But the points y_n do not τ-converge to z since $z \notin Y$ and

$$\overline{\{y_n\}}^{\tau} \subset \overline{\{y_n\}}^{\text{weak}} \subset Y.$$

Thus we close with the following.

4.6. Open problem. *Let X be a nonreflexive space that contains a τ-LUR body for a linear topology τ finer than the weak topology. Does then X contain two τ-LUR bodies C, D such that $C \mp D$ is not rotund?*

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Operators preserving orthogonality of polynomials

by

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Abstract. Let S be a degree preserving linear operator of $\mathbb{R}[X]$ into itself. The question is if, preserving orthogonality of some orthogonal polynomial sequences, S must necessarily be an operator of composition with some affine function of $\mathbb{R}$. In [2] this problem was considered for S mapping sequences of Laguerre polynomials onto sequences of orthogonal polynomials. Here we improve substantially the theorems of [2] as well as disprove the conjecture proposed there. We also consider the same questions for polynomials orthogonal on the unit circle.

Introduction. Call $\{p_n\}_{n=0}^{\infty} \subset \mathcal{P}$ where $\mathcal{P}$ is either $\mathbb{R}[X]$ or $\mathbb{C}[Z]$ a *polynomial system* (for short: PS) if $\deg p_n = n$, $n = 0, 1, \ldots$ A PS which is orthogonal with respect to a positive measure is here referred to as OGPS; if it is orthonormal the abbrevation is ONPS.

1. Let $\alpha \in \mathbb{R}$. Then, setting

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!},$$

the (*generalized*) *Laguerre polynomials* $L_n^{(\alpha)}$, $n = 0, 1, \ldots$, are defined as usual by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}, \quad x \in \mathbb{R}.$$

They satisfy the three-term recurrence relation

$$XL_n^{(\alpha)} = -(n+1)L_{n+1}^{(\alpha)} + (2n+1+\alpha)L_n^{(\alpha)} - (n+\alpha)L_{n-1}^{(\alpha)},$$
$$L_{-1}^{(\alpha)} = 0, \quad n = 0, 1, \ldots$$

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For $\alpha > -1$ the PS $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ is orthogonal with respect to a positive measure and its orthonormalization is given by

$$\int_0^\infty (L_n^{(\alpha)}(x))^2 e^{-x} x^\alpha\, dx = \frac{\Gamma(n+\alpha+1)}{n!}.$$

For $\alpha \le -1$, $\alpha \notin \{\ldots,-2,-1\}$, they are orthogonal with respect to a quasi-definite inner product while for $\alpha \in \{\ldots,-2,-1\}$ they are orthogonal with respect to a Sobolev type inner product (cf. [5], Proposition 3.3, and also [6]).

2. For $c,d \in \mathbb{R}$ define

$$\tau_{c,d}(x) = cx + d, \quad x \in \mathbb{R},$$

and

$$L_n^{(\alpha,c,d)} = L_n^{(\alpha)} \circ \tau_{c,d}, \quad n = 0,1,\ldots$$

Thus $L_n^{(\alpha)} = L_n^{(\alpha,1,0)}$.

Now let S be a linear operator of $\mathbb{R}[X]$ into itself preserving the degree of polynomials. The question is if preserving orthogonality of polynomials forces S to be of the form

$$Sp = sp \circ \tau_{a,b}, \quad p \in \mathbb{R}[X], \tag{1}$$

with some $s,a,b \in \mathbb{R}$. The results of [2] can be stated as follows.

THEOREM I (the orthonormal case). *If there is $\alpha \in \mathbb{R}$ not a negative integer* (1) *such that*

$$\left\{\left(\frac{n!}{\Gamma(n+\alpha+i+1)}\right)^{1/2} SL_n^{(\alpha+i)}\right\}_{n=0}^{\infty}$$

is an ONPS for any $i = 0,1,\ldots$, then S is of the form (1).

THEOREM II (the orthogonal case). *If there are $\alpha_1 < -1$ not a negative integer* (2) *and $\alpha_2 > -1$ such that $\{SL_n^{(\alpha,c,d)}\}_{n=0}^{\infty}$ is an orthogonal PS for $\alpha = \alpha_j + i$, $i = 0,1,\ldots$, $j = 1,2$, and for any $c,d \in \mathbb{R}$, $c \ne 0$, then S is of the form* (1).

CONJECTURE. *If there is α not a negative integer such that $\{SL_n^{(\alpha+i)}\}_{n=0}^{\infty}$ is an orthogonal PS for any $i = 0,1,\ldots$, then S is of the form* (1).

Our aim is to improve substantially Theorems I and II of Allaway (by the way, providing alternative proofs of those theorems as well as clarifying their circumstances) and to disprove the Conjecture. More precisely, both Theorems I and II require S to preserve orthonormality or orthogonality of an infinite number of ONPS's or, respectively, OGPS's. We are able to show that preserving orthonormality of 4 ONPS's is enough while in the the orthogonal case 16 of them do the job. In addition, we also bring the question over to the *unit circle* case (this question was raised in [1]) where the situation appears to be slightly different.

(1) It should have been $\alpha > -1$ so as to speak of orthonormality of the Laguerre polynomials in the commonly acceptable sense.

(2) Though the way in which orthogonality was defined in [2] might suggest that α would rather be greater than -1, cf. the previous footnote as well.

The real line case

3. In this section, as in the whole paper, S is a *degree preserving linear operator* of $\mathbb{R}[X]$ into itself. The observation which follows is the key to solving the problem.

