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Acyclic inductive spectra of Fréchet spaces

by

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Abstract. We provide new characterizations of acyclic inductive spectra of Fréchet spaces which improve the classical theorem of Palamodov and Retakh. It turns out that acyclicity, sequential retractivity (defined by Floret) and further strong regularity conditions (introduced e.g. by Bierstedt and Meise) are all equivalent. This solves a problem that was folklore since around 1970.

For inductive limits of Fréchet–Montel spaces we obtain even stronger results, in particular, Grothendieck's problem whether regular (LF)-spaces are complete has a positive solution in this case and we show that even the weakest regularity conditions already imply acyclicity.

One of the main benefits from our results is an improvement in the theory of projective spectra of (DFM)-spaces. We prove the missing implication in a theorem of Vogt and thus obtain evaluable conditions for vanishing of the derived projective limit functor which have direct applications to classical problems of analysis like surjectivity of partial differential operators on various classes of ultradifferentiable functions (as was explained e.g. by Braun, Meise and Vogt).

1. Introduction. Given an inductive spectrum $(E_n)_{n\in\mathbb{N}}$ of Fréchet spaces with inductive limit $E = \operatorname{ind}_n E_n$ Palamodov's [21] definition of acyclicity can be rephrased as stating that the short exact sequence

$$0 \to \bigoplus E_n \xrightarrow{\sigma} \bigoplus E_n \xrightarrow{q} E \to 0$$

is topologically exact, where $q((x_n)_{n\in\mathbb{N}}) = \sum x_n$ is the canonical quotient map and $\sigma((x_n)_{n\in\mathbb{N}}) = (x_n - x_{n-1})_{n\in\mathbb{N}}$ $(x_0 := 0)$.

The origin of this definition was the subspace problem in (LF)-spaces (which itself is closely related to surjectivity problems in classical analysis): which subspaces of (LF)-spaces are again inductive limits of Fréchet spaces? Such subspaces are called limit subspaces and there exists a long list of papers dealing with this subject (see e.g. [13] and its list of references). The subspace problem arises in a natural way if one is concerned with the

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question whether a partial differential or convolution operator on a space of distributions is surjective, which turned out to be equivalent to the fact that the range of its transposed operator is a limit subspace in a strict (LF)-space of test functions. This and related questions were studied by Hörmander [17] and his work was a starting point for Palamodov's investigations, because it is quite easy to see that a sequentially closed subspace L of an acyclic (LF)-space $E = \operatorname{ind}_n E_n$ is a limit subspace if and only if the inductive spectrum $(E_n/L \cap E_n)_{n\in\mathbb{N}}$ is again acyclic.

Palamodov [21] and Retakh [23] found the following characterization of acyclicity, which has been reproved by Vogt [29] avoiding the homological language and methods in the original proof.

THEOREM 1.1. *An inductive spectrum $(E_n)_{n\in\mathbb{N}}$ of Fréchet spaces is acyclic if and only if it satisfies Retakh's condition* (M), *i.e. in every space E_n there is an absolutely convex* 0*-neighbourhood U_n with*

1. *$U_n \subseteq U_{n+1}$ for every $n \in \mathbb{N}$ and*
2. *for every $n \in \mathbb{N}$ there is $m > n$ such that all topologies of the spaces E_k, $k > m$, coincide on U_n.*

Our main result (see Theorem 2.7, (4)⇒(2) below) states that the first requirement in Retakh's condition (M) can be dropped. This has been known only for (LF)-sequence spaces [29] and weighted inductive limits of continuous functions [2]. The proof for the general case is surprisingly elementary but it is based on a long development of the theory and has remarkable consequences for the general theory as well as for applications e.g. to distribution theory.

Besides acyclicity there are many other regularity conditions for (LF)-spaces, which were needed since, in general, many pathologies may appear, e.g. an inductive limit need not be Hausdorff even if the defining steps are Banach or nuclear Fréchet spaces.

