^p dx\end{aligned} \tag{20}$$

$$\le \sum_{j=0}^{K-1} 2^{jp} \sum_{i=j+1}^{K-1} 2^{i(p/2-1)} 2^{-Kp} + \sum_{j=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip/2} 2^{-K}$$

$$\le C_p 2^{-Kp} \sum_{j=0}^{K-1} 2^{j(3p/2-1)} + C_p 2^{-K} 2^{Kp} 2^{-Kp/2} \le C_p 2^{Kp/2-K}.$$

Similarly,

(21)
$$\int_{I^c}\Big(\sup_{n\le K}\int_I \sum_{j=0}^{n-1} 2^{j/2}\sum_{i=n}^{\infty} 2^{-i}\frac{D_{2^i}(x\dotplus t\dotplus 2^{-j-1})}{2^{K-i}\vee 1}\,dt\Big)^p dx$$

$$\le \int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}2^{j/2}\sum_{i=j+1}^{K-1}2^{-K}2^{i-K}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-i})}(x)\Big)^p dx$$

$$+\int_{I^c}\Big(\sup_{n\le K}\sum_{j=0}^{n-1}2^{j/2}\sum_{i=K}^{\infty}2^{-i}1_{[2^{-j-1},2^{-j-1}\dotplus 2^{-K})}(x)\Big)^p dx$$

$$\le \sum_{j=0}^{K-1}2^{jp/2}\sum_{i=j+1}^{K-1}2^{i(p-1)}2^{-2Kp}+\sum_{j=0}^{K-1}2^{jp/2}\sum_{i=K}^{\infty} 2^{-ip}2^{-K}$$

$$\le C_p 2^{-2Kp}\sum_{j=0}^{K-1}2^{j(3p/2-1)}+C_p2^{-K}2^{Kp/2}2^{-Kp}\le C_p2^{-Kp/2-K}$$

whenever $p<1$. If $p=1$ then

$$\sum_{j=0}^{K-1}2^{jp/2}\sum_{i=j+1}^{K-1}2^{i(p-1)}2^{-2Kp}=\sum_{j=0}^{K-1}2^{(j-K)/2}(K-j)2^{-3K/2}\le C2^{-3K/2}$$

and (21) is true in this case, too.

Obviously, if $x\notin I$ and $i\ge K$ then

(22)
$$\int_I D_{2^i}(x\dotplus t)\,dt=0.$$

This implies that

(23)
$$\int_{I^c}\Big(\sup_{n\le K}\int_I 2^{n/2}\sum_{k=0}^{\infty} 2^{-k}D_{2^{n+k}}(x\dotplus t)\,dt\Big)^p dx$$

$$= \int_{I^c}\Big(\sup_{n\le K}\sum_{k=0}^{K-n-1}2^{n/2-k}2^{n+k-K}1_{[2^{-K},2^{-n-k})}(x)\Big)^p dx$$

$$=2^{-Kp}\int_{I^c}\Big(\sum_{k=0}^{K-1}\sum_{n=0}^{K-k-2}2^{3n/2}1_{[2^{-n-k-1},2^{-n-k})}(x)$$

$$+ \sum_{k=0}^{K-1}2^{3(K-k-1)/2}1_{[2^{-K},2^{-K+1})}(x)\Big)^p dx$$

$$\le C_p2^{-Kp}\sum_{k=0}^{K-1}\sum_{n=0}^{K-k-2}2^{n(3p/2-1)}2^{-k}+C_p2^{-Kp}\sum_{k=0}^{K-1}2^{3Kp/2}2^{-3kp/2}2^{-K}$$

$$\le C_p2^{-Kp}\sum_{k=0}^{K-1}2^{(K-k)(3p/2-1)}2^{-k}+C_p2^{Kp/2-K}\le C_p2^{Kp/2-K}.$$