PROPOSITION. *Let s,a,b be real numbers and $\{p_n\}_{n=0}^{\infty}$ be a PS. Then the following conditions are equivalent*:

(i) $sS(Xp_n) = S(X)S(p_n)$, $n \ge 0$,
(ii) $sS(p_m p_n) = S(p_m)S(p_n)$, $m,n \ge 0$,
(iii) $sS(pq) = S(p)S(q)$, $p,q \in \mathbb{R}[X]$,
(iv) $sS(Xp) = S(X)S(p)$, $p \in \mathbb{R}[X]$,
(v) $SX^n = s(aX+b)^n$, $n = 0,1,2,\ldots$,
(vi) *if $\{p_n\}_{n=0}^{\infty}$ satisfies the three-term recurrence relation*

$$Xp_n = \alpha_n p_{n+1} + \beta_n p_n + \gamma_n p_{n-1},$$

then $\{Sp_n\}_{n=0}^{\infty}$ satisfies the three-term recurrence relation with coefficients $a_n = a^{-1}\alpha_n$, $b_n = a^{-1}(\beta_n - b)$ and $c_n = a^{-1}\gamma_n$ respectively.

Under the above circumstances $s = S1$ and $Sp = sp \circ \tau_{a,b}$, $p \in \mathbb{R}[X]$.

Proof. The implications (i)⇒(ii)⇒(iii)⇒(iv)⇒(v) follow straightforwardly. To prove the implication (v)⇒(vi) one has to notice that (v) implies $Sp = sp \circ \tau_{a,b}$. The proof of (vi)⇒(i) goes as follows: first we get

$$S(Xp_n) = (aX+b)Sp_n, \quad n = 0,1,\ldots,$$

from this we get $SX = s(aX+b)$ where $s = S1$ and finally (i). ∎

The Proposition shows us that

1° each operator S defined by

$$Sp = p \circ \tau_{a,b}, \quad s,a,b \in \mathbb{R},$$

preserves orthogonality of *all* orthogonal PS's,

2° instead of thinking of preserving orthogonality of PS's we can consider preserving (in the sense of (vi)) the three-term recurrence relation.

4. According to what the Proposition suggests we are going to find, for at least one α, the coefficients $a_n^{(\alpha)}$, $b_n^{(\alpha)}$ and $c_n^{(\alpha)}$ in the three-term recurrence relation of $\{SL_n^{(\alpha)}\}_{n=0}^{\infty}$ and compare them with those of $L_n^{(\alpha)}$. First we do this for the coefficients $a_n^{(\alpha,c,d)}$, $b_n^{(\alpha,c,d)}$ and $c_n^{(\alpha,c,d)}$ in

$$XSL_n^{(\alpha,c,d)} = a_n^{(\alpha,c,d)}SL_{n+1}^{(\alpha,c,d)} + b_n^{(\alpha,c,d)}SL_n^{(\alpha,c,d)} + c_n^{(\alpha,c,d)}SL_{n-1}^{(\alpha,c,d)}, \tag{2}$$

which is the three-term recurrence relation of $\{SL_n^{(\alpha,c,d)}\}_{n=0}^{\infty}$. The simple formula (cf. [7, p. 102])

$$L_n^{(\alpha)} = L_n^{(\alpha+1)} - L_{n-1}^{(\alpha+1)}$$

implies immediately

$$SL_n^{(\alpha,c,d)} = SL_n^{(\alpha+1,c,d)} - SL_{n-1}^{(\alpha+1,c,d)} \tag{3}$$

and this turns out to be essential in what follows.

We will also need the explicit forms of the first four Laguerre polynomials:

$$L_0^{(\alpha)} = 1, \quad L_1^{(\alpha)} = -X + \alpha + 1,$$
$$L_2^{(\alpha)} = \tfrac{1}{2}X^2 - (\alpha+2)X + \tfrac{1}{2}(\alpha+2)(\alpha+1),$$
$$L_3^{(\alpha)} = -\tfrac{1}{6}X^3 + \tfrac{1}{2}(\alpha+3)X^2 - \tfrac{1}{2}(\alpha+3)(\alpha+2)X + \tfrac{1}{6}(\alpha+3)(\alpha+2)(\alpha+1).$$

LEMMA. *Let*

$$SX = aX + b \quad \textit{and} \quad SX^2 = s_{22}X^2 + s_{12}X + s_{02}.$$

Fix c and d and suppose $\{SL_n^{(\alpha,c,d)}\}_{n=0}^{\infty}$ satisfies the three-term recurrence relation with the coefficients $a_n^{(\alpha,c,d)}$, $b_n^{(\alpha,c,d)}$ and $c_n^{(\alpha,c,d)}$ for $\alpha = \alpha_0 - 1, \alpha_0, \alpha_0 + 1, \alpha_0 + 2$ with some $\alpha_0 \in \mathbb{R}$. Then

$$s_{22} = a^2 \tag{4}$$

and, for $\alpha = \alpha_0, \alpha_0 + 1, \alpha_0 + 2$ and $n \ge 0$,

$$a_n^{(\alpha,c,d)} = -(ac)^{-1}(n+1), \tag{5}$$
$$b_n^{(\alpha,c,d)} = (ac)^{-1}(-bc - d + 1 + (1+\gamma)n + \alpha), \tag{6}$$
$$c_n^{(\alpha,c,d)} = -(ac)^{-1}(\beta + (n+\alpha)\gamma), \tag{7}$$

where β and γ are real parameters such that

$$cs_{12} = 2abc + a - a\gamma, \tag{8}$$
$$c^2 s_{02} = b^2c^2 + bc + d - bc\gamma - d\gamma - \beta. \tag{9}$$