K. Floret [11] studied properties of bases in inductive limits $E = \operatorname{ind}_n E_n$. To do this he defined E to be sequentially retractive if every null sequence in E already tends to 0 in some step. He investigated this property thoroughly in [12], where he proved a useful factorization theorem and showed that sequentially retractive (LF)-spaces are sequentially complete. As a consequence of our main result they are even complete.

A stronger regularity condition—bounded retractivity, see Definition 2.1 —was introduced by Bierstedt and Meise [3] who dealt with approximation properties of inductive limits.

For different purposes many other regularity conditions were defined; we refer to the appendix of Chapter 3 in Bierstedt's survey article [1].

Neus [19] studied all these conditions and he showed that they are equivalent for inductive limits of normed spaces.

In the second chapter of the present note we show that the equivalence of all the mentioned "strong regularity" conditions holds in (LF)-spaces because they are equivalent to acyclicity. This solves a problem that was explicitly stated e.g. in the book of Bonet and Pérez Carreras [22, Problem 13.8.7].

The third part deals with inductive limits of Fréchet–Montel spaces and we prove that in this setting already regularity implies acyclicity and completeness, in particular, Grothendieck's question [16, questions non résolues 9] whether quasi-complete (LF)-spaces are complete has a positive answer for this class. Moreover, these properties are characterized by a condition (wQ) due to Vogt, which is appropriate for evaluation in concrete cases.

As a main application we finally consider projective spectra of (DF)-spaces and in particular the derived projective limit functor, a construction due to Palamodov [20] and further developed by Vogt [27, 28]. Our Theorem 3.5 below is very helpful in the investigations of the surjectivity of convolution or partial differential operators defined on spaces of ultradifferentiable functions of Roumieu type (for example, on spaces of real-analytic functions, Gevrey classes etc.) as was shown e.g. in [5, 6], [7, Prop. 3.3], [8, Prop. 1.9].

In fact, the surjectivity of such an operator is equivalent to the vanishing of $\operatorname{Proj}^1 \mathcal{X}$, where $\mathcal{X}$ is a projective spectrum of (DFM)-spaces and the projective limit is the kernel of the considered operator. As a consequence of our result this condition is equivalent to a Phragmén–Lindelöf condition (introduced in this context, for example, in [18]) on the zero variety of the polynomial associated with the operator. Partial results of that type have been obtained so far with specialized "hard analysis" proofs (see e.g. [5]). The further evaluation of the Phragmén–Lindelöf conditions still requires more analytic work (see [5, 18]) but the result itself is immediate (for example, 3.5 below saves much of the work in [5]).

Furthermore, the derived projective limit functor is the main tool in the splitting theory for Fréchet spaces developed by Palamodov [21, §9] and Vogt [26]. However, in that situation the projective spectra never consist of (DFM)-spaces, so that our theorem is not directly applicable. We refer to [14, 15], where the methods of the present article are modified to get the desired splitting results.

Our notation for locally convex spaces (l.c.s.) is standard (like e.g. in [22]); note that $\mathcal{U}_0(E)$ always means the system of *absolutely convex* 0-neighbourhoods of a l.c.s. E.

2. Retractive (LF)-spaces. Recall that an (LF)-space is *regular* if every bounded set is contained and bounded in one of the steps. We now define some stronger regularity conditions.

DEFINITION 2.1. Let $(E,\mathcal{T}) = \operatorname{ind}_n(E_n,\mathcal{T}_n)$ be an inductive limit of l.c.s. The defining spectrum $(E_n)_{n\in\mathbb{N}}$ is said to be

1. *sequentially retractive* [11] if every null sequence in E converges to zero in some step,
2. *boundedly retractive* [3] if for every bounded set B in E there is $n \in \mathbb{N}$ such that B is contained in E_n and the topologies $\mathcal{T}$ and $\mathcal{T}_n$ coincide on B,
3. (*sequentially*) *compactly regular* [3] if every (sequentially) compact subset of the inductive limit is (sequentially) compact in some step.

By Grothendieck's factorization theorem [16, théorème A], for an (LF)-space all defining spectra of Fréchet spaces are equivalent. This justifies calling an (LF)-space sequentially or boundedly retractive or (sequentially) compactly regular if one (and then each) defining spectrum has this property.

