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Departamento de Matemáticas
Universidad Nacional de Mar del Plata
Mar del Plata, Bs. As.
Argentina
E-mail: ferrando@bart.uni-mdp.edu.ar

Department of Mathematics
DePaul University
2219 N. Kenmore
Chicago, Illinois 60614
U.S.A.
E-mail: rjones@condor.depaul.edu

Department of Mathematics
University at Albany, SUNY
1400 Western Ave.
Albany, New York 12222
U.S.A.
E-mail: reinhold@csc.albany.edu

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On invariant measures for power bounded positive operators

by

RYOTARO SATO (Okayama)

To the memory of Hisao Tominaga

Abstract. We give a counterexample showing that $\overline{(I-T^*)L_\infty}\cap L_\infty^+=\{0\}$ does not imply the existence of a strictly positive function u in L_1 with $Tu=u$, where T is a power bounded positive linear operator on L_1 of a σ-finite measure space. This settles a conjecture by Brunel, Horowitz, and Lin.

1. Introduction. Let (X,Σ,m) be a σ-finite measure space and T a positive linear operator in $L_1=L_1(X,\Sigma,m)$. T is called a *contraction* if $\|T\|\le 1$, *power bounded* if $\sup_n\|T^n\|<\infty$, and *Cesàro bounded* if $\sup_n\|n^{-1}\sum_{k=1}^n T^k\|<\infty$. Many ergodic theorems for positive L_1 contractions require the existence of a finite invariant measure equivalent to the original one, i.e., a strictly positive $u\in L_1$ with $Tu=u$. This problem has attracted many top researchers, and one of the conditions equivalent to the existence of such a $u\in L_1$, obtained by Brunel [1], is that

$$\overline{(I-T^*)L_\infty}\cap L_\infty^+=\{0\}. \tag{1}$$

For any T positive and Cesàro bounded, condition (1) is seen, by using the known fact that $n^{-1}\|T^n\|\to 0$ as $n\to\infty$, to be equivalent to the following condition:

$$\limsup_n\Big\|\frac{1}{n}\sum_{k=1}^n T^{*k}1_A\Big\|_\infty>0\quad\text{for any } A\in\Sigma \text{ with } m(A)>0. \tag{2}$$

Sucheston [7] started a systematic study of power bounded positive linear operators in L_1, and Fong [4] studied the problem of existence of strictly positive fixed points under an additional assumption of a null disappearing part. The problem in general was studied by Derriennic and Lin [3] (see also Sato [6]), who proved that for any T positive and Cesàro bounded, an

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equivalent condition for the existence of a strictly positive fixed point in L_1 is

$$(3)\quad m\Big(\Big\{\limsup_n \frac{1}{n}\sum_{k=1}^{n} T^{*k}1_A > 0\Big\}\Big) > 0 \quad \text{for any } A \in \Sigma \text{ with } m(A) > 0,$$

and implicitly asked if also the weaker condition (2) (hence (1)) is sufficient. For more detailed arguments in this topic we refer the reader to Krengel's book [5].

Recently Brunel, Horowitz and Lin [2] showed that (1) implies the existence of a strictly positive *subinvariant* function in L_1, but left the problem of invariance unsolved. Further they proved that the condition

$$(4)\qquad (\text{weak}^*\text{-closure}\,(I - T^*)L_\infty) \cap L_\infty^+ = \{0\}$$

is necessary and sufficient for the existence of a strictly positive invariant function in L_1. The purpose of this note is to construct a power bounded positive linear operator in L_1 satisfying (1) which has no $u \in L_1$ with $Tu = u \neq 0$.

2. The construction of an operator

THEOREM. *If L_1 has infinite dimension, then for any $\varepsilon > 0$ there exists a positive linear operator T in L_1 and $0 \le \nu \in L_\infty^*$, with $\nu = T^{**}\nu$, such that*

(i) $1 < \|T\| < \|T^2\| < \ldots < 2 + \varepsilon$,
(ii) $\lim_n \|T^n u\|_1 = 0$ *for every* $u \in L_1$,
(iii) $\nu(f) > 0$ *for all* $0 \neq f \in L_\infty^+$.