Proof. First notice that, because of the convention $L_{-1}^{(\alpha)} = 0$, $c_0^{(\alpha,c,d)}$ is unimportant. Suppose $S1 = 1$. Put (3) in (2) to get

$$\begin{aligned} XSL_n^{(\alpha,c,d)} &= a_n^{(\alpha,c,d)}SL_{n+1}^{(\alpha,c,d)} + b_n^{(\alpha,c,d)}SL_n^{(\alpha,c,d)} + c_n^{(\alpha,c,d)}SL_{n-1}^{(\alpha,c,d)} \\ &= a_n^{(\alpha,c,d)}SL_{n+1}^{(\alpha+1,c,d)} - a_n^{(\alpha,c,d)}SL_n^{(\alpha+1,c,d)} + b_n^{(\alpha,c,d)}SL_n^{(\alpha+1,c,d)} \\ &\quad - b_n^{(\alpha,c,d)}SL_{n-1}^{(\alpha+1,c,d)} + c_n^{(\alpha+1,c,d)}SL_{n-1}^{(\alpha+1,c,d)} - c_n^{(\alpha,c,d)}SL_{n-2}^{(\alpha+1,c,d)}. \end{aligned}$$

Doing the same in reverse order we get

$$\begin{aligned} XSL_n^{(\alpha,c,d)} &= XSL_n^{(\alpha+1,c,d)} - XSL_{n-1}^{(\alpha+1,c,d)} \\ &= a_n^{(\alpha+1,c,d)}SL_{n+1}^{(\alpha+1,c,d)} + b_n^{(\alpha+1,c,d)}SL_n^{(\alpha+1,c,d)} + c_n^{(\alpha+1,c,d)}SL_{n-1}^{(\alpha+1,c,d)} \\ &\quad - a_{n-1}^{(\alpha+1,c,d)}SL_n^{(\alpha+1,c,d)} - b_{n-1}^{(\alpha+1,c,d)}SL_{n-1}^{(\alpha+1,c,d)} + c_{n-1}^{(\alpha+1,c,d)}SL_{n-2}^{(\alpha+1,c,d)}. \end{aligned}$$

Comparing the coefficients of $SL_n^{(\alpha,c,d)}$'s (preserving degree!) we come to the following relations:

$$a_n^{(\alpha,c,d)} = a_n^{(\alpha+1,c,d)}, \qquad n > 0, \tag{10}$$
$$b_n^{(\alpha+1,c,d)} - b_n^{(\alpha,c,d)} = a_{n-1}^{(\alpha+1,c,d)} - a_n^{(\alpha,c,d)}, \qquad n > 0, \tag{11}$$
$$c_n^{(\alpha+1,c,d)} - c_n^{(\alpha,c,d)} = b_{n-1}^{(\alpha+1,c,d)} - b_n^{(\alpha,c,d)}, \qquad n > 0, \tag{12}$$
$$c_{n-1}^{(\alpha+1,c,d)} = c_n^{(\alpha,c,d)}, \qquad n > 1. \tag{13}$$

Set

$$A_n = a_{n-1}^{(\alpha,c,d)} - a_n^{(\alpha,c,d)} \quad \text{and} \quad C = b_0^{(\alpha,c,d)} - b_1^{(\alpha,c,d)} + A_1.$$

Then (11) and (12) imply

$$b_n^{(\alpha,c,d)} = b_n^{(\alpha_0,c,d)} + (\alpha - \alpha_0)A_n, \tag{14}$$
$$c_n^{(\alpha,c,d)} = c_n^{(\alpha_0,c,d)} + (\alpha - \alpha_0)(b_{n-1}^{(\alpha_0,c,d)} - b_n^{(\alpha_0,c,d)} + A_n). \tag{15}$$

Putting (15) into (13) and using the fact that the resulting equality is satisfied for (at least) two different α's (namely $\alpha = \alpha_0, \alpha_0 + 1$) we infer that, for $n > 1$,

$$b_{n-1}^{(\alpha_0,c,d)} - b_n^{(\alpha_0,c,d)} + A_n = b_{n-2}^{(\alpha_0,c,d)} - b_{n-1}^{(\alpha_0,c,d)} + A_{n-1}, \tag{16}$$
$$c_n^{(\alpha_0,c,d)} = c_{n-1}^{(\alpha_0,c,d)} + b_{n-1}^{(\alpha_0,c,d)} - b_n^{(\alpha_0,c,d)} + A_n. \tag{17}$$

Now (16) implies immediately

$$b_{n-1}^{(\alpha_0,c,d)} - b_n^{(\alpha_0,c,d)} + A_n = b_0^{(\alpha,c,d)} - b_1^{(\alpha,c,d)} + A_1 = C. \tag{18}$$

Consequently, (17) implies $c_n^{(\alpha,c,d)} = c_1^{(\alpha_0,c,d)} + (n-1)C$ and then (15) takes the form

$$c_n^{(\alpha_0,c,d)} = c_1^{(\alpha_0,c,d)} + (n-1)C + (\alpha - \alpha_0)C. \tag{19}$$

Inserting (14) into (12), using (18) and then (19), and comparing the coefficients of α we get

$$A_{n-1} = A_n, \quad n > 1. \tag{20}$$

Thus, from (18), we have

$$b_{n-1}^{(\alpha_0,c,d)} - b_n^{(\alpha_0,c,d)} = C - A_1, \quad b_n^{(\alpha_0,c,d)} = b_0^{(\alpha_0,c,d)} + n(A_1 - C)$$

and, finally, (14) can be written as

$$b_n^{(\alpha,c,d)} = b_0^{(\alpha_0,c,d)} + n(A_1 - C) + (\alpha - \alpha_0)A_1, \quad n \geq 0. \tag{21}$$