In the same way we conclude that

(24)
$$\int_{I^c}\Big(\sup_{n\le K}\int_I 2^{-n/2}\sum_{k=0}^{\infty} 2^{-k}\frac{D_{2^{n+k}}(x\dotplus t)}{2^{K-n-k}\vee 1}\,dt\Big)^p dx$$

$$= \int_{I^c}\Big(\sup_{n\le K}\sum_{k=0}^{K-n-1}2^{-n/2}2^{n-K}2^{n+k-K}1_{[2^{-K},2^{-n-k})}(x)\Big)^p dx$$

$$\le 2^{-2Kp}\int_{I^c}\Big(\sum_{k=0}^{K-1}\sum_{n=0}^{K-k-2}2^{3n/2+k}1_{[2^{-n-k-1},2^{-n-k})}(x)$$

$$+\sum_{k=0}^{K-1}2^{3(K-k-1)/2+k}1_{[2^{-K},2^{-K+1})}(x)\Big)^p dx$$

$$\le C_p2^{-Kp/2-K}.$$

Now we are ready to deal with the integrals of $(A_{i,j})^p$ over $I^c\times J^c$ ($i=1,\dots,5; j=1,\dots,4$). We investigate only three terms, $(A_{1,1})$, $(A_{1,3})$ and $(A_{3,1})$, because the others are all similar. Applying (18) twice we obtain

(25)
$$\int_{I^c}\int_{J^c}(A_{1,1})^p\,d\lambda$$

$$= \int_{I^c}\int_{J^c}\Big(\sup_{\substack{|n-m|\le r\\ n,m\le K}}\int_I\int_J |a(t,u)|2^{n-K}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1}(n-i)2^{j-n}D_{2^i}(x\dotplus t\dotplus 2^{-j-1})$$

$$\times \sum_{k=0}^{m-1}\sum_{l=k+1}^{m-1}(m-l)2^{k-m}D_{2^l}(y\dotplus u\dotplus 2^{-k-1})\,dt\,du\Big)^p dx\,dy$$

$$\leq C_p 2^{2K-Kp} \int_{I^c}\int_{J^c} \Big(\sup_{n,m\le K} \int_I 2^{n/2} \sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1} (n-i)2^{j-n} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt$$

$$\times \int_J 2^{r/2} 2^{m/2} \sum_{k=0}^{m-1}\sum_{l=k+1}^{m-1} (m-l)2^{k-m} D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1})\, du\Big)^p \,dx\,dy$$

$$\leq C_p 2^{2K-Kp} 2^{Kp/2-K} 2^{Kp/2-K} = C_p.$$

By (18) and (20),

$$(26)\qquad \int_{I^c}\int_{J^c} (A_{1,3})^p \, d\lambda$$

$$= \int_{I^c}\int_{J^c} \Big(\sup_{\substack{|n-m|\le r\\ n,m\le K}} \int_I\int_J |a(t,u)| 2^{n-K} \sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1} (n-i)2^{j-n} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})$$

$$\times \sum_{k=0}^{m-1} 2^k \sum_{l=m}^{\infty} 2^{-l} D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1})\, dt\, du\Big)^p \,dx\,dy$$

$$\leq C_p 2^{2K-Kp} \int_{I^c}\int_{J^c} \Big(\sup_{n,m\le K} \int_I 2^{n/2} \sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1} (n-i)2^{j-n} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt$$

$$\times \int_J 2^{r/2} \sum_{k=0}^{m-1} 2^k \sum_{l=m}^{\infty} 2^{-l/2} D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1})\, du\Big)^p \,dx\,dy$$

$$\leq C_p 2^{2K-Kp} 2^{Kp/2-K} 2^{Kp/2-K} = C_p.$$

Similarly, using (18) and (21) we can see that

$$(27)\qquad \int_{I^c}\int_{J^c} (A_{3,1})^p \, d\lambda$$

$$= \int_{I^c}\int_{J^c} \Big(\sup_{\substack{|n-m|\le r\\ n,m\le K}} \int_I\int_J |a(t,u)| \sum_{j=0}^{n-1} 2^j \sum_{i=n}^{\infty} 2^{-i} \frac{D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})}{2^{K-i}\vee 1}$$

$$\times \sum_{k=0}^{m-1}\sum_{l=k+1}^{m-1} (m-l)2^{k-m} D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1})\, dt\, du\Big)^p \,dx\,dy$$

