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On Dragilev type power Köthe spaces

by

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Abstract. A complete isomorphic classification is obtained for Köthe spaces $X = K(\exp[\chi(p-\kappa(i)) - 1/p]a_i)$ such that $X \overset{\text{qd}}{\simeq} X^2$; here χ is the characteristic function of the interval $[0,\infty)$, the function $\kappa : \mathbb{N} \to \mathbb{N}$ repeats its values infinitely many times, and $a_i \to \infty$. Any of these spaces has the quasi-equivalence property.

1. Introduction. For any matrix $(a_{ip})_{i\in I, p\in\mathbb{N}}$ of positive numbers (with countable index set I) we denote by $K(a_{ip})$ (or $K(a_{ip}, i \in I)$) the Köthe space generated by the matrix (a_{ip}).

M. M. Dragilev [1] proved that there exist Köthe spaces with regular bases which are not distinguished by the diametral dimension

$$\Gamma(X) = \{\gamma = (\gamma_n) : \ \forall p\ \exists q\ \gamma_n d_n(U_p, U_q) \to 0\},$$

considering the power Köthe spaces

$$D(\kappa, a) = K(\exp[\chi(p-\kappa(i)) - 1/p]a_i), \tag{1}$$

where $(\kappa(i)) = (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1,2,3,4,5,6,\ldots)$, $a = (a_i)$, $a_i \nearrow \infty$, $\chi(t) = 0$ for $t < 0$, $\chi(t) = 1$ for $t \ge 0$. We investigate here an analogous class of power Köthe spaces given by (1) for an arbitrary function $\kappa : \mathbb{N} \to \mathbb{N}$ that repeats its values infinitely many times and an arbitrary sequence of positive numbers $a_i \to \infty$ (not necessarily increasing).

Our aim is to study the structure and isomorphic classification of $D(\kappa, a)$ spaces for different κ and a. In order to distinguish non-isomorphic spaces of this class we first construct appropriate invariant characteristics (generalized linear topological invariants). The method of generalized linear topological invariants was developed in [6], [7], [9]–[11] (see the survey [12] for more details).

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In the following we denote by $E_0(a)$ and $E_\infty(a)$ respectively the finite and infinite type power series spaces generated by the sequence $a = (a_i)$ with $a_i \to \infty$, i.e.

$$E_0(a) = K(\exp(-a_i/p)), \qquad E_\infty(a) = K(\exp(pa_i)).$$

Every infinite subset ν of $\mathbb{N} = \{1, 2, 3, \ldots\}$ is identified with the corresponding increasing sequence of positive integers, i.e. $\nu = \{\nu_1, \nu_2, \nu_3, \ldots\}$. For any set A we denote by $|A|$ the number of elements of A if it is finite, and ∞ otherwise.

A subspace of a Köthe space generated by a subsequence of the natural basis is called a *basic subspace*. As shown by the next observation the space $X = D(\kappa, a)$ is similar to finite type power series spaces.

PROPOSITION 1. *Any basic subspace of $D(\kappa, a)$ contains a basic subspace which is isomorphic to a finite type power series space.*

Proof. Obviously any basic subspace of $X = D(\kappa, a)$ contains a basic subspace generated by a set $\nu \subset \mathbb{N}$ of indices having one of the following two properties:

$$\text{(a)}\ \sup\{\kappa(i) : i \in \nu\} < \infty, \qquad \text{(b)}\ \kappa(i) \to \infty \text{ as } i \to \infty,\ i \in \nu.$$

Then the basic subspace X_ν generated by the vectors e_i, $i \in \nu$, is isomorphic to the finite type power series space $E_0(a^\nu)$, where a^ν is the subsequence of a corresponding to ν, i.e. $a_i^\nu = a_{\nu_i}$. Indeed, in case (a) we have, for large enough p, $\chi(p - \kappa(i)) = 1$ for all $i \in \nu$, i.e. the Köthe matrix equals $\exp[(1 - 1/p)a_i]$, hence X_ν is diagonally isomorphic to $E_0(a^\nu)$. In case (b) we have $\chi(p - \kappa(i)) = 0$ for all large enough $i \in \nu$, i.e. the Köthe matrix equals $\exp(-a_i/p)$ for $i \in \nu$, $i > i_0$, hence $X_\nu \simeq E_0(a^\nu)$.

COROLLARY 1. *For any infinite subset $\nu \subset \mathbb{N}$ the basic subspace X_ν is not isomorphic to a power series space of infinite type.*

Indeed, by Proposition 1 the subspace X_ν contains a basic subspace X_μ (obviously complemented) that is isomorphic to a finite type power series space. On the other hand, it is known that a finite type power series space cannot be imbedded in an infinite type Schwartz power series space since any operator from a finite to an infinite type power series space is compact (see [8]).

Let us note that our interest in $D(\kappa, a)$ spaces was motivated by the problem of finding an appropriate "model" Köthe space for some spaces of analytic functions. More precisely, if

$$G = \overline{\mathbb{C}} \setminus \Big(\bigcup_{p=1}^{\infty} K_s \cup \{a\}\Big), \quad K_s = \{z \in \overline{\mathbb{C}} : |z - a_s| \le \delta_s\}, \quad a = \lim_s a_s,$$

then the space $A(G)$ has no complemented subspaces of infinite type and in case it has a basis it should be isomorphic to a Köthe space with the same property. So, it seems that $D(\kappa, a)$ spaces may be the desired model spaces.

2. Identity and quasi-diagonal isomorphisms. We begin with some general facts concerning quasi-diagonal operators between Köthe spaces. Suppose $X = K(a_{ip}, i \in I)$ and $Y = K(b_{jp}, j \in J)$ are Köthe spaces. An operator $T : X \to Y$ is called *quasi-diagonal* if there exist a function $\varphi : I \to J$ and constants r_i, $i \in I$, such that

$$Te_i = r_i \tilde{e}_{\varphi(i)}, \qquad i \in I,$$

where (e_i) and $(\tilde{e}_j)$ are the canonical bases in X and Y. We denote respectively by $X \overset{\text{qd}}{\hookrightarrow} Y$ and $X \overset{\text{qd}}{\simeq} Y$ a quasi-diagonal isomorphic imbedding and a quasi-diagonal isomorphism.

The next statement is well known (see, for example, [9]).

LEMMA 1. *If for Köthe spaces X and Y there are quasi-diagonal imbeddings $X \overset{\text{qd}}{\hookrightarrow} Y$ and $Y \overset{\text{qd}}{\hookrightarrow} X$ then $X \overset{\text{qd}}{\simeq} Y$.*

Proof. If the quasi-diagonal imbeddings $X \overset{\text{qd}}{\hookrightarrow} Y$ and $Y \overset{\text{qd}}{\hookrightarrow} X$ are defined respectively by $(r_i), \varphi : I \to J$ and $(\varrho_j), \psi : J \to I$ then by the Cantor–Bernstein theorem there exist complementary subsets $I_1, I_2 \subset I$ and $J_1, J_2 \subset J$ such that $\varphi(I_1) = J_1$ and $\psi(J_2) = I_2$. Then putting $Te_i = \gamma_i \tilde{e}_{g(i)}$, where $\gamma_i = r_i, g(i) = \varphi(i)$ for $i \in I_1$ and $\gamma_i = \varrho^{-1}_{\psi^{-1}(i)}, g(i) = \psi^{-1}(i)$ for $i \in I_2$ we obtain a quasi-diagonal isomorphism T between X and Y.

