\[(H_p, L_p)\text{-type inequalities for the two-dimensional dyadic derivative}\]

by

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Abstract. It is shown that the restricted maximal operator of the two-dimensional dyadic derivative of the dyadic integral is bounded from the two-dimensional dyadic Hardy–Lorentz space \(H_{p,q}\) to \(L_{p,q}\) \((2/3 < p < \infty, 0 < q \leq \infty)\) and is of weak type \((L_1, L_1)\). As a consequence we show that the dyadic integral of a two-dimensional function \(f \in L_1\) is dyadically differentiable and its derivative is \(f\) a.e.

1. Introduction. It is known that
\[
f(x) = \lim_{n \to \infty} \frac{1}{n+h} \int_{\frac{x}{h}}^{\frac{x+h}{h}} f(s) \, ds \quad \text{a.e.}
\]

if \(f \in L_1[0,1]\). The dyadic analogue of this result can be formulated as follows. Butzer and Wagner [5] introduced the dyadic derivative to be the limit of
\[
(d_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1} (f(x) - f(2^{-j}x)) \quad (x \in [0,1])
\]
as \(n \to \infty\) where \(+\) denotes the dyadic addition (see e.g. Schipp, Wade, Simon and Pál [13]). The dyadic integral \(I_f\) is defined by the convolution of \(f\) and the function \(W\) whose \(k\)th Walsh–Fourier coefficient is \(1/k\) (\(k \neq 0\)). The boundedness of \(I_f = \sup_{n \in \mathbb{N}} |d_n f|\) from \(L_p[0,1]\) to \(L_p[0,1]\) \((1 < p \leq \infty)\) and the weak type \((L_1, L_1)\) inequality
\[
\sup_{\gamma > 0} \gamma \lambda(\sup_{n \in \mathbb{N}} \gamma \lambda(I_f > \gamma)) \leq C \|f\|_1 \quad (f \in L_1[0,1])
\]

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are due to Schipp [9]. The dyadic analogue of the differentiation theorem follows easily from the last weak type inequality:

\[
\lim_{n \to \infty} d_n(If) = f \quad \text{a.e.}
\]

if \( f \in L_1[0,1] \) is of mean zero (see Schipp [9]).

The weak type inequality was extended by the author [15]. We proved that

\[
\|\Gamma f\|_{p,q} \leq C\|f\|_{H,p} \quad (1/2 < p < \infty, \ 0 < q \leq \infty)
\]

where \( H_{p,q} \) denotes the one-dimensional dyadic Hardy–Lorentz space. As a special case we obtain (1) from this by choosing \( p = 1 \) and \( q = \infty \).

The two-dimensional differentiation theorem

\[
f(x,y) = \lim_{h \to 0} \frac{1}{h^2} \int_{a}^{a+h} \int_{y}^{y+k} f(s,t) \, ds \, dt \quad \text{a.e.}
\]

if \( f \in L_1 \log L(0,1)^2 \) can be found in Zygmund [18]. The dyadic analogue of this result is

\[
\lim_{n,m \to \infty} d_{n,m}(If) = f \quad \text{a.e.} \quad (f \in L_1 \log L(0,1)^2)
\]

where \( If \) now denotes the convolution of \( f \) and \( W \times W \) and, moreover,

\[
(d_{n,m}f)(x,y) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} 2^{i+j-2} f(x,y) - f(x,y + 2^{-i-1}) - f(x + 2^{-i-1}, y) + f(x + 2^{-i-1}, y + 2^{-j-1})
\]

(see Schipp and Wale [12] and also Weiss [17]). Recently the author [15] generalized this convergence result for \( f \in H_1^2 \supset L_1(0,1)^2 \) where \( H_1^2 \) is the two-dimensional dyadic hybrid Hardy space.

In this paper the Hardy–Lorentz spaces \( H_{p,q} \) of dyadic martingales on the unit square are introduced with the \( L_{p,q} \) Lorentz norms of the maximal function \( \sup_{n \in \mathbb{N}} |f_n| \). Of course, \( H_p = H_{p,p} \) are the usual Hardy spaces \((0 < p \leq \infty)\).

We verify here the same results for the two-dimensional dyadic derivative as we proved in [14] for Cesàro means of two-dimensional Walsh–Fourier series. We denote the restricted maximal operator \( \sup_{n-m \leq \alpha} |d_{n,m}(If)| \) for any \( \alpha \geq 0 \) by \( L^\alpha f \) and prove inequality (2) for this operator \((2/3 < p < \infty, 0 < q \leq \infty)\). The two-dimensional version of (1) follows from this with \( p = 1 \) and \( q = \infty \). Note that the unrestricted maximal operator is investigated in Weiss [17].

It is known that if \( \alpha^{-1} \leq |h/k| \leq \alpha \) for any \( \alpha > 0 \) then (3) holds for all \( f \in L_1(0,1)^2 \). The dyadic analogue of this follows from the two-dimensional version of (1):

\[
\lim_{n,m \to \infty} d_{n,m}(If) = f \quad \text{a.e.} \quad (f \in L_1(0,1)^2).
\]

This convergence is also proved by Gát [7] with another method.