Now writing the three-term recurrence relation for $\{SL_n^{(\alpha,c,d)}\}_{n=0}^\infty$ in the case $n = 0$ we get

$$X = a_0^{(\alpha,c,d)}(-acX - bc - d + \alpha + 1) + b_0^{(\alpha,c,d)}.$$

Comparing the coefficients of X we get

$$a_0^{(\alpha,c,d)} = (ac)^{-1}.$$

Comparing the coefficients of X^0 and then those of α^0 we get

$$b_0^{(\alpha,c,d)} = (ac)^{-1}(-cb - d + 1 + \alpha_0).$$

Thus, after setting

$$\beta = ac(-c_1^{(\alpha,c,d)} + (1 + \alpha_0)C), \quad \gamma = -acC,$$

the formula (21) takes the form

$$b_n^{(\alpha,c,d)} = (ac)^{-1}(-cb - d + 1 + \alpha_0 + n(acA_1 + \gamma) + acA_1(\alpha - \alpha_0)), \tag{22}$$

while (19) becomes precisely (7). The only thing we have to do is to show (4)–(6), (8) and (9).

Now write the three-term recurrence relation for $\{SL_n^{(\alpha,c,d)}\}_{n=0}^\infty$ in the case $n = 1$. It leads to

$$\begin{aligned} X[-acX - bc - d + \alpha + 1] &= a_1^{(\alpha,c,d)}\left[\tfrac{1}{2}c^2(s_{22}X^2 + s_{12}X + s_{02}) + cd(aX + b)\right. \\ &\quad \left. + \tfrac{1}{2}d^2 - (\alpha + 2)(acX + bc + d) + \tfrac{1}{2}(\alpha + 2)(\alpha + 1)\right] \\ &\quad + (ac)^{-1}(-bc - d + 1 + \alpha_0 + acA_1 + \gamma + acA_1(\alpha - \alpha_0)) \\ &\quad \times [-acX - bc - d + \alpha + 1] - (ac)^{-1}[\beta + (1 + \alpha)\gamma]. \end{aligned}$$

Comparing the coefficients of X^0 and then of α^2 we get $0 = \frac{1}{2}a_1 + A_1$ and, invoking the definition of A_1, we get immediately $a_1 = 2a_0 = -2(ac)^{-1}$ and $A_1 = (ac)^{-1}$. Consequently, by (20), we get (5) and (4). Then also (22) becomes (6). Now comparing the coefficients of X^0 and then of α^0 we get (9) while comparing the coefficients of X and then of α^0 we get (8). ■

5. Now we are ready to prove our "orthonormal" result:

THEOREM 1. *Let $S : \mathbb{R}[X] \to \mathbb{R}[X]$ be a degree preserving linear operator $SX = aX + b$. Suppose $\{SL_n^{(\alpha)}\}_{n=0}^\infty$ satisfies the three-term recurrence relation for $\alpha = \alpha_0 - 1, \alpha_0, \alpha_0 + 1, \alpha_0 + 2$ for some $\alpha_0 \in \mathbb{R}$ and, moreover, for (at least) two of $\{\alpha_0, \alpha_0 + 1, \alpha_0 + 2\}$*

$$c_1^{(\alpha)} = 1 + \alpha.$$

Then S is of the form (1).

Proof. By (7) of the Lemma, $\beta + (1+\alpha)\gamma = 1 + \alpha$. Allowing two different α's we infer that $\beta = 0$ and $\gamma = 1$. Consequently,

$$a_n^{(\alpha)} = -a^{-1}(n + 1), \tag{23}$$

$$b_n^{(\alpha)} = a^{-1}(-b + 1 + 2n + \alpha), \tag{24}$$

$$c_n^{(\alpha)} = -a^{-1}(n + \alpha). \tag{25}$$

According to the Proposition, (vi) implies (iv), and this completes the proof. ■

The following is immediate:

COROLLARY 1. *Let $S : \mathbb{R}[X] \to \mathbb{R}[X]$ be a degree preserving linear operator with $SX = aX + b$. Suppose $\{SL_n^{(\alpha)}\}_{n=0}^\infty$ is an OGPS for $\alpha = \alpha_0 - 1, \alpha_0, \alpha_0 + 1, \alpha_0 + 2$ with some $\alpha_0 > 0$ and, moreover,*

$$\|SL_1^{(\alpha)}\|^2 / \|SL_0^{(\alpha)}\|^2 = 1 + \alpha$$

for (at least) two of $\{\alpha_0, \alpha_0 + 1, \alpha_0 + 2\}$. Then S is of the form (1).

The norm in the above is the image norm of $L_n^{(\alpha)}$'s under S.

Proof. Since, what is easy to verify, $\|SL_1^{(\alpha)}\|^2 / \|SL_0^{(\alpha)}\|^2 = c_1^{(\alpha)}/a_0^{(\alpha)}$, the conclusion follows from Theorem 1. ■

Our Corollary 1 improves substantially Theorem I which is, due to normalization of $\{L_n^{(\alpha)}\}_{n=0}^\infty$, precisely Theorem (3.1) of [2]. Now we pass to extending Theorem II.