The Cartesian product of m copies of a Köthe space $X = K(a_{ip}, i \in I)$ is denoted by X^m. The space X^m will be identified with the Köthe space $K(a_{\bar{\imath}p}, \bar{I})$, where

$$\bar{I} = \{\bar{\imath} = (i, \mu) : i \in I,\ \mu = 1, \ldots, m\},$$
$$a_{\bar{\imath}p} = a_{ip} \quad \text{if } \bar{\imath} = (i, \mu),\ \mu = 1, \ldots, m.$$

LEMMA 2. *Suppose $X = K(a_{ip}, I)$ and $Y = K(a_{jp}, J)$ are Köthe spaces and m is an integer. Then*

$$\text{(a)}\ X^m \overset{\text{qd}}{\hookrightarrow} Y^m \Rightarrow X \overset{\text{qd}}{\hookrightarrow} Y; \qquad \text{(b)}\ X^m \overset{\text{qd}}{\simeq} Y^m \Rightarrow X \overset{\text{qd}}{\simeq} Y.$$

Proof. (a) Let $T : K(a_{\bar{\imath}p}, \bar{I}) \to K(b_{\bar{\jmath}q}, \bar{J})$ be a quasi-diagonal imbedding; then $Te_{\bar{\imath}} = \varrho_i \tilde{e}_{\varphi(\bar{\imath})}$, where $\varphi : \bar{I} \to \bar{J}$ is an injection. Put

$$\pi(i, \mu) = i, \quad \tilde{\pi}(j, \mu) = j, \quad \mu = 1, \ldots, m.$$

Consider the multivalued mapping

$$G : I \to J, \qquad G(i) = \tilde{\pi}\varphi(\pi^{-1}(i)).$$

By the Hall–König theorem ([2], Ch. 3) there exists an injection $g : I \to J$ such that $g(i) \in G(i)$ for all i if and only if

$$|L| \le \Big|\bigcup_{i\in L} G(i)\Big| \quad \forall L \subset I, |L| < \infty.$$

Since the mapping $\widetilde{\pi} : \widetilde{J} \to J$ is m-sheeted we have $|B| \le m|\widetilde{\pi}(B)|$ for any $B \subset \widetilde{J}$. Therefore for any $L \subset I$,

$$|L| = \frac{1}{m}|\pi^{-1}(L)| = \frac{1}{m}|\varphi(\pi^{-1}(L))| \le |\widetilde{\pi}\varphi(\pi^{-1}(L))| = \Big|\bigcup_{i\in L} G(i)\Big|,$$

so there exists an injection $g : I \to J$ such that $g(i) \in \widetilde{\pi}\varphi(\pi^{-1}(i))$ for $i \in I$. For any $i \in I$ we fix some $\mu = 1, \ldots, m$ such that $\widetilde{\pi}(\varphi(i,\mu)) = g(i)$ and put $r_i = \varrho_{(i,\mu)}$. It is easy to see that the operator $S : X \to Y$ defined by $Se_i = r_i\widetilde{e}_{g(i)}, i \in I$, is an isomorphic imbedding because T is.

(b) follows immediately from (a) and Lemma 1.

The next proposition gives necessary and sufficient conditions for coincidence of two $D(\kappa, a)$ spaces.

PROPOSITION 2. *The spaces $D(\kappa, a)$ and $D(\widetilde{\kappa}, \widetilde{a})$ coincide as sets if and only if the following conditions hold:*

(i) *there exists $C > 0$ such that $a_i \le C\widetilde{a}_i \le C^2a_i$ for all $i \in \mathbb{N}$;*
(ii) *for all $\nu \subset \mathbb{N}$ with $|\nu| = \infty$, $\kappa(\nu_i) \to \infty \Leftrightarrow \widetilde{\kappa}(\nu_i) \to \infty$;*
(iii) *$\widetilde{a}_i/a_i \to 1$ as $i \to \infty$, and $\kappa(i) \le$ const.*

Proof. Since two Köthe spaces coincide as sets if and only if their matrices are equivalent (see [5], Lemma 4) we have $D(\kappa, a) = D(\widetilde{\kappa}, \widetilde{a})$ if and only if the following conditions hold:

$$(2) \quad \forall p\ \exists \widetilde{p}, C > 0 : (\chi(p - \kappa(i)) - 1/p)a_i \le \log C + (\chi(\widetilde{p} - \widetilde{\kappa}(i)) - 1/\widetilde{p})\widetilde{a}_i,$$

$$(3) \quad \forall \widetilde{p}\ \exists q, C > 0 : (\chi(\widetilde{p} - \widetilde{\kappa}(i)) - 1/\widetilde{p})\widetilde{a}_i \le \log C + (\chi(q - \kappa(i)) - 1/q)a_i.$$

(Here and in the following we denote by C any constant which does not depend on i.) By (2) and (3) it follows that for some indices $p_1, \widetilde{p}_1 < \widetilde{p}_2, q_2$,

$$[\chi(\widetilde{p}_2 - \widetilde{\kappa}(i)) - \chi(\widetilde{p}_1 - \widetilde{\kappa}(i)) + 1/\widetilde{p}_1 - 1/\widetilde{p}_2]\widetilde{a}_i \le [\chi(q_2 - \kappa(i)) - \chi(p_1 - \kappa(i)) + 1/p_1 - 1/q_2]a_i + \log C$$

and we obtain (since $a_i \to \infty$)

$$\limsup \widetilde{a}_i/a_i = C < \infty.$$

In an analogous way we get the symmetric relation, which proves (i).

Suppose $\nu \subset \mathbb{N}$, $|\nu| = \infty$ and $\kappa(\nu_i) \to \infty$. Then (3) implies that the sequence $\widetilde{\kappa}(\nu_i)$ is not bounded—otherwise taking $\widetilde{p} > \max\{\widetilde{\kappa}(\nu_i) : i \in \mathbb{N}\}$ we have for all i such that $\kappa(\nu_i) > q$,

$$(1 - 1/\widetilde{p})\widetilde{a}_{\nu_i} \le \log C,$$

which is impossible since $\widetilde{a}_i \to \infty$. If the sequence $\widetilde{\kappa}(\nu_i)$ does not tend to ∞ then passing to a subsequence if necessary we get a contradiction. Hence $\kappa(\nu_i) \to \infty$ implies $\widetilde{\kappa}(\nu_i) \to \infty$. Analogously by (2) it follows that $\widetilde{\kappa}(\nu_i) \to \infty$ implies $\kappa(\nu_i) \to \infty$, i.e. (ii) holds.

In order to check (iii) fix an arbitrary constant $K > 0$ and put $I_K = \{i \in \mathbb{N} : \kappa(i) \le K\}$. Then by (ii), $\sup\{\widetilde{\kappa}(i) : i \in I_K\} = \widetilde{K} < \infty$. Taking $\widetilde{p} > \widetilde{K}$ and $q > K$ in (3) we obtain

$$(1 - 1/\widetilde{p})\widetilde{a}_i \le \log C + (1 - 1/q)a_i \le a_i$$

for large enough $i \in I_K$. Therefore

$$\limsup_{i\to\infty, i\in I_K} \widetilde{a}_i/a_i \le \lim_{\widetilde{p}} \frac{\widetilde{p}}{\widetilde{p} - 1} = 1.$$

Analogously we obtain

$$\limsup_{i\to\infty, i\in I_K} a_i/\widetilde{a}_i \le 1, \quad \text{and therefore} \quad \liminf_{i\to\infty, i\in I_K} \widetilde{a}_i/a_i \ge 1$$

and we get

$$\lim_{i\to\infty, i\in I_K} \widetilde{a}_i/a_i = 1.$$

It is easy to see that the conditions (i)–(iii) are sufficient for the equality $D(\kappa, a) = D(\widetilde{\kappa}, \widetilde{a})$. Indeed, let us check that they imply (2) and (3). Fix p and choose $\widetilde{p} > Cp$ (where C is the constant appearing in (i)) such that $\kappa(i) \le p \Rightarrow \widetilde{\kappa}(i) \le \widetilde{p}$ (by (ii) this is possible). Then for i satisfying $\kappa(i) \le p$ the relation (2) is equivalent to

$$(1 - 1/p)a_i \le (1 - 1/\widetilde{p})\widetilde{a}_i + \log C,$$

which holds by (iii). If $\kappa(i) > p$ then by (i) we have

$$(-1/p)a_i \le (-1/\widetilde{p})\widetilde{a}_i,$$

which implies (2).