Hence $\overline{(I - T^*)L_\infty} \cap L_\infty^+ = \{0\}$, *and T has no $u \in L_1$ with $Tu = u \neq 0$.*

Proof. An easy standard argument shows that it is sufficient to construct T in $L_1 = L_1(X, \Sigma, m)$, where $X = \{0, 1, 2, \ldots\}$, $m(\{k\}) = 1$ for each $k \in X$, and Σ is the subsets of X. First of all, for $0 < \alpha < 1$ with $2\alpha/(1-\alpha) < \varepsilon$ let us choose a sequence $a_1, a_2, \ldots$ of positive reals such that

$$(5)\qquad \alpha = \sum_{k=1}^{\infty} k a_k,$$

$$(6)\qquad \frac{a_{k+1}}{a_k} \le \frac{1 - \beta_1}{2} \quad \text{for all } k \ge 1,$$

where

$$\beta_1 := \sum_{k=1}^{\infty} a_k < \alpha < 1.$$

For $u \in L_1$ define

$$(7)\qquad Tu(k) = \begin{cases} \sum_{n=1}^{\infty} u(n) & (k = 0), \\ a_k u(0) + u(k+1) & (k \ge 1). \end{cases}$$

Then T maps L_1 into itself, as

$$\|Tu\|_1 \le \sum_{k=1}^{\infty} |u(k)| + |u(0)| \sum_{k=1}^{\infty} a_k + \sum_{k=2}^{\infty} |u(k)| \le 2\|u\|_1.$$

From the relations $\int (Tu) f\, dm = \langle Tu, f\rangle = \langle u, T^* f\rangle = \int u(T^* f)\, dm$ for $u \in L_1$ and $f \in L_\infty$ it follows at once that

$$(8)\qquad T^* f(k) = \begin{cases} \sum_{n=1}^{\infty} a_n f(n) & (k = 0), \\ f(0) & (k = 1), \\ f(0) + f(k-1) & (k \ge 2), \end{cases}$$

so that $T^*1(0) = \sum_{n=1}^{\infty} a_n = \beta_1$, $T^*1(1) = 1$ and $T^*1(k) = 2$ for $k \ge 2$. If $n \ge 2$ then, upon writing $f = T^{*(n-1)}1$ so that $T^{*n}1 = T^*(T^{*(n-1)}1) = T^* f$, (8) gives

$$(9)\qquad T^{*n}1(k) = T^* f(k) = \begin{cases} T^{*(n-1)}1(0) & (k = 1), \\ T^{*(n-1)}1(0) + T^{*(n-1)}1(k-1) & (k \ge 2); \end{cases}$$

and since $T^*1 = T^*1_{\{0\}} + T^*(1 - 1_{\{0\}}) = (1 - 1_{\{0\}}) + T^*(1 - 1_{\{0\}})$ by (8), we have

$$\sum_{n=1}^{\infty} T^{*n}1(0) = 2\sum_{n=1}^{\infty} T^{*n}(1 - 1_{\{0\}})(0).$$

We now prove the fundamental equality

$$(*)\qquad \sum_{n=1}^{\infty} T^{*n}(1 - 1_{\{0\}})(0) = \alpha + \alpha^2 + \ldots = \frac{\alpha}{1 - \alpha}.$$

To do this, we set

$$f_n = 1_{\{n, n+1, \ldots\}} \quad \text{and} \quad \beta_n = \sum_{k=n}^{\infty} a_k \quad (n \ge 1).$$

Then $T^*1_{\{0\}} = f_1$, $T^* f_n = \beta_n 1_{\{0\}} + f_{n+1}$, and $\sum_{n=1}^{\infty} \beta_n = \alpha$. Thus

$$T^* f_1 = \beta_1 1_{\{0\}} + f_2,$$
$$T^{*2} f_1 = \beta_2 1_{\{0\}} + T^*(\beta_1 1_{\{0\}}) + f_3,$$
$$\vdots$$
$$T^{*n} f_1 = \beta_n 1_{\{0\}} + T^*(\beta_{n-1} 1_{\{0\}}) + \ldots + T^{*(n-1)}(\beta_1 1_{\{0\}}) + f_{n+1},$$

and

$$\sum_{k=1}^{n} T^{*k} f_1 = (\beta_1 + \ldots + \beta_n) 1_{\{0\}} + T^*((\beta_1 + \ldots + \beta_{n-1}) 1_{\{0\}}) + \ldots + T^{*(n-1)}(\beta_1 1_{\{0\}}) + (f_2 + \ldots + f_{n+1}).$$