2. Martingales and Hardy–Lorentz spaces. In this paper the unit square \([0,1]^2\) and the Lebesgue measure \( \lambda \) are considered. By a dyadic interval we mean one of the form \([k2^{-n}, (k+1)2^{-n})\) for some \( k, n \in \mathbb{N}, 0 \leq k < 2^n \).

Given \( n \in \mathbb{N} \) and \( x \in [0,1) \) let \( I_n(x) \) denote the dyadic interval of length \( 2^{-n} \) which contains \( x \). If \( I_1 \) and \( I_2 \) are dyadic intervals and \( \lambda(I_1) = \lambda(I_2) \) then the set

\[
I := I_1 \times I_2
\]

is a dyadic square. Clearly, the dyadic square of area \( 2^{-2n} \) containing \((x,y) \in [0,1]^2\) is given by

\[
I_{n,x}(x,y) := I_n(x) \times I_n(y).
\]

The \( \sigma \)-algebra generated by the dyadic squares \( \{I_{n,x} : x \in [0,1]^2\} \) will be denoted by \( F_{n,x} \) (\( n \in \mathbb{N} \)), more precisely,

\[
F_{n,x} = \sigma(\{k2^{-n}, (k+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n}) : 0 \leq k < 2^n, \ 0 \leq l < 2^n\}
\]

where \( \sigma(H) \) denotes the \( \sigma \)-algebra generated by an arbitrary set \( H \). We will investigate martingales of the form \( f = (f_n, n \in \mathbb{N}) \) with respect to \((F_{n,x}, n \in \mathbb{N})\). We briefly write \( L_p \) instead of the real \( L_p([0,1]^2, \lambda) \) space while the norm (or quasi-norm) of this space is defined by \( \|f\|_p := (\int_{[0,1]^2} |f|^p \, d\lambda)^{1/p} \) \((0 < p \leq \infty)\).

The distribution function of a Borel-measurable function \( f \) is defined by

\[
\lambda(\{ f > \gamma \}) := \lambda(\{ x : |f(x)| > \gamma \}) \quad (\gamma \geq 0).
\]

The weak \( L_p \) space \( L^*_p \) \((0 < p < \infty)\) consists of all measurable functions \( f \) for which

\[
\|f\|_{L^*_p} := \sup_{\gamma > 0} \lambda(\{ |f| > \gamma \})^{1/p} \leq \infty,
\]

while we set \( L^*_\infty = L_\infty \).

The spaces \( L^*_p \) are special cases of the more general Lorentz spaces \( L_{p,q} \).

In their definition another concept is used. For a measurable function \( f \) the non-increasing rearrangement is defined by

\[
\check{f}(t) := \inf(\gamma : \lambda(\{ |f| > \gamma \}) \leq t).
\]

The Lorentz space \( L_{p,q} \) is defined as follows: for \( 0 < p < \infty \) and \( 0 < q < \infty \),

\[
\|f\|_{p,q} := \left( \int_0^\infty (\int_t^\infty \check{f}(s)^q \, ds)^{p/q} \, dt^{1/q} \right)^{1/q},
\]
while for $0 < p \leq \infty$,
\[ \|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \tilde{f}(t). \]

Let
\[ L_{p,q} := L_{p,q}([0,1)^d, \lambda) := \{ f : \|f\|_{p,q} < \infty \}. \]

One can show the following equalities:
\[ L_{p,p} = L_p, \quad L_{p,\infty} = L_p^\ast \quad (0 < p \leq \infty) \]
(see e.g. Bennett and Sharpley [1] or Bergh and Lőrström [2]).

The maximal function of a martingale $f = (f_{n,n}, n \in \mathbb{N})$ is defined by
\[ f^\ast := \sup_{n \in \mathbb{N}} |f_{n,n}|. \]

It is easy to see that, in case $f \in L_1$, the maximal function can also be given by
\[ f^\ast(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{\lambda(I_{n,n}(x,y))} \int_{I_{n,n}(x,y)} f \, d\lambda. \]

For $0 < p, q \leq \infty$ the martingale Hardy–Lorentz space $H_{p,q}$ consists of all martingales $f = (f_{n,n}, n \in \mathbb{N})$ for which
\[ \|f\|_{H_{p,q}} := \|f^\ast\|_{p,q} < \infty. \]

Note that in case $p = q$ the usual definition of Hardy space $H_{p,p} = H_p$ is obtained.