Since the conditions (i)–(iii) are symmetric with respect to $D(\kappa, a)$ and $D(\widetilde{\kappa}, \widetilde{a})$, (3) follows analogously by the same argument.

PROPOSITION 3. *If $T : D(\kappa, a) \to D(\widetilde{\kappa}, \widetilde{a})$ is a diagonal operator defined by the formula $Te_i = \exp(r_i)\widetilde{e}_i$, then T is an isomorphism if and only if the following conditions hold:*

(i) *there exists $C > 0$ such that $a_i \le C\widetilde{a}_i \le C^2a_i$ for all $i \in \mathbb{N}$;*

(ii) $$\lim_{I_1} \frac{r_i}{a_i} = 1, \quad \lim_{I_2} \frac{r_i}{a_i} = 0, \quad \lim_{I_3} \frac{r_i + \widetilde{a}_i}{a_i} = 0, \quad \lim_{I_4} \frac{r_i + \widetilde{a}_i}{a_i} = 1,$$

where $I_1 := \kappa(i) \le$ const, $\widetilde{\kappa}(i) \to \infty$; $I_2 := \kappa(i) \to \infty$, $\widetilde{\kappa}(i) \to \infty$; $I_3 := \kappa(i) \to \infty$, $\widetilde{\kappa}(i) \le$ const; $I_4 := \kappa(i) \le$ const, $\widetilde{\kappa}(i) \le$ const.

Proof. Suppose T is an isomorphism. Then since T and T^{-1} are continuous there exist p_1, $\widetilde{p}_1$, $\widetilde{p}_2$, p_2 and $M > 0$ such that

$$|e_i|_{p_1} \le M|Te_i|_{\widetilde{p}_1}, \quad |Te_i|_{\widetilde{p}_2} \le M|e_i|_{p_2}.$$

Therefore

$$\frac{|Te_i|_{\widetilde{p}_2}}{|Te_i|_{\widetilde{p}_1}} \le M^2 \frac{|e_i|_{p_2}}{|e_i|_{p_1}},$$

which implies (after taking logarithms of both sides and estimating from below and above)

$$\left(\frac{1}{\widetilde{p}_1} - \frac{1}{\widetilde{p}_2}\right)\widetilde{a}_i \le \left(1 + \frac{1}{p_1}\right)a_i + 2\log M.$$

Hence there exists $C > 0$ such that $\limsup \widetilde{a}_i/a_i \le C$. Analogously it follows that $\limsup a_i/\widetilde{a}_i \le C$, which proves (i).

To prove (ii) we use the fact that T is continuous if and only if

(4) $\quad \forall \widetilde{p}\ \exists p, M > 0:$

$$r_i + (\chi[\widetilde{p} - \widetilde{\kappa}(i)] - 1/\widetilde{p})\widetilde{a}_i \le (\chi[p - \kappa(i)] - 1/p)a_i + \log M.$$

If $\kappa(i) \le \text{const} = C_1$ and $\widetilde{\kappa}(i) \to \infty$ then taking $p > C_1$ we obtain $r_i - \widetilde{a}_i/\widetilde{p} \le a_i + \log M$ for large enough i, and therefore

$$\limsup r_i/a_i \le 1 + C/\widetilde{p}.$$

It follows (since $\widetilde{p}$ is arbitrary) that $\limsup r_i/a_i \le 1$. Using the continuity of T^{-1} we get by the same argument $\liminf r_i/a_i \ge 1$, hence the first relation in (ii) is proved. The proof of the other three is analogous.

Conversely, suppose (i) and (ii) hold. Fix an arbitrary $\widetilde{p}$. By the second relation in (ii) there exist p_1 and $\widetilde{p}_1 > \widetilde{p}$ such that

$$\frac{r_i}{a_i} < \frac{1}{2\widetilde{p}C} \quad \text{if } \kappa(i) \ge p_1,\ \widetilde{\kappa}(i) \ge \widetilde{p}_1.$$

Analogously by the third relation in (ii) there exists p_2 such that

$$\frac{r_i + \widetilde{a}_i}{a_i} < \frac{1}{2\widetilde{p}C} \quad \text{if } \kappa(i) > p_2,\ \widetilde{\kappa}(i) \le \widetilde{p}_1.$$

Choose $p > \max(p_1, p_2, 2\widetilde{p}C)$; then (4) holds, i.e. T is continuous. Indeed:

1) if $\kappa(i) > p$ and $\widetilde{\kappa}(i) > \widetilde{p}_1$ then

$$r_i - \frac{\widetilde{a}_i}{\widetilde{p}} \le \left(\frac{r_i}{a_i} - \frac{1}{C\widetilde{p}}\right)a_i \le -\frac{a_i}{p},$$

i.e. (4) is true with $M = 1$;

2) if $\kappa(i) > p$ and $\widetilde{\kappa}(i) \le \widetilde{p}_1$ then

$$r_i + \left(1 - \frac{1}{\widetilde{p}}\right)\widetilde{a}_i \le \left(\frac{r_i + \widetilde{a}_i}{a_i} - \frac{1}{C\widetilde{p}}\right)a_i \le -\frac{a_i}{p},$$

i.e. (4) is true with $M = 1$;

3) if $\kappa(i) \le p$ then by the first relation in (ii) there exists $\widetilde{p}_2$ such that $r_i/a_i - 1 < 1/(2C\widetilde{p})$ for $\kappa(i) \le p$ and $\widetilde{\kappa}(i) > \widetilde{p}_2$, hence

$$r_i - \frac{\widetilde{a}_i}{\widetilde{p}} \le \left(\frac{r_i}{a_i} - \frac{1}{C\widetilde{p}}\right)a_i \le \left(1 - \frac{1}{2C\widetilde{p}}\right)a_i \le \left(1 - \frac{1}{p}\right)a_i,$$

i.e. for $\kappa(i) \le p$ and $\widetilde{\kappa}(i) > \widetilde{p}_2$, (4) holds with $M = 1$;

4) if $\kappa(i) \le p$ and $\widetilde{\kappa}(i) \le \widetilde{p}_2$ then by the fourth relation in (ii) we have $(r_i + \widetilde{a}_i)/a_i \le 1 + 1/(2C\widetilde{p})$ for $i > i_0$, hence

$$r_i + \left(1 - \frac{1}{\widetilde{p}}\right)\widetilde{a}_i \le \left(\frac{r_i + \widetilde{a}_i}{a_i} - \frac{1}{C\widetilde{p}}\right)a_i \le \left(1 - \frac{1}{p}\right)a_i$$

for $i > i_0$, i.e. (4) holds with some constant M.