THEOREM 2. *Let $S : \mathbb{R}[X] \to \mathbb{R}[X]$ be a degree preserving linear operator with $SX = aX + b$. Suppose $\{SL_n^{(\alpha,c,d)}\}_{n=0}^\infty$ satisfies the three-term recurrence relation for $\alpha = \alpha_0 - 1, \alpha_0, \alpha_0 + 1, \alpha_0 + 2$ with some $\alpha_0 \in \mathbb{R}$, two different nonzero c's and two different d's. Then S is of the form (1).*

Proof. Set $SX^3 = s_{33}X^3 + s_{23}X^2 + s_{13}X + s_{03}$ and write the three-term recurrence relation for $\{L_n^{(\alpha,c,d)}\}_{n=0}^\infty$ in the case $n = 2$:

$$X\left[\tfrac{1}{2}c^2(a^2X^2+s_{12}X+s_{02})+cd(aX+b)+\tfrac{1}{2}d^2\right.$$
$$\left.-(\alpha+2)(acX+bc+d)+\tfrac{1}{2}(\alpha+2)(\alpha+1)\right]$$
$$=-3(ac)^{-1}\left[-\tfrac{1}{6}(c^3(s_{33}X^3+s_{23}X^2+s_{13}X+s_{03})\right.$$
$$+3c^2d(a^2X^2+s_{12}X+s_{02})+3cd^2(aX+b)+d^2)$$
$$+\tfrac{1}{2}(\alpha+3)(c^2(a^2X^2+s_{12}X+s_{02})+2cd(aX+b)+d^2)$$
$$\left.-\tfrac{1}{2}(\alpha+3)(\alpha+2)(acX+bc+d)+\tfrac{1}{6}(\alpha+3)(\alpha+2)(\alpha+1)\right]$$
$$+(ac)^{-1}(-bc-d+3+2\gamma+\alpha)$$
$$\times\left[\tfrac{1}{2}c^2(a^2X^2+s_{12}X+s_{02})+cd(aX+b)+\tfrac{1}{2}d^2\right.$$
$$\left.-(\alpha+2)(acX+bc+d)+\tfrac{1}{2}(\alpha+2)(\alpha+1)\right]$$
$$-(ac)^{-1}(\beta+2\gamma+\gamma\alpha)[-ac-bc-d+\alpha+1].$$

Comparing the coefficients of X^0 and then of α^0 we get

$$0=\tfrac{1}{2}c^3s_{03}+c^2ds_{02}-3c^2s_{02}-\tfrac{1}{2}bc^3s_{02}-2bcd-d^2$$
$$+2bc+2d-b^2c^2d^2+2b^2c^2+c^2s_{02}\gamma+2bcd\gamma$$
$$+d^2\gamma-2bc\gamma-2d\gamma+bc\beta+d\beta-\beta.$$

Inserting β from (9) into the above we simplify it as

$$0=\tfrac{1}{2}c^3s_{03}-\tfrac{3}{2}bc^3s_{02}-2c^2s_{02}+b^3c^3+2b^2c^2$$
$$+bc+d+c^2s_{02}\gamma-b^2c^2\gamma-bc\gamma-d\gamma.$$

Comparing the coefficients of d we get $\gamma=1$, and from those of d^0, after dividing by c^2, we get $s_{02}=b^2$ and, consequently, $\beta=0$. Thus $\{b_n^{(\alpha)}\}_{n=0}^{\infty}$ and $\{c_n^{(\alpha)}\}_{n=0}^{\infty}$ satisfy (24) and (25). Finally, due to the Lemma again, we come to the conclusion. ∎

Again our result betters significantly Theorem (4.1) of [2] (here reported as Theorem II).

6. The result which follows is related to Allaway's Conjecture.

THEOREM 3. *For any $\beta\in\mathbb{R}$ the operator S_β defined as*

$$S_\beta L_n^{(\alpha)}=L_n^{(\alpha+\beta)}\circ\tau_{1,\beta},\qquad n=0,1,\ldots,$$

is independent of α.

Proof. Fixing β we define a linear operator S_β^α as

$$S_\beta^\alpha L_n^{(\alpha)}=L_n^{(\alpha+\beta)}\circ\tau_{1,\beta},\qquad n=0,1,\ldots$$

We show that for $\alpha,\alpha'\in\mathbb{R}$,

$$S_\beta^{\alpha'}L_n^{(\alpha)}=S_\beta^{\alpha}L_n^{(\alpha)},\qquad n=0,1,\ldots \tag{26}$$

This will follow from the formula (cf. [3, p. 57])

$$L_n^{(\gamma+\delta+1)}=\sum_{j=0}^{n}A_{n-j}^{(\gamma)}L_j^{(\delta)}\quad\text{where}\quad A_k^{(\gamma)}=\binom{\gamma+k}{k}.$$

Indeed, using this formula, we have

$$S_\beta^{\alpha'}L_n^{(\alpha)}=\sum_{j=0}^{n}A_{n-j}^{(\alpha-\alpha'-1)}S_\beta^{\alpha'}L_j^{(\alpha')}=\sum_{j=0}^{n}A_{n-j}^{(\alpha-\alpha'-1)}L_j^{(\alpha'+\beta)}\circ\tau_{1,\beta}$$
$$=L_n^{(\alpha-\alpha'-1+\alpha'+\beta+1)}\circ\tau_{1,\beta}=S_\beta^{\alpha}L_n^{(\alpha)}.$$

This is (26) and, because $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ is a basis of $\mathbb{R}[X]$, we infer that

$$S_\beta^{\alpha'}p=S_\beta^{\alpha}p,\qquad p\in\mathbb{R}[X],$$

which gives us the conclusion. ∎

It is a matter of direct verification to check that $S_\beta L_n^{(\alpha)}$'s satisfy the three-term recurrence relation

$$XS_\beta L_n^{(\alpha)}=-(n+1)S_\beta L_{n+1}^{(\alpha)}+(2n+1+\alpha)S_\beta L_n^{(\alpha)}-(n+\alpha+\beta)S_\beta L_{n-1}^{(\alpha)},$$
$$S_\beta L_{-1}^{(\alpha)}=0,\qquad n=0,1,\ldots$$