$$\leq C_p 2^{2K} \int_{I^c}\int_{J^c} \Big(\sup_{n,m\le K} \int_I \sum_{j=0}^{n-1} 2^{j/2} \sum_{i=n}^{\infty} 2^{-i} \frac{D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})}{2^{K-i}\vee 1}\, dt$$

$$\times \int_J 2^{r/2} 2^{m/2} \sum_{k=0}^{m-1}\sum_{l=k+1}^{m-1} (m-l)2^{k-m} D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1})\, du\Big)^p \,dx\,dy$$

$$\leq C_p 2^{2K} 2^{-Kp/2-K} 2^{Kp/2-K} = C_p.$$

Observe that $(A_{5,j})(x,y) = 0$ $(j = 1,\ldots,4)$ follows from the definition.

(13) and (9) imply that

$$(28)\qquad \int_{I^c}\int_{J^c} (B_{i,j})^p \, d\lambda \le C_p 2^{2K} \int_{I^c}\int_{J^c} \Big(\sup_{n,m\ge K-r} \int_I F^i_{K,n}(x \dot{+} t)\, dt \times \int_J 2^r F^j_{K,m}(y \dot{+} u)\, du\Big)^p \,dx\,dy \le C_p$$

for each $i = 1,\ldots,5$ and $j = 1,\ldots,4$.

Notice that the terms $|a| * F^i_{0,n} \times F^j_{K,m}$ $(i = 1,\ldots,4;\, j = 1,\ldots,5)$ can be handled similarly. Finally, by (22), $|a| * F^5_{K,n} \times F^5_{K,m}(x,y) = 0$ when $x \in I^c$ or $y \in J^c$. Combining this and (10), (12), (14) and (25)–(28) we can establish that

$$\int_{[0,1)^2\setminus Q} (\mathbf{J}^*_\alpha a)^p \, dx \le C_p,$$

which proves the theorem for $f \in H_p \cap L_1$.

If $f \in H_p$ $(2/3 < p < 1)$ then $f_{k,k} \in L_1$ and $f_{k,k} \to f$ in H_p norm as $k \to \infty$. We have

$$\|\mathbf{J}^*_\alpha f_{j,j} - \mathbf{J}^*_\alpha f_{k,k}\|_p \le \|\mathbf{J}^*_\alpha (f_{j,j} - f_{k,k})\|_p \le C_p \|f_{j,j} - f_{k,k}\|_{H_p} \to 0$$

as $j,k \to \infty$. For $f \in H_p$ we define $\mathbf{J}^*_\alpha f \in L_p$ by

$$\mathbf{J}^*_\alpha f := \lim_{k\to\infty} \mathbf{J}^*_\alpha f_{k,k} \quad \text{in } L_p \text{ norm},$$

which finishes the proof of the theorem. ■

The next corollary follows from (5) and from Theorems B and 2.

COROLLARY 1. *There are absolute constants C_1 and $C_{p,q}$ such that*

$$\|\mathbf{J}^*_\alpha f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $2/3 < p < \infty$ and $0 < q \le \infty$. In particular, $\mathbf{J}^*_\alpha$ is of weak type (L_1, L_1), i.e. if $f \in L_1$ then

$$\|\mathbf{J}^*_\alpha f\|_{1,\infty} = \sup_{\gamma>0} \gamma\lambda(\mathbf{J}^*_\alpha f > \gamma) \le C_1 \|f\|_{H_{1,\infty}} = C_1 \sup_{\gamma>0} \gamma\lambda(f^* > \gamma) \le C_1 \|f\|_1.$$

Now we can state our main result.

COROLLARY 2. *Suppose that for a martingale $f = (f_{n,n}, n \in \mathbb{N}) \in H_{p,q}$ we have $\int_0^1 f_{n,n}(x,y_0)\,dx = \int_0^1 f_{n,n}(x_0,y)\,dy = 0$ for each $n \in \mathbb{N}$ and almost every $x_0, y_0 \in [0,1)$. Then*

$$\|\mathbf{I}^*_\alpha f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}}$$

for every $2/3 < p < \infty$ and $0 < q \le \infty$. In particular, $\mathbf{I}_\alpha^*$ is of weak type (L_1, L_1), i.e. if $f \in L_1$ such that $\int_0^1 f(x,y_0)\,dx = \int_0^1 f(x_0,y)\,dy = 0$ for almost every $x_0, y_0 \in [0,1)$ then

$$\sup_{\gamma>0} \gamma\lambda(\mathbf{I}_\alpha^* f > \gamma) \le C_1\|f\|_{H_{1,\infty}} \le C_1\|f\|_1.$$