It is well known that for a martingale $f = (f_{n,n}, n \in \mathbb{N})$,
\[ \sup_{\gamma > 0} \gamma \lambda(f^\ast > \gamma) \leq \sup_{n \in \mathbb{N}} \|f_{n,n}\|_1 \]
and
\[ \|f^\ast\|_p \leq \frac{p}{p-1} \|f\|_p \quad (1 < p \leq \infty), \]
hence $H_p \sim L_p$ whenever $1 < p \leq \infty$ (see Neveu [8]), where $\sim$ denotes the equivalence of the norms and spaces. Moreover, it is proved in Weisz [16] that
\[ H_{p,q} \sim L_{p,q} \quad (1 < p \leq \infty, 0 < q \leq \infty). \]

A bounded measurable function $a$ is a $p$-atom if $a = 1$ or there exists a dyadic square $Q$ such that
(i) $\int_Q a \, d\lambda = 0$,
(ii) $\|a\|_\infty \leq \lambda(Q)^{-1/p}$,
(iii) $\{a \neq 0\} \subset Q$.

Using the atomic decomposition we verified the next theorem in [14].

**Theorem A.** Suppose that the operator $T$ is sublinear and, for each $p_0 \leq p \leq 1$, there exists a constant $C_p > 0$ such that
\[ \int_{Q} |Ta|^p \, d\lambda \leq C_p \]
for every $p$-atom $a$ where the support of $a$ is contained in $Q$ as in (i)–(iii). If $T$ is bounded from $L_\infty$ to $L_\infty$ then for every $p_0 \leq p \leq 1$,
\[ \|Tf\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p \cap L_1). \]

The following interpolation result concerning Hardy–Lorentz spaces will be used in this paper (see Weisz [16]).

**Theorem B.** If a sublinear operator $T$ is bounded from $H_{p_0}$ to $L_{p_0}$ and from $L_{\infty}$ to $L_{\infty}$ then it is also bounded from $H_{p,q}$ to $L_{p,q}$ if $p_0 < p < \infty$ and $0 < q \leq \infty$.

3. The two-dimensional dyadic derivative. First we introduce the Walsh system. Every point $x \in [0,1)$ can be written in the following way:
\[ x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \quad 0 \leq x_k < 2, \quad x_k \in \mathbb{N}. \]

In case there are two different forms, we choose the one for which $\lim_{k \to \infty} x_k = 0$.

The functions
\[ r_n(x) := \exp(\pi x_n \sqrt{-1}) \quad (n \in \mathbb{N}) \]
are called Rademacher functions. The product system generated by these functions is the one-dimensional Walsh system:
\[ w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \]
where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \leq n_k < 2$ and $n_k \in \mathbb{N}$.

The Kronecker product $(w_{n,m}; n, m \in \mathbb{N})$ of two Walsh systems is said to be the two-dimensional Walsh system. Thus
\[ w_{n,m}(x,y) := w_n(x)w_m(y). \]

Recall that the Walsh Dirichlet kernels
\[ D_n := \sum_{k=0}^{n-1} w_k \]
satisfy
\[ D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0, 2^{-n}), \\
0 & \text{if } x \in [2^{-n}, 1), \end{cases} \]
for $n \in \mathbb{N}$ (see e.g. Schipp, Wade, Simon and Pál [13]).
For each function \( f \) defined on \([0, 1]^2\) Butzer and Engels [3] introduced the concept of the two-dimensional dyadic derivative by (4). Then \( f \) is said to be \textit{dyadically differentiable} at \( x, y \in [0, 1] \) if \( (d_{n,m} f)(x, y) \) converges as \( n, m \to \infty \). It was verified by Butzer and Wagner [4] that every Walsh function is dyadically differentiable and

\[
\lim_{n, m \to \infty} d_{n,m}(w_k \times w_l)(x, y) = kl(w_k \times w_l)(x, y)
\]

for all \( x, y \in [0, 1] \) and \( k, l \in \mathbb{N} \). Let \( W \) be the function whose Walsh-Fourier coefficients satisfy

\[
\overline{W}(k) := \int_0^1 W w_k \, d\lambda := \begin{cases} 1 & \text{if } k = 0, \\
1/k & \text{if } k \in \mathbb{N}, \ k \neq 0.
\end{cases}
\]

The two-dimensional dyadic integral of \( f \in L_1 \) is introduced by

\[
I_f(x, y) := f \ast (W \times W)(x, y) := \int_0^1 \int_0^1 f(t, u)W(x + t)W(y + u) \, dt \, du.
\]

Notice that \( W \in L_2 \subset L_1 \), so \( I \) is well defined on \( L_1 \).

Set

\[
W_k := \sum_{n=2^k}^{\infty} w_n/n
\]

and let us estimate \( d_nW \) and \( d_nW_K \). The following theorem can be proved with the help of the ideas in Schipp, Wade, Simon and Pál [13] (pp. 272–275) and in Weiss [16].