Cascales and Orihuela [9] proved that precompact sets in (LF)-spaces are metrizable and therefore sequentially compactly regular, and sequentially retractive (LF)-spaces are already compactly regular.

It is an important result due to Palamodov [21, Theorem 6.2] and Valdivia [25, Chap. 1, §9, 5 (3)] (the latter for inductive limits of arbitrary l.c.s.) that if an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods satisfies the requirements of condition (M) in Theorem 1.1 then even the limit topology coincides with almost all step topologies on these neighbourhoods. However, the assumption that the sequence $(U_n)_{n\in\mathbb{N}}$ is increasing is essential for their proofs.

The following result—essentially due to Palamodov [21, Cor. 7.1], cf. [29, Thm. 3.2, Cor. 3.3]—is of particular importance for the theory.

PROPOSITION 2.2. *Acyclic (LF)-spaces are complete and boundedly retractive.*

Combined with 1.1 this implies that (LF)-spaces with (M) are sequentially retractive. On the other hand, the following contribution of C. Fernández [10] shows that sequentially retractive (LF)-spaces "nearly" satisfy (M).

PROPOSITION 2.3. *Every sequentially retractive (LF)-space $E = \operatorname{ind}_n E_n$ satisfies* (Q), *i.e. for every $n \in \mathbb{N}$ there are $U_n \in \mathcal{U}_0(E_n)$ and $m > n$ such that E_m and E_k induce the same topology on U_n for all $k > m$.*

Proof. For the sake of completeness we give a slightly simplified version of the original proof. We show that for every $n \in \mathbb{N}$ there are $U_n \in \mathcal{U}_0(E_n)$ and $m > n$ such that the topologies of E_m and E have the same convergent sequences in U_n. Fix $n \in \mathbb{N}$ and a decreasing basis $(V_k)_{k\in\mathbb{N}}$ of $\mathcal{U}_0(E_n)$ and assume that for every $k \in \mathbb{N}$ there are a sequence $(x_{k,l})_{l\in\mathbb{N}} \in V_k^{\mathbb{N}}$ and $x_k \in V_k$ such that $x_{k,l}$ converges to x_k for $l \to \infty$ in E but not in E_k. Arranging the double indexed sequence $y_{k,l} = x_{k,l} - x_k$ in an arbitrary way into a single indexed sequence it is easy to check that the latter converges to 0 in E and therefore also in some step E_{k_0}. But then also the subsequence $y_{k_0,l}$ converges to 0 in E_{k_0}, a contradiction. ■

To show that condition (Q) in the previous proposition (which also appeared in [29, Proposition 2.3]) already implies (M) we need the following

LEMMA 2.4. *Let X be a vector space and $\mathcal{S}$, $\mathcal{T}$ two locally convex topologies on X. Let A be an absolutely convex subset of X and assume that there is $U \in \mathcal{U}_0(X,\mathcal{S})$ such that $\mathcal{S}|_{U\cap A}$ is coarser than $\mathcal{T}|_{U\cap A}$. Then $\mathcal{S}|_A$ is coarser than $\mathcal{T}|_A$.*

Proof. Since A is absolutely convex we only have to show that $\mathcal{T}$ induces a finer filter of 0-neighbourhoods in A than $\mathcal{S}$. Let $V \in \mathcal{U}_0(X,\mathcal{S})$ be given. Since $V \cap \frac{1}{2}U \in \mathcal{U}_0(X,\mathcal{S})$ there is $W \in \mathcal{U}_0(X,\mathcal{T})$ with $W\cap A\cap U \subseteq V \cap \frac{1}{2}U$. We show that $W \cap A \subseteq V$ (which gives the conclusion).