Thus we conclude that T is continuous. On the other hand, it is easy to see that the relations (ii) are symmetric with respect to T and T^{-1}. Hence T^{-1} is also continuous, which completes the proof.

COROLLARY 2. *$D(\kappa, a) \overset{\text{qd}}{\simeq} E_0(a)$ if and only if there exists $N_1 \subset \mathbb{N}$ such that $\kappa(i)$ is bounded on N_1 and $\kappa(i) \to \infty$ as $i \to \infty, i \in \mathbb{N} \setminus N_1$.*

3. Invariant characteristics. In this section we construct some invariant characteristics suitable for investigation of isomorphisms between $D(\kappa, a)$ spaces. Our construction is based on the geometric argument developed in [11], which makes it much easier compared to similar earlier constructions (see e.g. [9]).

Characteristic β. Suppose E is a linear space, U and V are absolutely convex sets in E and $\mathcal{E}_V$ is the set of all finite-dimensional subspaces of E that are spanned by elements of V. We put

$$\beta(V, U) = \sup\{\dim L : L \in \mathcal{E}_V,\ L \cap U \subset V\}.$$

It is obvious that

$$\widetilde{V} \subset V,\ U \subset \widetilde{U} \Rightarrow \beta(\widetilde{V}, \widetilde{U}) \le \beta(V, U)$$

and of course if T is an injective linear operator defined on E then

$$\beta(T(V), T(U)) = \beta(V, U).$$

Let E be a sequence space with the property

$$x = (x_n) \in E,\ |y_n| \le |x_n|\ \forall n \ \Rightarrow\ y = (y_n) \in E,$$

and A be the set of all sequences with positive terms. For any $a, b \in A$ we put

$$a.b = (a_i b_i), \quad a^\alpha = (a_i^\alpha), \quad a \wedge b = (\min(a_i, b_i)), \quad a \vee b = (\max(a_i, b_i)).$$

For any $x = (x_i) \in E$ and $a \in A$ we also put

$$\|x\|_a = \sum_i |x_i| a_i, \quad B_a = \{x \in E : \|x\|_a < 1\}.$$

LEMMA 3. *If* $a, b \in A$ *then* $\beta(B_a, B_b) = |\{i : a_i/b_i \le 1\}|$.

Proof. Put

$$J = \{i : a_i \le b_i\}, \quad Px = \sum_{i \in J} x_i e_i,$$

and let M be the linear span of $\{e_i : i \in J\}$. Then obviously $\|x\|_a \le \|x\|_b$ for $x \in M$, hence $M \cap B_b \subset B_a$ and $\beta(B_a, B_b) \ge \dim M = |J|$.

Conversely, suppose L is a finite-dimensional subspace in X satisfying $L \cap B_b \subset B_a$ (i.e. $\|x\|_a \le \|x\|_b$ for all $x \in L$). If $\dim L > |J|$, then obviously there exists $x \in L$ with $x \ne 0$ such that $Px = 0$. But then $x_i = 0$ for $i \in J$ and $a_i > b_i$ for $i \notin J$, and therefore $\|x\|_a > \|x\|_b$, which is a contradiction. Hence $\beta(B_a, B_b) = |J|$.

COROLLARY 3. *For all* $a, b, c, d \in A$,

$$\left|\left\{i : \frac{\max(a_i, b_i)}{\min(c_i, d_i)} \le 1\right\}\right| \le \beta(B_a \cap B_b, \operatorname{conv}(B_c \cup B_d)) \le \left|\left\{i : \frac{\max(a_i, b_i)}{\min(c_i, d_i)} \le 2\right\}\right|.$$

Indeed, it is easy to see that

$$(5) \qquad B_{a\vee b} \subset B_a \cap B_b \subset 2B_{a\vee b}, \quad B_{a\wedge b} = \operatorname{conv}(B_a \cup B_b),$$

hence

$$\beta(B_{a\vee b}, B_{c\wedge d}) \le \beta(B_a \cap B_b, \operatorname{conv}(B_c \cup B_d)) \le \beta(2B_{a\vee b}, B_{c\wedge d}).$$

For convenience we put $B_a^\alpha B_b^{1-\alpha} = B_{a^\alpha b^{1-\alpha}}$. It is well known that sets of the type $B_a^\alpha B_b^{1-\alpha}$ have a natural interpolation property; it is formulated in the next lemma in the form appropriate for us.

LEMMA 4. *Suppose* E *and* $\widetilde{E}$ *are Köthe spaces,* (e_i) *and* $(\widetilde{e}_j)$ *are their canonical bases and* $T : E \to \widetilde{E}$ *is a linear operator. If* $a, b, \widetilde{a}, \widetilde{b} \in A$ *and*

$$T(B_a) \subset B_{\widetilde{a}}, \quad T(B_b) \subset B_{\widetilde{b}},$$

then for any $\alpha \in (0, 1)$ *we have*

$$T(B_a^\alpha B_b^{1-\alpha}) \subset B_{\widetilde{a}}^\alpha B_{\widetilde{b}}^{1-\alpha}.$$

Proof. Put

$$Te_i = \sum_j t_{ij} \widetilde{e}_j, \quad i = 1, 2, \ldots;$$

then since $\|Tx\|_{\widetilde{a}} \le \|x\|_a$ and $\|Tx\|_{\widetilde{b}} \le \|x\|_b$ we have for any i,

$$\|Te_i\|_{\widetilde{a}} = \sum_j |t_{ij}| \widetilde{a}_j \le \|e_i\|_a = a_i, \quad \|Te_i\|_{\widetilde{b}} = \sum_j |t_{ij}| \widetilde{b}_j \le \|e_i\|_b = b_i.$$

Therefore by the Hölder inequality it follows that

$$\|Te_i\|_{\widetilde{a}^\alpha \widetilde{b}^{1-\alpha}} = \sum_j |t_{ij}| \widetilde{a}_j^\alpha \widetilde{b}_j^{1-\alpha} \le \Big(\sum_j |t_{ij}| \widetilde{a}_j\Big)^\alpha \Big(\sum_j |t_{ij}| \widetilde{b}_j\Big)^{1-\alpha} \le a_i^\alpha b_i^{1-\alpha},$$

hence

$$\|Tx\|_{\widetilde{a}^\alpha \widetilde{b}^{1-\alpha}} \le \sum_i |x_i| \cdot \|Te_i\|_{\widetilde{a}^\alpha \widetilde{b}^{1-\alpha}} \le \sum_i |x_i| a_i^\alpha b_i^{1-\alpha} = \|x\|_{a^\alpha b^{1-\alpha}}.$$

If $E = K(a_{ip})$ is a Köthe space and $U_p = \{x \in E : |x|_p = \sum_i |x_i| a_{ip} < 1\}$, $p = 1, 2, \ldots$, are the corresponding unit balls then $U_p = B_{a_p}$, where $a_p = (a_{ip})$. We also need to consider bounded subsets of Köthe spaces of the type

$$U_{(p_i)} = B_a, \quad a = (a_{ip_i}),$$

where (p_i) is an increasing sequence of indices.

LEMMA 5. *If* $E = K(a_{ip})$ *is a Schwartz Köthe space and* $B \subset E$ *is a bounded set then there exists an increasing sequence* (p_i) *of indices such that* $B \subset CU_{(p_i)}$ *for some constant* $C > 0$.