So for any α, $\{S_\beta L_n^{(\alpha)}\}_{n=0}^{\infty}$ is an orthogonal PS (cf. Introduction, Sec. 2). On the other hand,

$$S_\beta L_0^{(\alpha)}=L_0^{(\alpha)},\qquad S_\beta L_1^{(\alpha)}=L_1^{(\alpha)}\quad\text{while}\quad S_\beta L_2^{(\alpha)}=L_2^{(\alpha)}-\beta/2.$$

This gives us immediately

COROLLARY 2. *The operator S_β of Theorem 3 preserves orthogonality of all PS's $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$, $\alpha\in\mathbb{R}$, and it is not of the form (1) if $\beta\neq0$.*

This *disproves* Allaway's Conjecture.

The unit circle case

7. For $\varphi\in\mathbb{C}[Z]$ of the form $\varphi=\sum_{i=0}^{n}a_iZ^i$, $a_n\neq0$, define φ^* as

$$\varphi^*=\sum_{i=0}^{n}\overline{a}_iZ^{n-i}=\sum_{i=0}^{n}\overline{a}_{n-i}Z^i.$$

The following classical characterization (which is implicit in [4, pp. 3–5]) of polynomials orthogonal on the unit circle will be used:

(*) *The sequence* $\{\varphi_n\}_{n=0}^\infty \subset \mathbb{C}[Z]$ *of monic polynomials is orthogonal with respect to a real (positive resp.) measure* μ *supported on the unit circle, that is,*

$$\int_0^{2\pi} \varphi_m(e^{it})\overline{\varphi_n(e^{it})}\,\mu(dt) = c_m\delta_{mn},$$

$$c_m \neq 0 \quad (c_m > 0 \text{ resp.}), \quad m, n = 0, 1, \ldots,$$

if and only if, for $n = 0, 1, \ldots$, *it satisfies the recurrence relation*

$$\varphi_{n+1} = Z\varphi_n + \varphi_{n+1}(0)\varphi_n^*$$

and $|\varphi_{n+1}(0)| \neq 1$ $(|\varphi_{n+1}(0)| < 1$ *resp.*$)$.

Notice that the above recurrence relation for not necessarily monic polynomials reads as

$$\alpha_n\varphi_{n+1} = \alpha_{n+1}Z\varphi_n + \varphi_{n+1}(0)\frac{\alpha_n}{\overline{\alpha}_n}\varphi_n^* \tag{27}$$

where α_n is the leading coefficient of φ_n allowed to be complex. For $\alpha \in \mathbb{C}$ set

$$\tau_\alpha = \alpha Z \quad \text{and} \quad S_\alpha\varphi = \varphi \circ \tau_\alpha, \quad \varphi \in \mathbb{C}[Z].$$

Let, in what follows, $|w| < 1$. Define

$$\varphi_0^{(w,1)} = (1 - |w^2|)^{1/2}, \quad \varphi_n^{(w,1)} = Z^{n-1}(Z - w), \quad n = 1, 2, \ldots,$$

and

$$\varphi_0^{(w,2)} = 1, \quad \varphi_1^{(w,2)} = Z - \frac{2\overline{w}}{1 + |w|^2},$$

$$\varphi_n^{(w,2)} = Z^{n-2}(Z - w)^2, \quad n = 2, 3, \ldots$$

Then the sequence $\{\varphi_n^{(w,1)}\}_{n=0}^\infty$ is an ONPS on the unit circle with respect to the measure

$$\frac{1}{2\pi}|e^{it} - w|^{-2}\,dt$$

while $\{\varphi_n^{(w,2)}\}_{n=0}^\infty$ is an OGPS on the unit circle with respect to the measure

$$\frac{1}{2\pi}|e^{it} - w|^{-4}\,dt.$$

8. Let S be a degree preserving linear operator of $\mathbb{C}[Z]$ into itself. Then we have the following

LEMMA 2. *Suppose we are given three different real numbers* w_1, w_2, w_3, *with* $|w_k| < 1$, $k = 1, 2, 3$. *If, for any* $k = 1, 2, 3$, $\{S\varphi_n^{(w_k)}\}_{n=0}^\infty$ *satisfies the recurrence relation* (27), *then* $S = sS_\alpha$, *for some* $s, \alpha \in \mathbb{C}$.

Proof. There is no loss of generality if we assume $S(1) = 1$. Set

$$SZ^n = \sum_{i=0}^n s_{i,n}Z^i.$$

Then, with the notation $s_{n,n-1} = 0$, we have

$$S\varphi_n^{(w,1)} = \sum_{i=0}^n (s_{i,n} - ws_{i,n-1})Z^i, \quad n \geq 1,$$

$(S\varphi_n^{(w,1)})(0) = s_{0,n} - ws_{0,n-1}$ and

$$(S\varphi_n^{(w,1)})^* = \sum_{i=0}^n (\overline{s}_{n-i,n} - w\overline{s}_{n-i,n-1})Z^i.$$