Let $x \in W \cap A$. If $x \notin U$ there is $n \in \mathbb{N}$ with $\frac{1}{2^n}x \in U$ and $\frac{1}{2^{n-1}}x \notin U$. Since W and A are absolutely convex this implies $\frac{1}{2^n}x \in W\cap A\cap U \subseteq \frac{1}{2}U$. Therefore, $\frac{1}{2^{n-1}}x \in U$, which contradicts the choice of n. We have shown $x \in W\cap A\cap U \subseteq V$. ■

Note that for $A = X$ the previous lemma was obtained already in the seventies by Roelcke [24]. Now we are ready to prove

PROPOSITION 2.5. *Let $E = \operatorname{ind}_n(E_n,\mathcal{T}_n)$ be an inductive limit of l.c.s. with* (Q), *i.e. for every $n \in \mathbb{N}$ there is $U_n \in \mathcal{U}_0(E_n)$ on which residually all topologies $\mathcal{T}_k$ coincide. Then $(E_n)_{n\in\mathbb{N}}$ satisfies* (M) *and is therefore acyclic.*

Proof. It is known and very easy to see that it is enough to establish (M) for a defining spectrum which is a subsequence of the given one. So, we may assume that for every $n, k \in \mathbb{N}$ already $\mathcal{T}_{n+1}|_{U_n} = \mathcal{T}_{n+k}|_{U_n}$ holds.

The coincidence of $\mathcal{T}_2$ and $\mathcal{T}_3$ on U_1 implies that there is $V_3 \in \mathcal{U}_0(E_3)$ with $2V_3 \cap U_1 \subseteq \frac{1}{2}U_2$. Define $\widetilde{U}_2 = U_1 + \left(\frac{1}{2}U_2 \cap V_3\right)$, which is an absolutely convex 0-neighbourhood in E_2 containing U_1.

To show $\mathcal{T}_3|_{\widetilde{U}_2} = \mathcal{T}_{3+k}|_{\widetilde{U}_2}$ for every $k \in \mathbb{N}$ it is—by the previous lemma—enough to show the coincidence of the topologies on $\widetilde{U}_2 \cap V_3$. But

$$\begin{aligned}\widetilde{U}_2 \cap V_3 = \left(U_1 + \left(\tfrac{1}{2}U_2 \cap V_3\right)\right) \cap V_3 &\subseteq (U_1 \cap 2V_3) + \left(\tfrac{1}{2}U_2 \cap V_3\right)\\ &\subseteq \tfrac{1}{2}U_2 + \tfrac{1}{2}U_2 = U_2\end{aligned}$$

and the topologies coincide on U_2 (hence also on the subset $\widetilde{U}_2 \cap V_3$) by assumption. Thus, we have shown $\mathcal{T}_3|_{\widetilde{U}_2} = \mathcal{T}_{3+k}|_{\widetilde{U}_2}$ for every $k \in \mathbb{N}$. Proceeding by induction yields that the inductive spectrum satisfies (M), which implies acyclicity (note that this implication in Theorem 1.1 holds for arbitrary inductive spectra of l.c.s.). ■

Combining 2.3 and 2.5 we arrive at

THEOREM 2.6. *Sequentially retractive* (*LF*)*-spaces are acyclic.*

Collecting all known facts (namely 1.1, 2.2, 2.3 and 2.5) we finally get the following

THEOREM 2.7. *For an* (*LF*)*-space E the following conditions are equivalent*:

(1) *E is acyclic*,
(2) *E satisfies* (M),
(3) *E is sequentially retractive*,
(4) *E satisfies* (Q),
(5) *E is boundedly retractive*,
(6) *E is compactly regular*,
(7) *E is sequentially compactly regular.*

If the steps are only metrizable one has to add in (1), (2) and (4) the requirement that E is regular (because sequentially retractive inductive limits are always regular but even strict inductive limits of normed spaces need not be regular [12]; note that in 2.3 completeness of the steps was not needed). An immediate consequence of the theorem (together with 2.2) is the following improvement of a result due to Floret [12, Korollar 5.3] who showed that sequentially retractive (LF)-spaces are sequentially complete.

COROLLARY 2.8. *Sequentially retractive* (*LF*)*-spaces are complete.*

3. Inductive limits of Fréchet–Montel spaces. In this section we consider the situation where the steps are Fréchet–Montel spaces, which is the most interesting case for applications in distribution theory. By duality we characterize all relevant properties of projective spectra of (DFM)-spaces (which are exactly the strong duals of Fréchet–Montel spaces).