Proof. Since E is a Schwartz space we can assume without loss of generality that for any $p = 1, 2, \ldots$ we have $a_{ip}/a_{i,p+1} \to 0$ as $i \to \infty$. Since B is bounded, for any p we have

$$\sup_{x \in B} \Big(\sum_i |x_i| a_{ip}\Big) = C_p < \infty.$$

Choose integers m_k, $k = 1, 2, \ldots$, in such a way that $m_k < m_{k+1}$ and

$$a_{ik}/a_{i,k+1} < 2^{-k} C_{k+1}^{-1} \quad \text{for } i > m_k.$$

Put $p_i = 1$ for $i = 1, \ldots, m_1$ and $p_i = k$ for $i = m_k + 1, \ldots, m_{k+1}$, $k = 1, 2, \ldots$ Then for $x \in B$ we obtain

$$\sum_{i=1}^\infty |x_i| a_{ip_i} = \sum_{i=1}^{m_1} |x_i| a_{ip_1} + \sum_{k=1}^\infty \sum_{i=m_k+1}^{m_{k+1}} |x_i| a_{ik} \le C_1 + \sum_{k=1}^\infty 2^{-k} C_{k+1}^{-1} \sum_{i=m_k+1}^{m_{k+1}} |x_i| a_{i,k+1} \le C,$$

where $C = C_1 + 1$.

COROLLARY 4. *The sets* $U_{(p_i)}$ *form a basis of bounded sets in* E.

THEOREM 1. *If* $D(\kappa, a) \simeq D(\widetilde{\kappa}, \widetilde{a})$ *then the following relations hold:*

(a) $\forall \widetilde{p}\ \exists C_{\widetilde{p}}, p\ \forall q\ \exists \widetilde{q}, \tau_0 > 0\ \forall \tau > \tau_0, t > \tau$:

$$(6)\qquad |\{i : p \le \kappa(i) \le q,\ \tau \le a_i \le t\}| \le |\{j : \widetilde{p} \le \widetilde{\kappa}(j) \le \widetilde{q},\ \tau/C_{\widetilde{p}} \le \widetilde{a}_j \le C_{\widetilde{p}} t\}|,$$

where $C_{\widetilde{p}} \to 1$ *as* $\widetilde{p} \to \infty$;

(b) $\forall p\ \exists \widetilde{p}\ \forall (q_i)\ \exists (\widetilde{q}_j), c > 0, \tau_0 > 0\ \forall \tau > \tau_0, t > \tau$:

$$(7)\qquad |\{i : \kappa(i) \le p \text{ or } \kappa(i) \ge q_i;\ \tau \le a_i \le t\}| \le |\{j : \widetilde{\kappa}(j) \le \widetilde{p} \text{ or } \widetilde{\kappa}_j \ge \widetilde{q}_j;\ \tau/c \le \widetilde{a}_j \le ct\}|.$$

Proof. (a) For convenience we write $V \prec W$ if $V \subset \mathrm{const} W$. Suppose $T : D(\kappa, a) \to D(\widetilde{\kappa}, \widetilde{a})$ is an isomorphism. Let $\varphi : (1, \infty) \to (1, \infty)$ be an increasing function such that $\varphi(k) > 4k$ and

$$T(U_{\varphi(k)}) \prec \widetilde{U}_k, \quad \widetilde{U}_{\varphi(k)} \prec T(U_k).$$

Then for any $\widetilde{p} \ge \varphi^3(1)$ put

$$\widetilde{m} = \varphi^{-1}(\widetilde{p}), \quad m = \varphi^{-2}(\widetilde{p}), \quad \widetilde{m}_0 = \varphi^{-3}(\widetilde{p}), p = \varphi(\widetilde{p}), \quad \widetilde{q}_0 = \varphi(p).$$

Further for any $q \ge \varphi^2(\widetilde{p})$ put

$$\widetilde{q} = \varphi(q), \quad \widetilde{r} = \varphi(\widetilde{q}), \quad r = \varphi(\widetilde{r}), \quad s = \varphi(r), \quad \widetilde{s}_1 = \varphi(s).$$

Then each of the indices $\widetilde{m}_0, m, \widetilde{m}, \widetilde{p}, p, \widetilde{q}_0, q, \widetilde{q}, \widetilde{r}, r, s, \widetilde{s}$ is at least four times the previous one and $\widetilde{U}_{\widetilde{s}} \prec T(U_s) \prec T(U_r) \prec \widetilde{U}_{\widetilde{r}} \prec \widetilde{U}_{\widetilde{q}} \prec T(U_q) \prec \widetilde{U}_{\widetilde{q}_0} \prec T(U_p) \prec \widetilde{U}_{\widetilde{p}} \prec \widetilde{U}_{\widetilde{m}} \prec T(U_m) \prec \widetilde{U}_{\widetilde{m}_0}$. By Lemma 4 and the elementary properties of β it follows that there exists a constant $C > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\beta(U_p \cap e^t U_r \cap U_m^{1-\varepsilon} U_q^{\varepsilon}, \mathrm{conv}(U_m^{1-\varepsilon} U_q^{\varepsilon} \cup e^{\tau} U_s)) \le \beta(C\widetilde{U}_{\widetilde{p}} \cap e^t \widetilde{U}_{\widetilde{r}} \cap \widetilde{U}_{\widetilde{m}_0}^{1-\varepsilon} \widetilde{U}_{\widetilde{q}_0}^{\varepsilon}, \mathrm{conv}(\widetilde{U}_{\widetilde{m}}^{1-\varepsilon} \widetilde{U}_{\widetilde{q}}^{\varepsilon} \cup e^{\tau} \widetilde{U}_{\widetilde{s}})).$$

Let us estimate both sides of this inequality respectively from below and above using the same argument as in the proof of Corollary 3 but with

$$B_{a \vee b \vee c} \subset B_a \cap B_b \cap B_c \subset 3B_{a \vee b \vee c}$$

instead of the first formula in (5). Then we get

$$(8)\qquad \left|\left\{i : \frac{\max(a_{ip}, e^{-t} a_{ir}, a_{im}^{1-\varepsilon} a_{iq}^{\varepsilon})}{\min(a_{im}^{1-\varepsilon} a_{iq}^{\varepsilon}, e^{-\tau} a_{is})} \le 1\right\}\right| \le \left|\left\{j : \frac{\max(\widetilde{a}_{j\widetilde{p}}, e^{-t} \widetilde{a}_{j\widetilde{r}}, \widetilde{a}_{j\widetilde{m}_0}^{1-\varepsilon} \widetilde{a}_{j\widetilde{q}_0}^{\varepsilon})}{\min(\widetilde{a}_{j\widetilde{m}}^{1-\varepsilon} \widetilde{a}_{j\widetilde{q}}^{\varepsilon}, e^{-\tau} \widetilde{a}_{j\widetilde{s}})} \le 3C\right\}\right|.$$

Obviously the left-hand side of this inequality equals

$$\left|\left\{i : \frac{a_{ip}}{a_{im}^{1-\varepsilon} a_{iq}^{\varepsilon}} \le 1,\ \frac{e^{-t} a_{ir}}{a_{im}^{1-\varepsilon} a_{iq}^{\varepsilon}} \le 1,\ \frac{a_{im}^{1-\varepsilon} a_{iq}^{\varepsilon}}{e^{-\tau} a_{is}} \le 1\right\}\right|.$$

Since $a_{ip} = \exp([\chi(p - \kappa(i)) - 1/p] a_i)$ the first inequality in the last expression is equivalent to

$$\chi(p - \kappa(i)) - (1 - \varepsilon)\chi(m - \kappa(i)) - \varepsilon\chi(q - \kappa(i)) - \frac{1}{p} + \frac{1 - \varepsilon}{m} + \frac{\varepsilon}{q} \le 0.$$