Proof. By the proof of Theorem 2 it is enough to verify the corollary for integrable functions. Let $f \in L_1$ such that $\int_0^1 f(x,y_0)\,dx = \int_0^1 f(x_0,y)\,dy = 0$ for almost every $x_0, y_0 \in [0,1)$. Then it is easy to see that

$$\begin{aligned}\mathbf{d}_{n,m}(\mathbf{I}f)(x,y) &= \mathbf{d}_{n,m}\Big(\int_0^1\int_0^1 f(t,u)W(x \dot{+} t)W(y \dot{+} u)\,dt\,du\Big)\\ &= \int_0^1\int_0^1 f(t,u)\mathbf{d}_n W(x \dot{+} t)\mathbf{d}_m W(y \dot{+} u)\,dt\,du\\ &= \delta_{n,m}(x,y).\end{aligned}$$

Hence $\mathbf{I}_\alpha^* f = \mathbf{J}_\alpha^* f$ and the result follows from Corollary 1. ■

The next corollary follows from the weak type inequality in Corollary 2 and from the fact that the Walsh polynomials are dense in L_1.

COROLLARY 3. *If $\alpha \ge 0$ is arbitrary and if $f \in L_1$ is such that*

$$\int_0^1 f(x,y_0)\,dx = \int_0^1 f(x_0,y)\,dy = 0$$

for almost every $x_0, y_0 \in [0,1)$ then

$$\mathbf{d}_{n,m}(\mathbf{I}f) \to f \quad \text{a.e. as } n,m \to \infty \text{ and } |n-m| \le \alpha.$$

We remark that this corollary is also proved by Gát [7].

Finally, we note that without the condition $\int_0^1 f(x,y_0)\,dx = \int_0^1 f(x_0,y)\,dy = 0$ we can prove Corollary 2 only for $p \ge 1$; more exactly:

THEOREM 3. *There are absolute constants C_1 and $C_{p,q}$ such that*

$$\|\mathbf{I}_\alpha^* f\|_1 \le C_1\|f\|_{H_1} \quad (f \in H_1)$$

and

$$\|\mathbf{I}_\alpha^* f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \quad (f \in H_{p,q})$$

for every $1 < p < \infty$ and $0 < q \le \infty$.

Proof. We can apply only the second inequality of Theorem 1. That is to say, we have to investigate the terms $\sup_{|n-m|\le\alpha} |a| * F_{K,n}^i \times F_{0,m}^j$ $(i,j = 1,\dots,5)$. If $j \ne 5$ then they are considered in the proof of Theorem 2. If $j = 5$ then

$$\begin{aligned}|a| * F_{K,n}^i \times F_{0,m}^5(x,y) &= \int_I\int_J |a(t,u)|F_{K,n}^i(x \dot{+} t)F_{0,m}^5(y \dot{+} u)\,dt\,du\\ &\le 2^{2K}2^{-K}\int_I F_{K,n}^i(x \dot{+} t)\,dt\end{aligned}$$

where a is a 1-atom with support $Q = I \times J$, $I = J = [0, 2^{-K})$. Applying (9), we get

$$\begin{aligned}&\int_{I^c}\int_{J^c} \sup_{|n-m|\le\alpha} |a| * F_{K,n}^i \times F_{0,m}^5(x,y)\,dx\,dy\\ &\qquad \le 2^K \int_{I^c} \sup_{n\in\mathbb{N}} \int_I F_{K,n}^i(x \dot{+} t)\,dt\,dx \le C.\end{aligned}$$

We get the same result if we integrate over $I^c \times J$ or $I \times J^c$. Hence the condition (6) is verified for $p = 1$; this means that the first inequality in Theorem 3 is proved. The second inequality follows by interpolation. ■

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