The following notation was introduced by Vogt in [29].

DEFINITION 3.1. An inductive limit $E = \operatorname{ind}_n E_n$ is said to satisfy (wQ) if for every $n \in \mathbb{N}$ there are $U_n \in \mathcal{U}_0(E_n)$ and $m > n$ such that for every $k > m$ and $W \in \mathcal{U}_0(E_m)$ there are $V \in \mathcal{U}_0(E_k)$ and $S > 0$ with $V \cap U_n \subseteq SW$.

Note that without the factor S the condition would mean that E_k and E_m induce the same topology in U_n (i.e. condition (Q)).

If $\| \|_{n,l}$ is a fundamental system of seminorms in the space E_n, (wQ) is equivalent to the following inequalities, which are appropriate for calculations in concrete cases:

$$\forall n\, \exists m > n, N \in \mathbb{N}\ \forall k > m, M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0\ \forall x \in E_n$$
$$\|x\|_{m,M} \le S\,(\|x\|_{k,K} + \|x\|_{n,N}).$$

It was noticed by Bonet and Fernández [4] that in (LF)-spaces, (wQ) is equivalent to the fact that the topologies of E_k and E_m have the same bounded sets in U_n. Using this remark one can easily see that regular (LF)-spaces satisfy (wQ), which was first proved by Vogt [29, Thm. 4.7]. The next proposition improves this result. Recall that an inductive limit is *α-regular* if every bounded set is contained (but not necessarily bounded) in some step, and *β-regular* if every bounded set which is contained in a step is also bounded in some step.

PROPOSITION 3.2. *An* (*LF*)*-space which is either α- or β-regular satisfies* (wQ).

Proof. Let $E = \operatorname{ind}_n E_n$ be an α-regular (LF)-space and $n \in \mathbb{N}$ fixed. Then there is $U_n \in \mathcal{U}_0(E_n)$ and $m > n$ such that $\overline{U}_n^E \subset E_m$. Indeed, let $(V_k)_{k\in\mathbb{N}}$ be a decreasing basis of $\mathcal{U}_0(E_n)$ and assume that for every $k \in \mathbb{N}$ there is x_k in the E-closure of V_k but not contained in E_k. Given $W \in \mathcal{U}_0(E)$ there is $k \in \mathbb{N}$ with $\overline{V}_k^E \subset W$ (because every l.c.s. has a 0-basis of closed sets). This means $x_k \to 0$ in E without being contained in any step, a contradiction. Let now $k > m$ and $B \subseteq U_n$ bounded in E_k. Then $D = \overline{\Gamma(B)}^{E_k}$ is a Banach disc contained in E_m and the identity map $([D], p_D) \to E_m$ has closed graph since both spaces are continuously included in E_k. The closed graph theorem implies that this identity map is continuous and therefore B is bounded in E_m. The above mentioned remark of Bonet and Fernández yields that E satisfies (wQ).

Let now E be β-regular and as before $(V_k)_{k\in\mathbb{N}}$ be a decreasing basis of $\mathcal{U}_0(E_n)$. Assume that for every $k \in \mathbb{N}$ there is $B_k \subseteq V_k$ which is bounded in E but not bounded in E_k. The union $B = \bigcup_{k\in\mathbb{N}} B_k$ is contained in E_n and again bounded in E. Indeed, if $W \in \mathcal{U}_0(E)$ there is $k \in \mathbb{N}$ with $V_k \subseteq W$ and $0 < \lambda \le 1$ with $\lambda \bigcup_{1\le j\le k} B_j \subseteq W$. Hence, $\lambda B \subseteq W$. Now β-regularity implies that B is bounded in some step E_k, a contradiction since $B_k \subseteq B$ is unbounded in E_k. Again [4] implies that E satisfies (wQ). ■

In addition to Theorem 2.7 we now get the following result, which means that for inductive limits of (FM)-spaces all regularity conditions considered in the literature are equivalent.