**Theorem 1.** For all \( n, K \in \mathbb{N} \) we have

\[
|d_nW(x) + 1| \leq C \sum_{i=1}^4 F_{0,n}^i(x) \quad \text{and} \quad |d_nW_K(x)| \leq C \sum_{i=1}^5 F_{K,n}^i(x)
\]

where

\[
F_{K,n}^1(x) := \frac{1}{2^{K-n} \sqrt{1}} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i)2^{j-n}D_{2^j}(x + 2^{-j-i-1}),
\]

\[
F_{K,n}^2(x) := \frac{1}{2^{K-n} \sqrt{1}} \sum_{j=0}^{n-1} (n-i)2^{i-n}D_{2^i}(x),
\]

\[
F_{K,n}^3(x) := \sum_{j=0}^{n-1} 2^j \sum_{i=n}^{\infty} 2^{-i}D_{2^i}(x + 2^{-j-i-1}) \frac{1}{2^{K-n} \sqrt{1}},
\]

\[
F_{K,n}^0(x) := \sum_{k=0}^{\infty} 2^{-k}D_{2^{k+n}}(x) \frac{1}{2^{K-n} \sqrt{1}}.
\]

4. **Inequalities concerning the two-dimensional dyadic derivative.** Before considering the operator

\[
I_{\alpha, f} := \sup_{|n-m| \leq \alpha} |d_{n,m}(I f)| \quad (f \in L_1)
\]

for any \( \alpha \geq 0 \) let us modify slightly the dyadic derivative. Set

\[
\delta_{n,m} f(x, y) := \int_0^1 \int_0^1 f(t, u)[d_nW(x + t) + 1][d_mW(y + u) + 1] \, dt \, du
\]

and

\[
J_{\alpha} f := \sup_{|n-m| \leq \alpha} |\delta_{n,m} f| \quad (f \in L_1).
\]

First we can prove that \( J_{\alpha} f \) is bounded from \( H_p \) to \( L_p \).

**Theorem 2.** There exist constants \( C_p \) depending only on \( p \) and \( \alpha \) such that for each \( 2/3 < p \leq 1 \),

\[
\|J_{\alpha} f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p)
\]

where \( J_{\alpha} f \) will be defined for \( f \in H_p \setminus L_1 \) in the proof.

**Proof.** First assume that \( f \in H_p \cap L_1 \). By Theorem A the proof of Theorem 2 will be complete if we show that the operator \( J_{\alpha} f \) satisfies (6) and is bounded from \( L_{\infty} \) to \( L_1 \).

Since \( \|D_{2^n}\|_1 = 1 \), we can show that

\[
\|F_{0,0}\|_1 \leq C \quad (i = 1, \ldots, 4; \ n \in \mathbb{N}).
\]

From this it follows that \( \|d_nW + 1\|_1 \leq C \) for all \( n \in \mathbb{N} \), which verifies that \( J_{\alpha} f \) is bounded on \( L_{\infty} \).

If \( n = 1 \) then the left hand side of (6) is zero. Let \( \alpha \neq 1 \) be an arbitrary \( \alpha \)-atom with support \( Q = J \times J \) and \( \lambda(J) = \lambda(J) = 2^{-2J} \) \( \lambda \in \mathbb{N} \). Without loss of generality we can suppose that \( f = f = [0, 2^{-2J}] \). If \( k < 2^K \) and \( I < 2^K \) then \( w_{n, i} \) is constant on \( Q \) and so

\[
\int_0^1 \int_0^1 a(t, u)w_k(x + t)w_l(y + u) \, dt \, du = 0.
\]

Since

\[
d_n(w_{2^{2n} + k}) = k w_{2^{2n} + k} \quad (0 \leq k < 2^n; \ n \in \mathbb{N})
\]
(see Schipp, Wade, Simon and Pál [13], p. 272) it is not hard to see that
\[
\epsilon_{n,m} a(x, y) = \int_0^1 \int_0^1 a(t, u)[d_n W_K(x + t)(d_m W(y + u) + 1) + (d_n W(x + t) + 1)d_m W_K(y + u) - d_n W_K(x + t)d_m W_K(y + u)] dt du.
\]

By the fact that \(F_{i,n}^1 \leq F_{0,n}^i (i = 1, \ldots, 4; n, K \in \mathbb{N})\) and by Theorem 1 we obtain
\[
J_{a}^* \leq \sup_{|n-m| \leq \alpha} \sup_{|n-m| \leq \alpha} |a| * F_{i,n}^1 \times F_{0,m}^i
\]
\[
+ \sup_{|n-m| \leq \alpha} \sup_{|n-m| \leq \alpha} |a| * F_{i,n}^1 \times F_{K,m}^i
\]
\[
+ \sup_{|n-m| \leq \alpha} \sup_{|n-m| \leq \alpha} |a| * F_{K,n}^i \times F_{i,m}^1
\]
\[
\leq 2 \sup_{|n-m| \leq \alpha} \sup_{|n-m| \leq \alpha} |a| * F_{0,n}^i \times F_{0,m}^i
\]
\[
+ 2 \sup_{|n-m| \leq \alpha} \sup_{|n-m| \leq \alpha} |a| * F_{i,n}^1 \times F_{K,m}^i + \sup_{|n-m| \leq \alpha} |a| * F_{K,n}^i \times F_{K,m}^i.
\]

Now we investigate the first term, the integral of \(|\sup_{|n-m| \leq \alpha} |a| * F_{K,n}^i \times F_{0,m}^i| \) over \([0, 1)^2 \setminus Q\) for all \(i = 1, \ldots, 5\) and \(j = 1, \ldots, 4\).