We take $\varepsilon = 1/m$. Then the last inequality is true if and only if $p < \kappa(i) \le q$. The other two inequalities give for $\kappa(i) \in (p, q]$ respectively

$$\left(1 - \varepsilon - \frac{1}{r} + \frac{1 - \varepsilon}{m} + \frac{\varepsilon}{q}\right) a_i \le t, \qquad \left(1 - \varepsilon - \frac{1}{s} + \frac{1 - \varepsilon}{m} + \frac{\varepsilon}{q}\right) a_i \ge \tau,$$

hence, taking into account our choice of ε, we see that the left-hand side of (8) is greater than

$$(9)\qquad \left|\left\{i : p < \kappa(i) \le q,\ \frac{m}{m - 1}\tau \le a_i \le t\right\}\right|.$$

It is easy to see that the right-hand side of (8) is less than

$$(10)\qquad \left|\left\{j : \frac{\widetilde{a}_{j\widetilde{p}}}{\widetilde{a}_{j\widetilde{m}}^{1-\varepsilon} \widetilde{a}_{j\widetilde{q}}^{\varepsilon}} \le 3C,\ \frac{e^{-t} \widetilde{a}_{j\widetilde{r}}}{\widetilde{a}_{j\widetilde{m}}^{1-\varepsilon} \widetilde{a}_{j\widetilde{q}}^{\varepsilon}} \le 3C,\ \frac{\widetilde{a}_{j\widetilde{m}_0}^{1-\varepsilon} \widetilde{a}_{j\widetilde{q}_0}^{\varepsilon}}{e^{-\tau} \widetilde{a}_{j\widetilde{s}}} \le 3C\right\}\right|.$$

Here the first inequality is equivalent to

$$\chi(\widetilde{p} - \widetilde{\kappa}(j)) - (1 - \varepsilon)\chi(\widetilde{m} - \widetilde{\kappa}(j)) - \varepsilon\chi(\widetilde{q} - \widetilde{\kappa}(j)) - \frac{1}{\widetilde{p}} + \frac{1 - \varepsilon}{\widetilde{m}} + \frac{\varepsilon}{\widetilde{q}} \le \frac{\log(3C)}{\widetilde{a}_j}.$$

Note that in the case $\widetilde{\kappa}(j) \notin (\widetilde{p}, \widetilde{q}]$ this inequality implies

$$\left(-\frac{1}{\widetilde{p}} + \frac{1 - \varepsilon}{\widetilde{m}} + \frac{\varepsilon}{\widetilde{q}}\right)\widetilde{a}_j \le \log(3C),$$

therefore (since $\varepsilon = 1/m$) $\widetilde{a}_j \le 2\widetilde{m}\log(3C)$, and by the third inequality of (10) we get

$$\tau \le \log(3C) + 2\widetilde{a}_j \le \tau_0 := (1 + 4\widetilde{m})\log(3C).$$

Thus for $\tau > \tau_0$ the triple of inequalities in (10) is equivalent to

$$\widetilde{p} < \widetilde{\kappa}(j) \le \widetilde{q}, \quad \left(1 - \varepsilon - \frac{1}{\widetilde{r}} + \frac{1 - \varepsilon}{\widetilde{m}} + \frac{\varepsilon}{\widetilde{q}}\right)\widetilde{a}_j \le t + \log(3C),$$

$$\tau - \log(3C) \le \left(1 - \varepsilon\chi(\widetilde{q}_0 - \widetilde{\kappa}(j)) - \frac{1}{\widetilde{s}} + \frac{1 - \varepsilon}{\widetilde{m}_0} + \frac{\varepsilon}{\widetilde{q}_0}\right)\widetilde{a}_j.$$

Hence it is easy to see that for $\varepsilon = 1/m$ and $\tau > \tau_0$ the right-hand side of (8) is less than

$$(11)\qquad \left|\left\{j : \widetilde{p} < \widetilde{\kappa}(j) \le \widetilde{q};\ \frac{4\widetilde{m}}{4\widetilde{m} + 1} \cdot \frac{\widetilde{m}_0}{\widetilde{m}_0 + 1}\tau \le \widetilde{a}_j \le \frac{m}{m - 1} \cdot \frac{4\widetilde{m} + 2}{4\widetilde{m} + 1} t\right\}\right|.$$

Now it follows from the bounds (9) and (11) that (6) holds with

$$C_{\widetilde p} = \frac{m}{m-1}\cdot\frac{4\widetilde m+1}{4\widetilde m}\cdot\frac{\widetilde m_0+1}{\widetilde m_0}.$$

Hence $C_{\widetilde p}\to 1$ as $\widetilde p\to\infty$ because the function φ^{-1} diverges to ∞ together with its argument.

(b) As in (a) put for any $p\ge\varphi^3(1)$,

$$\widetilde m_1=\varphi^{-1}(p),\quad m=\varphi^{-2}(p),\quad \widetilde m=\varphi^{-3}(p),\quad \widetilde p=\varphi(p);$$

then

$$\widetilde U_{\widetilde p}\prec T(U_p)\prec \widetilde U_{\widetilde m_1}\prec T(U_m)\prec \widetilde U_{\widetilde m}.$$

Further choose successively sequences $(\widetilde s_j)$, (s_i), (r_i), $(\widetilde r_j)$, $(\widetilde q_j^1)$ with

$$\widetilde U_{(\widetilde s_j)}\prec T(U_{(s_i)})\prec T(U_{(r_i)})\prec \widetilde U_{(\widetilde r_j)}\prec \widetilde U_{(\widetilde q_j^1)}.$$

Since T is an isomorphism, by Lemma 5 such a choice is possible. Finally for any sequence (q_i) such that $\widetilde U_{(\widetilde q_j^1)}\prec T(U_{(q_i)})$ choose a sequence $(\widetilde q_j)$ such that $T(U_{(q_i)})\prec \widetilde U_{(\widetilde q_j)}$. Then by Lemma 4 and the elementary properties of β it follows that there exists a constant $C>0$ such that

$$\beta(e^tU_{(r_i)}\cap U_m^{1/2}U_{(q_i)}^{1/2},\operatorname{conv}(U_m^{1/2}U_{(q_i)}^{1/2}\cup e^\tau U_{(s_i)}\cup U_p))$$
$$\le \beta(Ce^t\widetilde U_{(\widetilde r_j)}\cap \widetilde U_{\widetilde m}^{1/2}\widetilde U_{(\widetilde q_j)}^{1/2},\operatorname{conv}(\widetilde U_{\widetilde m_1}^{1/2}\widetilde U_{(\widetilde q_j^1)}^{1/2}\cup e^\tau\widetilde U_{(\widetilde s_j)}\cup\widetilde U_{\widetilde p})).$$

Estimating from below and above as in (a), but using

$$U_{a\wedge b\wedge c}=\operatorname{conv}(U_a\cup U_b\cup U_c)$$

instead of the second formula in (5), we get

$$\text{(12)}\qquad \left|\left\{i:\frac{\max(e^{-t}a_{ir_i},a_{im}^{1/2}a_{iq_i}^{1/2})}{\min(a_{im}^{1/2}a_{iq_i}^{1/2},e^{-\tau}a_{is_i},a_{ip})}\le 1\right\}\right|$$
$$\le\left|\left\{j:\frac{\max(e^{-t}\widetilde a_{j\widetilde r_j},\widetilde a_{j\widetilde m}^{1/2}\widetilde a_{j\widetilde q_j}^{1/2})}{\min(\widetilde a_{j\widetilde m_1}^{1/2}\widetilde a_{j\widetilde q_j^1}^{1/2},e^{-\tau}\widetilde a_{j\widetilde s_j},\widetilde a_{j\widetilde p})}\le 2C\right\}\right|.$$