If w is any of w_1, w_2, w_3, the recurrence relation (27) for $\{S\varphi_n^{(w,1))}\}_{n=0}^\infty$ leads to

$$\begin{aligned} s_{n,n}&\sum_{i=0}^{n+1}(s_{i,n+1} - ws_{i,n})Z^i \\ &= s_{n+1,n+1}\sum_{i=0}^n (s_{i,n} - ws_{i,n-1})Z^{i+1} \\ &\quad + (s_{0,n+1} - ws_{0,n})\sum_{i=0}^n (\overline{s}_{n-i,n} - w\overline{s}_{n-i,n-1})Z^i\frac{s_{n,n}}{\overline{s}_{n+1,n+1}}. \end{aligned}$$

Comparing the coefficients of Z^i, $i = 1, \ldots, n$, we get

$$\begin{aligned} s_{n,n}(s_{i,n+1} - ws_{i,n}) &= s_{n+1,n+1}(s_{i-1,n} - ws_{i-1,n-1}) \\ &\quad + (s_{0,n+1} - ws_{0,n})(\overline{s}_{n-i,n} - w\overline{s}_{n-i,n-1})\frac{s_{n,n}}{\overline{s}_{n+1,n+1}}. \end{aligned}$$

Now comparing the coefficients of w^2, because always $s_{n,n} \neq 0$, we get

$$s_{0,n}\overline{s}_{n-i,n-1} = 0, \quad 1 \leq i \leq n,$$

and putting $i = 1$ implies

$$s_{0,n} = 0, \quad n \geq 1.$$

After comparing the coefficients of w, we have

$$\begin{aligned} s_{n,n}s_{i,n} &= s_{n+1,n+1}s_{i-1,n-1} \\ &\quad + s_{0,n}\overline{s}_{n-i,n} + s_{0,n+1}\overline{s}_{n-i,n-1}, \quad 1 \leq i \leq n. \end{aligned} \tag{28}$$

Putting $i = n$ and using $s_{0,n} = 0$ this gives us

$$(s_{n,n})^2 = s_{n+1,n+1}s_{n-1,n-1}$$

and, consequently, since $s_{0,0} = 1$, we get $s_{n,n} = \alpha^n$ where $\alpha = s_{1,1}$. Now (28) simplifies to

$$s_{i,n} = \alpha s_{i-1,n-1}, \quad n \geq 1,$$

and, since $s_{0,n} = 0$, this leads to $s_{k,n+k} = 0$, $k = 0, 1, \ldots$ Thus we finally obtain $S = S_\alpha$. Removing the assumption $S1 = 1$ we get $s = S1$. ∎

Because the operator S_α does not affect the numbers $\varphi_{n+1}^{(w,i)}(0)$, $n \geq 0$, it preserves the kind of orthogonality of $\{\varphi_n^{(w,i)}\}_{n=0}^\infty$ (cf. (∗)). In particular, invoking the characterization (∗), we get immediately

THEOREM 4. *Suppose we are given three different real numbers* w_1, w_2, w_3, *with* $|w_k| < 1$, $k = 1, 2, 3$. *If, for any* $k = 1, 2, 3$, $\{S\varphi_n^{(w_k)}\}_{n=0}^\infty$ *is an OGPS, then* $S = sS_\alpha$ *with* $|\alpha| > \max\{|w_1|, |w_2|, |w_3|\}$.

In order to get more information about α we have to allow S to preserve orthogonality of more ONPS's. For instance we have

COROLLARY 3. *Suppose* $W = \{w_k\}$ *is a sequence such that* $|w_k| < 1$, $k = 1, 2, \ldots$, *and* $\sup_k |w_k| = 1$. *Suppose that among* w_k*'s there are at least three different real numbers. If for any* $w \in W$, $\{S\varphi_n^w\}_{n=0}^\infty$ *is an OGPS, then* $|\alpha| \geq 1$.

On the other hand, for any w with $|w| < 1$ and α with $|\alpha| \geq 1$ we have

$$S_\alpha \varphi_n^{(w,1)} = \alpha^n \varphi_n^{(\alpha^{-1}w,1)}, \quad n = 1, 2, \ldots, \quad S_\alpha \varphi_0^{w,1} = \varphi_0^{(w,1)},$$

which means that, while $\{\varphi_n^{(w,1)}\}$ is an ONPS, $\{S_\alpha \varphi_n^{(w,1)}\}_n$ is an OGPS. However, it is not an ONPS unless $|\alpha| = 1$. This is reflected in the following

THEOREM 5. *Suppose we are given three different real numbers* w_1, w_2, w_3, *with* $|w_k| < 1$, $k = 1, 2, 3$. *If for any* $k = 1, 2, 3$, $\{S\varphi_n^{(w_k)}\}_{n=0}^\infty$ *is an OGPS and if for at least one* $w = w_k$,

$$\|S\varphi_0^{(w,1)}\| = \|S\varphi_1^{(w,1)}\|,$$

then $S = sS_\alpha$ *for some* $s, \alpha \in \mathbb{C}$ *with* $|\alpha| = 1$.

In the above the norm is the image norm of $\{\varphi_n^{(w,1)}\}$ under S.