Step 1: Integrating over \(((0, 1) \setminus I) \times J\). We proved in [15] that for all \(n, K \in \mathbb{N}\) and \(i = 1, \ldots, 5,\)
\[
(9) \quad \left( \sup_{j \in J} \left( \int_{m \in \mathbb{N}} F_{K,n}^i(x + t) dt \right)^p \right) \leq C_p 2^{-K}
\]
where \(I^c := [0, 1) \setminus I\). Taking into account (8) and the definition of the \(p\)-atom, we can establish that, for all \(i = 1, \ldots, 5\) and \(j = 1, \ldots, 4,\)
\[
(10) \quad \int_{J \times J} \left( \sup_{j \in J} \left( \int_{m \in \mathbb{N}} F_{K,n}^i(x + t) F_{i,m}^j(y + u) dt du \right)^p \right) dx dy
\]
\[
\leq C_p 2^{2K} \int_{J \times J} \left( \sup_{j \in J} \left( \int_{m \in \mathbb{N}} F_{K,n}^i(x + t) dt \right)^p \right) dx dy \leq C_p.
\]

Step 2: Integrating over \(I \times ((0, 1) \setminus J)\). If \(j < K\) and \(x \in I\) then \(x + 2^{j-1} \notin I\). Hence, it follows from (7) that
\[
\int_I D_2(x + t + 2^{j-1}) dt = 0
\]
whenever \(x \in I\) and \(j \geq j\). Using this and (7) we can calculate the integrals
\[
\int_I F_{K,n}^i(x + t) dt = \frac{1}{2^{K-n}} \int_I \left( \sum_{j=0}^{n-1} \sum_{\alpha=0}^{n-1} (n-i) 2^{j-\alpha} \right) D_2(x + t + 2^{j-1}) dt
\]
\[
\leq \left\{ \begin{array}{ll}
0 & \text{if } n \leq K, \\
C & \text{if } n > K,
\end{array} \right.
\]
\[
\leq \left\{ \begin{array}{ll}
C 2^{n-K} & \text{if } n \leq K, \\
C & \text{if } n > K,
\end{array} \right.
\]
\[
\int_I F_{K,n}^i(x + t) dt = \frac{1}{2^{K-n}} \int_I \sum_{j=0}^{n-1} \sum_{\alpha=0}^{n-1} (n-i) 2^{j-\alpha} D_2(x + t + 2^{j-1}) dt
\]
\[
\leq \left\{ \begin{array}{ll}
0 & \text{if } n \leq K, \\
C & \text{if } n > K,
\end{array} \right.
\]
Let \(r \in \mathbb{N}\) satisfy \(r - 1 < \alpha \leq r\) and observe that
\[
\sup_{|n-m| \leq \alpha} |a| * F_{K,n}^i \times F_{0,m}^j \leq \sup_{|n-m| \leq \alpha} |a| * F_{K,n}^i \times F_{0,m}^j
\]
\[
+ \sup_{n,m \geq K-r} |a| * F_{K,n}^i \times F_{0,m}^j =: (A_{i,j}) + (B_{i,j}).
\]
for all \(i = 1, \ldots, 5\) and \(j = 1, \ldots, 4\). Of course, \((A_{i,j})(x,y) = 0\) if \(i = 1, 3, 5\) and \(x \in I\). So suppose that \(i = 2, 4\) and \(j = 1, \ldots, 4\). It is easy to see that
\[
2^{m-K} F_{0,m}^j \leq F_{K,m}^j \quad (m \leq K; \ j = 1, \ldots, 4).
\]
Consequently,
\[
(A_{i,j})(x, y) = \sup_{|n-m| \leq r} \int \int |a(t, u)| F_{K, n}(x + t) F_{m, n}(y + u) \, dt \, du
\leq C_p 2^{2K/p} \sup_{|n-m| \leq r} 2^{n-K} \int F_{m, n}(y + u) \, du
\leq C_p 2^{2K/p} \sup_{m \leq K} 2^{m-K} \int F_{m, n}(y + u) \, du
\leq C_p 2^{2K/p} \sup_{m \leq K} \int F_{m, n}(y + u) \, du.
\]

Then the inequality
\[
\left( \int \int (A_{i,j})^p \, d\lambda \right)^{1/p} \leq C_p
\]
can be proved as in (10) where \( i = 1, \ldots, 5 \) and \( j = 1, \ldots, 4 \).