Obviously the left-hand side of this inequality equals

$$\left|\left\{i:\frac{e^{-t}a_{ir_i}}{a_{im}^{1/2}a_{iq_i}^{1/2}}\le 1,\ \frac{a_{im}^{1/2}a_{iq_i}^{1/2}}{e^{-\tau}a_{is_i}}\le 1,\ \frac{a_{im}^{1/2}a_{iq_i}^{1/2}}{a_{ip}}\le 1\right\}\right|.$$

The last inequality in the above expression is equivalent to

$$\frac12[\chi(m-\kappa(i))+\chi(q_i-\kappa(i))]-\chi(p-\kappa(i))+\frac1p-\frac1{2m}-\frac1{2q}\le 0,$$

which is true if and only if $\kappa(i)\le p$ or $\kappa(i)>q_i$. Then the other two inequalities are equivalent respectively to

$$\left(\gamma_i-\frac1{r_i}+\frac1{2m}+\frac1{2q_i}\right)a_i\le t,\qquad \left(\delta_i-\frac1{s_i}+\frac1{2m}+\frac1{2q_i}\right)a_i\ge\tau,$$

where γ_i and δ_i take values 0, 1/2, 1. Hence the left-hand side of (12) is greater than

$$\text{(13)}\qquad |\{i:\kappa(i)\le p\text{ or }\kappa(i)>q_i,\ 4m\tau\le a_i\le t/2\}|.$$

Analogously the right-hand side of (12) is less than

$$\left|\left\{j:\frac{e^{-t}\widetilde a_{j\widetilde r_j}}{\widetilde a_{j\widetilde m_1}^{1/2}\widetilde a_{j\widetilde q_j^1}^{1/2}}\le 2C,\ \frac{\widetilde a_{j\widetilde m}^{1/2}\widetilde a_{j\widetilde q_j}^{1/2}}{e^{-\tau}\widetilde a_{j\widetilde s_j}}\le 2C,\ \frac{\widetilde a_{j\widetilde m_1}^{1/2}\widetilde a_{j\widetilde q_j}^{1/2}}{\widetilde a_{j\widetilde p}}\le 2C\right\}\right|.$$

Here the last inequality is equivalent to

$$\frac12[\chi(\widetilde m_1-\widetilde\kappa(j))+\chi(\widetilde q_j-\widetilde\kappa(j))]-\chi(\widetilde p-\widetilde\kappa(j))+\frac1{\widetilde p}-\frac1{2\widetilde m_1}-\frac1{2\widetilde q_j}\le\frac{\log(2C)}{\widetilde a_j}.$$

Since $\widetilde a_j\to\infty$ as $j\to\infty$ this inequality holds for large enough j if and only if $\widetilde\kappa(j)\le\widetilde p$ or $\widetilde\kappa(j)>\widetilde q_j$. In that case the other two inequalities are equivalent to

$$\left(\widetilde\gamma_j-\frac1{\widetilde r_j}+\frac1{2\widetilde m_1}+\frac1{2\widetilde q_j^1}\right)\widetilde a_j\le t+\log(2C),$$
$$\left(\widetilde\delta_j-\frac1{\widetilde s_j}+\frac1{2\widetilde m}+\frac1{2\widetilde q_j}\right)\widetilde a_j\ge\tau-\log(2C),$$

where $\widetilde\gamma_j$ and $\widetilde\delta_j$ take values $0,1/2,1$. Obviously for large enough j the left-hand side of the first inequality is greater than $\widetilde a_j/(3\widetilde m_1)$, while the left-hand side of the second is less than $2\widetilde a_j$. Therefore there exists $\tau_0>3\log(2C)$ such that for $\tau\ge\tau_0$ the right-hand side of (12) is less than

$$\text{(14)}\qquad |\{j:\widetilde\kappa(j)<\widetilde p\text{ or }\widetilde\kappa(j)\ge\widetilde q_j;\ \tau/3\le\widetilde a_j\le 4\widetilde m_1t\}|.$$

Now (b) follows from the bounds (13) and (14).

4. Main results

THEOREM 2. *If* $X=D(\kappa,a)$ *and* $Y=D(\widetilde\kappa,\widetilde a)$ *are isomorphic and* $X\overset{\text{qd}}{\simeq}X^2$ *then* $X\overset{\text{qd}}{\simeq}Y$.

Proof. If X and Y are isomorphic then the conditions (a), (b) of Theorem 1 hold. Using them we construct a quasi-diagonal imbedding of X into Y^{10}. Analogously the corresponding symmetric conditions imply the existence of a quasi-diagonal imbedding of Y into X^{10}. Therefore since $X\overset{\text{qd}}{\simeq}X^2$ we obtain $X^{10}\overset{\text{qd}}{\hookrightarrow}Y^{10}$, so by Lemma 2, $X\overset{\text{qd}}{\hookrightarrow}Y$, and also $Y\overset{\text{qd}}{\hookrightarrow}X^{10}\overset{\text{qd}}{\simeq}X$.

Hence by Lemma 1, $X \overset{\mathrm{qd}}{\simeq} Y$. So, by symmetry we only have to prove that $X \overset{\mathrm{qd}}{\hookrightarrow} Y^{10}$.

Let I and J denote respectively the sets of indices of the canonical bases in X and Y.

By Theorem 1 there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that the condition (a) of Theorem 1 holds with $p = \varphi(\widetilde{p})$ and $\widetilde{q} = \varphi(q)$. In fact, one can consider the function φ used in the proof of Theorem 1. We put

$$p_n = \varphi^{n+4}(1), \quad n = 1, 2, \ldots$$

Then by Theorem 1 there exist constants $\tau_n > 0$, $c_n > 0, n = 1, 2, \ldots$, such that for $\tau > \tau_n$,

$$(15) \quad |\{i : p_n < \kappa(i) \le p_{n+1},\ \tau < a_i \le t\}| \le |\{j : p_{n-1} < \widetilde{\kappa}(j) \le p_{n+2},\ \tau/c_n < a_j \le c_n t\}|$$

and $c_n \to 1$ as $n \to \infty$.