THEOREM 3.3. *Let $E = \operatorname{ind}_n E_n$ be an inductive limit of Fréchet–Montel spaces. The following conditions are equivalent*:

(1) *E is complete*,
(2) *E is regular*,
(3) *E is α-regular*,
(4) *E is β-regular*,
(5) *E satisfies* (wQ),
(6) *E is acyclic.*

Proof. By Grothendieck's factorization theorem, complete (LF)-spaces are regular. The implications (2)⇒(3) and (2)⇒(4) are trivial, and (3)⇒(5) and (4)⇒(5) are proved above. Let now (5) be satisfied. Using the remark of Bonet and Fernández above we know that there is a sequence of 0-neighbourhoods of the steps in which almost all step topologies have the same bounded sets. Let U_n be an absolutely convex 0-neighbourhood such that all topologies of E_k, $k > m$, have the same bounded sets in U_n. Fix $k > m$ and let $(x_r)_{r\in\mathbb{N}}$ be a sequence in U_n which converges to 0 in the topology of E_k. This sequence is then bounded and hence relatively compact in the topology of E_m. Since on a compact space no coarser Hausdorff topology exists we conclude that the sequence tends to 0 also in E_m. Thus, the topologies of E_k and E_m coincide in U_n. Theorem 2.7 implies acyclicity, which yields completeness by 2.2. ■

The previous theorem has an immediate consequence for the subspace problem described in the introduction.

COROLLARY 3.4. *Let $E = \operatorname{ind}_n E_n$ be an acyclic inductive limit of Fréchet–Schwartz spaces and L a stepwise closed subspace (i.e. $L_n := L \cap E_n$ is closed in E_n for each n). The following conditions are equivalent*:

(1) *L is a limit subspace of E*,

(2) *the spectrum (E_n/L_n) satisfies* (wQ),

(3) *the quotient map $q : E \to E/L$ lifts bounded sets, i.e. every bounded set in the quotient is contained in the image of some bounded set in E.*

Proof. Since quotients of Fréchet–Schwartz spaces are Montel, the equivalence of (1) and (2) follows from 3.3 and the remarks preceding 1.1.

(2)⇒(3). By 3.3 the bounded sets of $E/L = \operatorname{ind}_n E_n/L_n$ are contained and bounded, hence compact, in some step E_n/L_n and thus can be lifted to E_n.

(3)⇒(2). Since (3) easily implies regularity of E/L, again 3.3 implies (2). ■

As promised in the introduction, we now turn to projective spectra of (DF)-spaces. We want to recall the main definitions.

A *projective spectrum* is a sequence $\mathcal{X} = (X_n, \varrho^n_{n+1})_{n\in\mathbb{N}}$ of l.c.s. X_n and continuous linear maps $\varrho^n_{n+1} : X_{n+1} \to X_n$. We set

$$X = \operatorname{Proj}^0 \mathcal{X} = \Big\{(x_n)_n \in \prod_n X_n : \varrho^n_{n+1}(x_{n+1}) = x_n \text{ for all } n\Big\},$$

$$\operatorname{Proj}^1 \mathcal{X} = \prod_n X_n / B(\mathcal{X}),$$

where

$$B(\mathcal{X}) = \Big\{(x_n)_n \in \prod_n X_n : \exists (y_n)_n \in \prod_n X_n \text{ with } x_n = y_n - \varrho^n_{n+1}(y_{n+1})\ \forall n\Big\}.$$

This definition of Proj^1 is taken from [28] and it has been obtained by Palamodov [20] as a characterization of his original definition in terms of homological algebra. It is quite easy to compute that if

$$0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$$

is a short exact sequence of projective spectra such that $\operatorname{Proj}^1 \mathcal{X} = 0$, then the sequence

$$0 \to \operatorname{Proj}^0 \mathcal{X} \to \operatorname{Proj}^0 \mathcal{Y} \to \operatorname{Proj}^0 \mathcal{Z} \to 0$$

is again exact. This fact reflects the homological origin of the definition and is the typical situation in applications (like the one explained in the introduction).