Since \( F_{m, n} = F_{n, m} \) for \( m > K \), (11) yields that
\[
F_{i, j} \leq 2^{n-i} F_{K, n} \quad (m \geq K - r; \ j = 1, \ldots, 4).
\]
Then, for each \( i = 1, \ldots, 5 \) and \( j = 1, \ldots, 4 \),
\[
\int \int (B_{i,j})^p \, d\lambda \leq C_p 2^{2K/p} \int \int \left( \sup_{|n-m| \leq r} 2^{n-K} \int F_{m, n}(y + u) \, du \right)^p \, dx \, du \leq C_p
\]
as we have seen in (10).

Step 3: Integrating over \((0, 1) \setminus I \times (0, 1) \setminus J\). By (7) it is easy to verify that, for \( x \not\in I \),
\[
\int D_{2^i}(x + t + 2^{-j-1}) \, dt = 2^{n-K} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x)
\]
if \( j < i \leq K - 1 \),
\[
\int D_{2^i}(x + t) \, dt = 2^{n-K} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x)
\]
if \( i \in \mathbb{N} \) and
\[
\int D_{2^i}(x + t + 2^{j-1}) \, dt = 2^{n-K} 1_{[2^{j-1} - 2^{j-1} - 1]}(x)
\]
if \( i \geq K \).

Now we modify slightly the kernel functions \( F_{K, m} \) \((i = 1, \ldots, 4)\) and calculate their integrals like (9). By (15),
\[
\left( \sup_{|n-m| \leq r} \int 2^{n/2} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i)2^{j-n} D_{2^i}(x + t + 2^{-j-1}) \, dt \right)^p \, dx
\]
\[
= C_p 2^{-Kp} \int \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx
\]
\[
\leq C_p 2^{-Kp} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx.
\]

Since the function \( f(n) = (n/2)^{2-n/2} \) is decreasing for \( n \geq 3 \), we obtain
\[
\left( \sup_{|n-m| \leq r} 2^{n/2} \int \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i)2^{j-n} D_{2^i}(x + t + 2^{-j-1}) \, dt \right)^p \, dx
\]
\[
\leq C_p 2^{-Kp} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx
\]
\[
\leq C_p 2^{-Kp} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} 2^{n/2} \int \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx.
\]

Provided that \( 2/3 < p \leq 1 \).

Using (16) we get
\[
\int \int (B_{i,j})^p \, d\lambda \leq C_p 2^{2K/p} \int \int \left( \sup_{|n-m| \leq r} 2^{n-K} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx \, du
\]
\[
\leq C_p 2^{-Kp} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} 2^{n/2} \int \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx.
\]

It follows from (15) and (17) that
\[
\int \int (A_{i,j})^p \, d\lambda \leq C_p 2^{2K/p} \int \int \left( \sup_{|n-m| \leq r} 2^{n-K} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx \, du
\]
\[
\leq C_p 2^{-Kp} \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} 2^{n/2} \int \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{i=j+1}^{n-1} \frac{(n-i)2^{j-n}2^{i-K}}{2} 1_{[2^{-j-1} + 2^{-j-1} - 1]}(x) \right)^p \, dx.
\]
\[ \leq \sum_{j=0}^{K-1} 2^{jp} \sum_{i=0}^{K-1} 2^{(p/2-1)j-Kp} + \sum_{i=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip/2-K} \]
\[ \leq C_p 2^{Kp} \sum_{j=0}^{K-1} 2^{3(p/2-1)} + C_p 2^{Kp} 2^{Kp} 2^{-Kp/2} \leq C_p 2^{Kp}/2 - K. \]

Similarly,
\[ \left( \sup_{I \subset \alpha \leq K} \left( \sum_{\substack{n \leq K \ \text{even} \ \text{or} \ \text{odd}}} \sum_{j=0}^{n-1} 2^{n/2} \sum_{i=0}^{\infty} 2^{-i-K} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ \leq \left( \sup_{I \subset \alpha \leq K} \left( \sum_{\substack{n \leq K \ \text{even} \ \text{or} \ \text{odd}}} \sum_{j=0}^{n-1} 2^{n/2} \sum_{i=0}^{\infty} 2^{-i-K} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ + \left( \sup_{I \subset \alpha \leq K} \left( \sum_{\substack{n \leq K \ \text{even} \ \text{or} \ \text{odd}}} \sum_{j=0}^{n-1} 2^{n/2} \sum_{i=0}^{\infty} 2^{-i} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ \leq C_p 2^{Kp} \sum_{j=0}^{K-1} 2^{3(p/2-1)} + C_p 2^{Kp} 2^{Kp} 2^{-Kp/2} \leq C_p 2^{Kp}/2 - K \]

whenever \( p < 1 \). If \( p = 1 \) then
\[ \sum_{j=0}^{K-1} 2^{jp/2} - Kp = \sum_{j=0}^{K-1} 2^{(j-K)/2} (K-j)2^{-3K/2} \leq C 2^{2-K} \]
and (21) is true in this case, too.