Assume for convenience that $\tau_n = c_n^{3m_n-2}$, $n = 1, 2, \ldots$, where m_n are integers, and put

$$I_{n,m} = \{i : p_n < \kappa(i) \le p_{n+1},\ c_n^m < a_i \le c_n^{m+1}\},$$
$$J_{n,m} = \{j : p_{n-1} < \widetilde{\kappa}(j) \le p_{n+2},\ c_n^{m-1} < a_j \le c_n^{m+2}\},$$
$$I^{\beta,\gamma} = \bigcup_{r=1}^{\infty} \bigcup_{s>m_n} I_{3r-\beta,3s-\gamma}, \quad \beta, \gamma = 0, 1, 2,$$
$$K = \{i : \kappa(i) \le p_1\} \cup \bigcup_{n=1}^{\infty} \{i : p_n < \kappa(i) \le p_{n+1},\ a_i \le c_n^{3m_n-2}\}.$$

Then obviously we have

$$|I_{3r-\beta,3s-\gamma}| \le |J_{3r-\beta,3s-\gamma}|, \quad r \in \mathbb{N},\ s > m_n,$$

as a consequence of (15). Therefore, since the sets $J_{3r-\beta,3s-\gamma}, r, s = 1, 2, \ldots$, (for fixed β, γ) are disjoint, we deduce that for every $\beta, \gamma = 0, 1, 2$ there exists an injection $\sigma_{\beta,\gamma} : I^{\beta,\gamma} \to J$ such that

$$\sigma_{\beta,\gamma}(I_{3r-\beta,3s-\gamma}) \subset J_{3r-\beta,3s-\gamma}, \quad r \in \mathbb{N},\ s > m_n.$$

It is easy to see that

$$\frac{1}{c_{3r-\beta}^2} a_i \le \widetilde{a}_{\sigma_{\beta,\gamma}(i)} \le c_{3r-\beta}^2 a_i, \quad |\widetilde{\kappa}(\sigma_{\beta,\gamma}(i)) - \kappa(i)| \le p_{r+2} - p_{r-1},$$

for $i \in I_{3r-\beta,3s-\gamma}$, $r \in \mathbb{N}$, $s > m_n$, $\beta, \gamma = 0, 1, 2$. Therefore by Proposition 3 the formula

$$T_{\beta,\gamma}(e_i) = [\exp(a_i - \widetilde{a}_{\sigma_{\beta,\gamma}(i)})]\widetilde{e}_{\sigma_{\beta,\gamma}(i)}$$

defines a quasi-diagonal isomorphic imbedding $T_{\beta,\gamma} : X_{\beta,\gamma} \to Y$, where $X_{\beta,\gamma} = \overline{\operatorname{span}}_X\{e_i : i \in I^{\beta,\gamma}\}$, $\beta, \gamma = 0, 1, 2$.

On the other hand,

$$K \subset \{i : \kappa(i) \le p \text{ or } \kappa(i) > q_i\}$$

for some $p \in \mathbb{N}$ and a sequence of indices (q_i) such that $q_i \uparrow \infty$. By Theorem 1 there exist $\widetilde{p}, (q_j)$, $C > 0$ and $\tau_0 > 0$ such that (7) holds for $t > \tau > \tau_0$. Let

$$K_1 = \{i \in K : a_i > \tau_0\} = \{i_k : k \in \mathbb{N}\},$$
$$L = \{j \in \mathbb{N} : \widetilde{\kappa}(j) \le \widetilde{p} \text{ or } \widetilde{\kappa}(j) > \widetilde{q}_j\} = \{j_k : k \in \mathbb{N}\},$$
$$E = \overline{\operatorname{span}}_X\{e_i : i \in K_1\}, \quad F = \overline{\operatorname{span}}_Y\{\widetilde{e}_j : j \in L\}.$$

Then by Corollary 1,

$$E \overset{\mathrm{qd}}{\simeq} E_0(c), \quad F \overset{\mathrm{qd}}{\simeq} E_0(d),$$

where $c = (c_k) = (a_{i_k})$ and $d = (d_k) = (\widetilde{a}_{j_k})$. In this notation (7) means that

$$|\{k : \tau \le c_k \le t\}| \le |\{k : \tau/C \le d_k \le Ct\}|.$$

Then the result of Mityagin [7] (see also [12] for a simple proof, without using the Hall–König theorem) implies that $E_0(c) \overset{\mathrm{qd}}{\hookrightarrow} E_0(d)$, and therefore

$$E \overset{\mathrm{qd}}{\hookrightarrow} F \overset{\mathrm{qd}}{\hookrightarrow} Y.$$

Since the space $G = \overline{\operatorname{span}}\{e_i : i \in K,\ a_i \le \tau_0\}$ is finite-dimensional, we have $E \oplus G \overset{\mathrm{qd}}{\hookrightarrow} Y$. Finally, taking into account that X is the direct sum of $X_{\beta,\gamma}$, $\beta, \gamma = 0, 1, 2$, and E, G we get $X \overset{\mathrm{qd}}{\hookrightarrow} Y^{10}$.

COROLLARY 5. *Conditions* (a), (b) *of Theorem* 1, *together with the symmetric conditions obtained by interchanging the roles of* a, b *and* $\widetilde{a}, \widetilde{b}$, *determine a complete linear topological invariant in the class of* $D(\kappa, a)$ *spaces which are quasi-diagonally isomorphic to their Cartesian square.*

Finally, we consider the question of quasi-equivalence of absolute bases in $D(\kappa, a)$ spaces. Recall that two absolute bases (x_i) and (y_j) are *quasi-equivalent* if and only if the corresponding Köthe spaces $K(|x_i|_p)$ and $K(|y_j|_p)$ are quasi-diagonally isomorphic.

THEOREM 3. *If* $X = D(\kappa, a)$ *is quasi-diagonally isomorphic to its Cartesian square then any two absolute bases in* X *are quasi-equivalent.*

Proof. Of course it is enough to show that any absolute basis (x_j) in X is quasi-equivalent to the canonical basis (e_i). By [4] (see also [3]), the bases (e_i) and (x_j) are weakly quasi-equivalent, i.e. there exist constants $r_j > 0$ and a finite-to-one function $i(j) : J \to I$ such that the Köthe matrices $(|r_j x_j|_p)$ and $(|e_{i(j)}|_p)$ are equivalent. Hence $K(|r_j x_j|_p)$ and $K(|e_{i(j)}|_p)$ coincide and we obtain

$$K(|r_j x_j|_p) = D(\widetilde{\kappa}, \widetilde{a}),$$

where $\widetilde{\kappa}(j) = \kappa(i(j))$ and $\widetilde{a}_j = a_{i(j)}$. Since the spaces $D(\kappa, a)$ and $D(\widetilde{\kappa}, \widetilde{a})$ are isomorphic it follows by Theorem 1 that they are quasi-diagonally isomorphic. This proves the theorem.

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A non-regular Toeplitz flow with preset pure point spectrum

by

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Abstract. Given an arbitrary countable subgroup σ_0 of the torus, containing infinitely many rationals, we construct a strictly ergodic 0-1 Toeplitz flow with pure point spectrum equal to σ_0. For a large class of Toeplitz flows certain eigenvalues are induced by eigenvalues of the flow Y which can be seen along the aperiodic parts.

Introduction. In this paper we continue the study of Toeplitz flows initiated in 1984 by S. Williams in her work [W]. Toeplitz sequences have been known earlier (e.g. [O], [G-H], [J-K]), but it is the construction of Williams that is exploited in most of later works on Toeplitz sequences (e.g. [B-K1], [D], [B-K2], [I-L], [D-K-L], [I]). Spectral properties of Toeplitz flows have been studied in [I-L] and [I]. In this note we develop the method introduced by A. Iwanik in [I]. Each eigenvalue γ obtained there satisfies a certain equation formulated in Section I of this paper as (3). In [I], however, this equation remains unsolved, and an irrational γ is obtained by constructing uncountably many Toeplitz flows with different eigenvalues.

We have succeeded in solving the equation (3) simultaneously for an arbitrary countable set of γ's. This enables us to prove the existence of strictly ergodic Toeplitz flows with an arbitrarily preset pure point spectrum containing infinitely many rationals.

Section I contains slightly modified formulations of the results of [I]. We rid the constructions of technical details used in [I] to produce uncountably many sequences. For a large class of Toeplitz flows we identify certain eigenvalues not arising from the maximal uniformly continuous factor. We also adapt the cohomology statement of [I] to the countable product of tori.

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