$\mathcal{X}$ is called *reduced* if the canonical map $\varrho^n : X \to X_n$, $(x_k)_{k\in\mathbb{N}} \mapsto x_n$, has dense range for every $n \in \mathbb{N}$. In this case, the strong duals X_n' form an inductive spectrum where the transposed maps of ϱ^n_{n+1} may be considered as continuous inclusions. We denote by X^* the inductive limit of these strong duals. If X_n is a regular (LB)-space with fundamental sequence $(B_{n,l})_{l\in\mathbb{N}}$ of bounded sets then the strong topology on X_n' is determined by the seminorms

$$\|y\|^*_{n,l} = \sup\{|y(x)| : x \in B_{n,l}\}, \quad l \in \mathbb{N}.$$

For $n < m$ the transposed of the map $\varrho^n_m = \varrho^n_{n+1} \circ \ldots \circ \varrho^{m-1}_m : X_m \to X_n$ is denoted by ι^m_n. Vogt called the spectrum $\mathcal{X}$ to be of type (P_2^*) if the inductive spectrum of the strong duals satisfies (wQ), i.e.

$$\forall n\ \exists m > n, N \in \mathbb{N}\ \forall k > m, M \in \mathbb{N}\ \exists K \in \mathbb{N}, S > 0\ \forall y \in X_n',$$
$$\|\iota^m_n y\|^*_{m,M} \le S\,(\|\iota^k_n y\|^*_{k,K} + \|y\|^*_{n,N}).$$

In concrete cases this condition is very useful (even if it looks complicated at first sight), and it is much easier to check these inequalities than to show $\operatorname{Proj}^1 \mathcal{X} = 0$ by other means. We remark that in the applications to partial differential equations it was exactly condition (P_2^*) that characterized $\operatorname{Proj}^1 \mathcal{X} = 0$ and thus the desired surjectivity of the partial differential operators considered there. Moreover, it could be reformulated as a Phragmén–Lindelöf condition similar to that in Hörmander's work [18], but hard analysis had to be used to show sufficiency. The following theorem gives the missing implication in a result of Vogt [28, Theorem 3.4] and thus saves large parts of work in the applications.

THEOREM 3.5. *Let $\mathcal{X}$ be a reduced projective spectrum of (DFM)-spaces and $X = \operatorname{Proj}^0 \mathcal{X}$. The following conditions are equivalent*:

(1) $\operatorname{Proj}^1 \mathcal{X} = 0$,
(2) X *is bornological*,
(3) X'_b *is complete*,
(4) X *is barrelled*,
(5) X^* *is regular*,
(6) $\mathcal{X}$ *is of type* (P_2^*).

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6) were proved in [28, Theorem 3.4]. To show (6)⇒(1) note that (P_2^*) is a reformulation of condition (wQ) for the inductive spectrum of the strong duals and apply 3.3 to conclude that X^* is acyclic. By the very definition of acyclicity this implies that the transposed of the acyclicity map σ is surjective, which gives $\operatorname{Proj}^1 \mathcal{X} = 0$. ∎

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The multiplicity of solutions and geometry of a nonlinear elliptic equation

by

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let L denote a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity, $0 < \lambda_1 < \lambda_2 < \lambda_3 \le \ldots \le \lambda_i \le \ldots \to \infty$. We consider a semilinear elliptic Dirichlet problem $Lu + bu^+ - au^- = f(x)$ in Ω, $u = 0$ on $\partial\Omega$. We assume that $a < \lambda_1$, $\lambda_2 < b < \lambda_3$ and f is generated by ϕ_1 and ϕ_2. We show a relation between the multiplicity of solutions and source terms in the equation.

0. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1\le i,j\le n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big),$$

where $a_{ij} = a_{ji} \in C^\infty(\overline{\Omega})$. We consider the semilinear elliptic Dirichlet boundary value problem

$$\begin{aligned} Lu + bu^+ - au^- &= f(x) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{0.1}$$

Here L is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated acccording to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \le \ldots \le \lambda_i \le \ldots \to \infty.$$

In [3, 4, 8, 10, 15], the authors have investigated the multiplicity of solutions of (0.1) when the forcing term f is supposed to be a multiple of

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