Obviously, if \( x \notin I \) and \( i \geq K \) then
\[ \int_I D_{2^i}(x + t) dt = 0. \]

This implies that
\[ \int_I \left( \sup_{n \leq K} \left( \sum_{k=0}^{K-n-1} 2^{n/2} 2^{-(k+1)} 2^{K-n+1} K \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ = \int_I \left( \sup_{n \leq K} \left( \sum_{k=0}^{K-n-1} 2^{n/2} 2^{-K-n+1} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ \times \sum_{k=0}^{n-1} \sum_{m=1}^{n-1} (m-l)2^{-m} D_{2^l}(y + u + 2^{-l-1}) dt du \]
\[ = 2^{-Kp} \int_I \left( \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3K-n-1} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p dx \]
\[ + \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3(3K-n-1)} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p dx \]
\[ \leq C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{n(3K-n-1)2^{-k}} + C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3K-n-1} 2^{3K-n-1} 2^{-Kp/2} \leq C_p 2^{Kp}/2 - K. \]

In the same way we conclude that
\[ \int I \left( \sup_{n \leq K} \left( \sum_{k=0}^{K-n-1} 2^{n/2} 2^{-(k+1)} 2^{K-n+1} K \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ = \int_I \left( \sup_{n \leq K} \left( \sum_{k=0}^{K-n-1} 2^{n/2} 2^{-K-n+1} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ \leq C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{n(3K-n-1)2^{-k}} + C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3K-n-1} 2^{3K-n-1} 2^{-Kp/2} \leq C_p 2^{Kp}/2 - K. \]

Now we are ready to deal with the integrals of \((A_{1,1})^p\) over \(I \times J\)
\((i = 1, \ldots, 5; j = 1, \ldots, 4)\). We investigate only three terms, \((A_{1,1}), (A_{1,2}), \text{and} (A_{3,1}),\) because the others are all similar. Applying (18) twice we obtain
\[ \int_{I \times J} (A_{1,1})^p d\lambda \]
\[ = \int_I \left( \sup_{n \leq K} \left( \sum_{k=0}^{n-n-1} \sum_{m=1}^{n-1} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p \right) dx \]
\[ \times \sum_{k=0}^{m-1} \sum_{m=1}^{n-1} (m-l)2^{n-m} D_{2^l}(y + u + 2^{-l-1}) dt du \]
\[ = 2^{-Kp} \int_I \left( \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3K-n-1} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p dx \]
\[ + \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3(3K-n-1)} \int_0^1 \frac{d^2t}{2^{K-i} \sqrt{1}} \right)^p dx \]
\[ \leq C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{n(3K-n-1)2^{-k}} + C_p 2^{2-K} \sum_{k=0}^{K-n-1} \sum_{m=1}^{n-1} 2^{3K-n-1} 2^{3K-n-1} 2^{-Kp/2} \leq C_p 2^{Kp}/2 - K. \]
Two-dimensional dyadic derivative

Observe that \((A_{k,j})(x, y) = 0 \) (\(j = 1, \ldots, 4\)) follows from the definition.

(13) and (9) imply that

\[
\int_{j_1} \int_{j_2} (B_{i,j})^p \, d\lambda 
\leq C_p \int_{j_1} \int_{j_2} \left( \sup_{n, m \leq K_i} \sum_{n, m \leq K_i} -1 \sum_{j=0}^{n-2} \sum_{j=0}^{m-2} (m - l) 2^{j-1} \ldots \right) dt \, dx \, dy
\]

\[
\times \left( \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} (m - l) 2^{j-1} \ldots \right) dy
\]

\[
\leq C_p \frac{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}}{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}} = C_p.
\]

By (18) and (20),

\[
\int_{j_1} \int_{j_2} (A_{k,j})^p \, d\lambda
= \int_{j_1} \int_{j_2} \left( \sup_{n, m \leq K_i} \sum_{n, m \leq K_i} (m - l) 2^{j-1} \ldots \right) dt \, dx \, dy
\]

\[
\times \left( \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} (m - l) 2^{j-1} \ldots \right) dy
\]

\[
= C_p \frac{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}}{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}} = C_p.
\]

Similarly, using (18) and (21) we can see that

\[
\int_{j_1} \int_{j_2} (A_{k,j})^p \, d\lambda
= \int_{j_1} \int_{j_2} \left( \sup_{n, m \leq K_i} \sum_{n, m \leq K_i} (m - l) 2^{j-1} \ldots \right) dt \, dx \, dy
\]

\[
\times \left( \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} (m - l) 2^{j-1} \ldots \right) dy
\]

\[
\leq C_p \frac{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}}{2^{K_1} \cdot 2^{K_2} \cdot 2^{K_3}} = C_p.
\]

The next corollary follows from (5) and from Theorems B and 2.