Proof. Again $S1 = 1$. Since $0 = \langle S\varphi_1^{(w,1)}, 1\rangle = \langle \alpha Z - w, 1\rangle = \alpha\langle Z, 1\rangle - w\|1\|^2$, we get

$$\begin{aligned}\|S\varphi_1^{(w,1)}\|^2 &= \|\alpha Z - w\|^2 = |\alpha|^2\|1\|^2 - 2\Re(\alpha\overline{w}\langle Z, 1\rangle) + |w|^2\|1\|^2 \\ &= (|\alpha|^2 - |w|^2)\|1\|^2.\end{aligned}$$

On the other hand, we have $\|S\varphi_0^{(w,1)}\|^2 = (1 - |w|^2)\|1\|^2$. This provides the argument in the proof of $|\alpha| = 1$. ∎

9. Consider now the sequence $\{\varphi_n^{(w,2)}\}_{n=0}^\infty$. We get the following

LEMMA 3. *Suppose we are given five different real numbers* $w_1, \ldots, w_5$, *with* $|w_k| < 1$, $k = 1, \ldots, 5$. *If, for any* $k = 1, \ldots, 5$, $\{S\varphi_n^{(w_k)}\}_{n=0}^\infty$ *satisfies the recurrence relation* (27), *then* $S = sS_\alpha$ *for some* $s, \alpha \in \mathbb{C}$.

Proof. The proof is much the same as that of Lemma 2. Here we point out the major steps. Assume again $S(1) = 1$ and set

$$SZ^n = \sum_{i=0}^n s_{i,n} Z^i.$$

Then, the recurrence relation (27) for $\{S\varphi_n^{(w,2)}\}_{n=0}^\infty$ leads to ($n \geq 2$)

$$\begin{aligned}&s_{n,n} \sum_{i=0}^{n+1} (s_{i,n+1} - 2ws_{i,n} + w^2 s_{i,n-1}) Z^i \\ &= s_{n+1,n+1} \sum_{i=0}^{n} (s_{i,n} - 2ws_{i,n-1} + w^2 s_{i,n-2}) Z^{i+1} \\ &\quad + (s_{0,n+1} - 2ws_{0,n} + w^2 s_{0,n-1}) \\ &\quad \times \sum_{i=0}^{n} (\overline{s}_{n-i,n} - 2w\overline{s}_{n-i,n-1} + w^2 \overline{s}_{n-i,n-2}) Z^i \frac{s_{n,n}}{\overline{s}_{n+1,n+1}}.\end{aligned}$$

Comparing the coefficients of Z^i, $i = 1, \ldots, n$, and then of w^4 we get $s_{0,n} = 0$, $n \geq 1$. Then comparing the coefficients of w we come to

$$s_{k,n+k} = 0, \quad k = 0, 1, \ldots, \quad n = 1, 2, \ldots,$$

and $(s_{n,n})^2 = s_{n+1,n+1} s_{n-1,n-1}$. Consequently, we get the final conclusion. ∎

Once we know that $S = S_\alpha$ we can apply S to (27) and compare this with the recurrence relation for $\{S\varphi_n\}_{n=0}^\infty$ to get

$$\varphi_{n+1}(0)(\varphi_n(\overline{\alpha}^{-1}z) - \varphi(\alpha z)) = 0, \quad n = 0, 1, \ldots, \quad z \in \mathbb{C}.$$

While for $\{\varphi_n^{(w,1)}\}_{n=0}^\infty$ this condition provides no additional information, for the sequence $\{\varphi_n^{(w,2)}\}_{n=0}^\infty$ it implies immediately (putting $n = 1$) that $|\alpha| = 1$. Thus we arrive at the following

THEOREM 6. *Suppose we are given five different real numbers* $w_1, \ldots, w_5$, *with* $|w_k| < 1$, $k = 1, \ldots, 5$. *If, for any* $k = 1, \ldots, 5$, *the PS* $\{S\varphi_n^{(w_k,2)}\}_{n=0}^\infty$ *satisfies the recurrence relation* (27), *then* $S = sS_\alpha$ *with* $|\alpha| = 1$.

Remark. Assuming in all the above that w_k's are real is a matter of convenience, not of necessity.

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On Dragilev type power Köthe spaces

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Abstract. A complete isomorphic classification is obtained for Köthe spaces $X = K(\exp[\chi(p-\kappa(i)) - 1/p]a_i)$ such that $X \overset{\text{qd}}{\simeq} X^2$; here χ is the characteristic function of the interval $[0,\infty)$, the function $\kappa : \mathbb{N} \to \mathbb{N}$ repeats its values infinitely many times, and $a_i \to \infty$. Any of these spaces has the quasi-equivalence property.

1. Introduction. For any matrix $(a_{ip})_{i\in I, p\in\mathbb{N}}$ of positive numbers (with countable index set I) we denote by $K(a_{ip})$ (or $K(a_{ip}, i \in I)$) the Köthe space generated by the matrix (a_{ip}).

M. M. Dragilev [1] proved that there exist Köthe spaces with regular bases which are not distinguished by the diametral dimension

$$\Gamma(X) = \{\gamma = (\gamma_n) : \ \forall p\ \exists q\ \gamma_n d_n(U_p, U_q) \to 0\},$$

considering the power Köthe spaces

$$D(\kappa, a) = K(\exp[\chi(p-\kappa(i)) - 1/p]a_i), \tag{1}$$

where $(\kappa(i)) = (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1,2,3,4,5,6,\ldots)$, $a = (a_i)$, $a_i \nearrow \infty$, $\chi(t) = 0$ for $t < 0$, $\chi(t) = 1$ for $t \geq 0$. We investigate here an analogous class of power Köthe spaces given by (1) for an arbitrary function $\kappa : \mathbb{N} \to \mathbb{N}$ that repeats its values infinitely many times and an arbitrary sequence of positive numbers $a_i \to \infty$ (not necessarily increasing).

Our aim is to study the structure and isomorphic classification of $D(\kappa, a)$ spaces for different κ and a. In order to distinguish non-isomorphic spaces of this class we first construct appropriate invariant characteristics (generalized linear topological invariants). The method of generalized linear topological invariants was developed in [6], [7], [9]–[11] (see the survey [12] for more details).

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