**COROLLARY 1.** There are absolute constants \(C_1\) and \(C_{p,q}\) such that

\[
\|J_\alpha F\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}} \quad (f \in H_{p,q})
\]

for every \(2/3 < p < \infty\) and \(0 < q \leq \infty\). In particular, \(J_\alpha F\) is of weak type \((L_1, L_1)\), i.e. if \(f \in L_1\) then

\[
\|J_\alpha F\|_{1,\infty} = \sup_{\gamma > 0} \gamma \lambda_{(J_\alpha F) > \gamma} \leq C_1 \|f\|_{H_{1,\infty}}
\]

\[
= C_1 \sup_{\gamma > 0} \gamma \lambda(f > \gamma) \leq C_1 \|f\|_{1,\infty}.
\]

Now we can state our main result.

**COROLLARY 2.** Suppose that for a martingale \(f = (f_{n,n}, n \in \mathbb{N}) \in H_{p,q}\) we have \(\int_0^1 f_{n,n} (x, y) \, dx \, dy = \int_0^1 f_{n,n} (x_0, y) \, dy = 0\) for each \(n \in \mathbb{N}\) and almost every \(x_0, y \in [0, 1]\). Then

\[
\|J_\alpha f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}}
\]
for every $2/3 < p < \infty$ and $0 < q \leq \infty$. In particular, $\mathbf{I}_p^*$ is of weak type $(L_1, L_1)$, i.e. if $f \in L_1$ such that $\int_0^1 f(x, y_0) \, dx = \int_0^1 f(x_0, y) \, dy = 0$ for almost every $x_0, y_0 \in [0, 1]$ then

$$\sup_{\gamma > 0} \gamma \lambda(\mathbf{I}_p^* f > \gamma) \leq C_1 \|f\|_{L_1} \leq C_1 \|f\|_p.$$  

Proof. By the proof of Theorem 2 it is enough to verify the corollary for integrable functions. Let $f \in L_1$ such that $\int_0^1 f(x, y_0) \, dx = \int_0^1 f(x_0, y) \, dy = 0$ for almost every $x_0, y_0 \in [0, 1]$. Then it is easy to see that

$$d_{n,m}(f)(x, y) = d_{n,m} \left( \int_0^1 f(t, u) W(x + t) W(y + u) \, dt \, du \right)$$

$$= \int_0^1 \int_0^1 f(t, u) d_n W(x + t) d_m W(y + u) \, dt \, du$$

$$= \delta_{n,m}(x, y).$$

Hence $\mathbf{I}_p^* f = J_p^* f$ and the result follows from Corollary 1. 

The next corollary follows from the weak type inequality in Corollary 2 and from the fact that the Walsh polynomials are dense in $L_1$.

**Corollary 3.** If $\alpha \geq 0$ is arbitrary and if $f \in L_1$ is such that

$$\int_0^1 f(x, y_0) \, dx = \int_0^1 f(x_0, y) \, dy = 0$$

for almost every $x_0, y_0 \in [0, 1]$ then

$$d_{n,m}(f) \rightharpoonup f \quad \text{a.e. as } n, m \to \infty \text{ and } |n - m| \leq \alpha.$$

We remark that this corollary is also proved by Gáv [7].

Finally, we note that without the condition $\int_0^1 f(x, y_0) \, dx = \int_0^1 f(x_0, y) \, dy = 0$ we can prove Corollary 2 only for $p \geq 1$; more exactly:

**Theorem 3.** There are absolute constants $C_1$ and $C_{p,q}$ such that

$$\|\mathbf{I}_p^* f\|_1 \leq C_1 \|f\|_{H_1} \quad (f \in H_1)$$

and

$$\|\mathbf{I}_p^* f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}} \quad (f \in H_{p,q})$$

for every $1 < p < \infty$ and $0 < q \leq \infty$.

Proof. We can apply only the second inequality of Theorem 1. That is to say, we have to investigate the terms $\sup_{|n - m| \leq \alpha} |a| \ast F_{K,n} \times F_{0,m}$ 

$(i, j = 1, \ldots, 5).$ If $j \neq 5$ then they are considered in the proof of Theorem 2. If $j = 5$ then

$$|a| \ast F_{K,n} \times F_{0,m}(x, y) = \int \int |a(t, u)| F_{K,n}(x + t) F_{0,m}(y + u) \, dt \, du$$

$$\leq 2^{2K} 2^{-K} \int F_{K,n}(x + t) \, dt$$

where $a$ is a 1-atom with support $Q = I \times J$, $I = J = [0, 2^{-K})$. Applying (9), we get

$$\int \int \sup_{|n - m| \leq \alpha} |a| \ast F_{K,n} \times F_{0,m}(x, y) \, dx \, dy$$

$$\leq 2^K \int \int \sup_{n \in \mathbb{N}} F_{K,n}(x + t) \, dt \, dz \leq C.$$

We get the same result if we integrate over $I^c \times J$ or $I \times J^c$. Hence the condition (6) is verified for $p = 1$; this means that the first inequality in Theorem 3 is proved. The second inequality follows by interpolation. 

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Index of Volumes 111–120

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