Since $\beta_1 + \ldots + \beta_{n-j} < \sum_{i=1}^{\infty} \beta_i = \alpha$, putting $g_N = \sum_{i=2}^{N+1} f_i$ for $N > 1$ we obtain

$$\sum_{k=1}^{N} T^{*k} f_1 \le \alpha \sum_{j=0}^{N-1} T^{*j} 1_{\{0\}} + g_N = \alpha 1_{\{0\}} + \alpha \sum_{j=0}^{N-2} T^{*j} f_1 + g_N \le \alpha 1_{\{0\}} + \alpha \sum_{j=0}^{N-1} T^{*j} f_1 + g_N.$$

By definition $g_N(0) = 0 = f_1(0)$, so

$$\sum_{k=1}^{N} T^{*k} f_1(0) \le \alpha + \alpha \sum_{j=0}^{N-1} T^{*j} f_1(0).$$

Define $c_N = \sum_{k=1}^{N} T^{*k} f_1(0)$. Then $c_N \le \alpha + \alpha c_{N-1}$ for $N > 1$, with

$$c_1 = \beta_1 = \sum_{i=1}^{\infty} a_i < \alpha < \frac{\alpha}{1-\alpha},$$

since $0 < \alpha < 1$. Hence by induction we have $c_N < \alpha/(1-\alpha)$ for every $N \ge 1$, so that

$$\sum_{k=1}^{\infty} T^{*k} f_1(0) \le \frac{\alpha}{1-\alpha}.$$

We have the equality

$$\sum_{k=1}^{n} T^{*k} f_1 = \sum_{j=0}^{n-1} \Big(\sum_{k=1}^{n-j} \beta_k T^{*j} 1_{\{0\}} \Big) + g_n \quad (n \ge 1).$$

Evaluating at 0, using $g_n(0) = 0$, and by the convergence already proved of $\sum_{k=1}^{\infty} T^{*k} f_1(0)$ we obtain, for $n \ge 2$,

$$\sum_{k=1}^{\infty} T^{*k} f_1(0) \ge \alpha + \sum_{j=1}^{n-1} \Big(\sum_{k=1}^{n-j} \beta_k \Big) T^{*(j-1)} f_1(0) = \alpha + \sum_{j=2}^{n-1} \Big(\sum_{k=1}^{n-j} \beta_k \Big) T^{*(j-1)} f_1(0) \to \alpha + \sum_{j=2}^{\infty} \alpha T^{*(j-1)} f_1(0)$$

as n tends to infinity. Hence

$$(1-\alpha) \sum_{k=1}^{\infty} T^{*k} f_1(0) \ge \alpha,$$

which completes the proof of $(*)$.

We then notice by induction that

$$\|T^{*n} 1\|_\infty = \sup_{k \ge 2} T^{*n} 1(k) \quad (n \ge 1).$$

In fact, for $n = 1$ it follows from $T^* 1(0) = \beta_1 < 1$, $T^* 1(1) = 1$, and $T^* 1(k) = 2$ for $k \ge 2$. The induction step follows from (8) and (9). Using this together with $(*)$ and (9) we see that for $n \ge 2$,

$$\|T^{*n} 1\|_\infty = T^{*(n-1)} 1(0) + \|T^{*(n-1)} 1\|_\infty = \ldots = \sum_{k=1}^{n-1} T^{*k} 1(0) + \|T^* 1\|_\infty < \sum_{k=1}^{\infty} T^{*k} 1(0) + \|T^* 1\|_\infty = \frac{2\alpha}{1-\alpha} + 2 < \varepsilon + 2,$$

which proves (i).

To prove (ii), take a positive real γ so that

$$\beta_1 = \sum_{k=1}^{\infty} a_k < \gamma < \frac{1+\beta_1}{2}, \tag{10}$$

and define a function w in L_1^+ by

$$w(0) = \gamma \quad \text{and} \quad w(k) = a_k \quad \text{for } k \ge 1. \tag{11}$$

By (7), (6) and (10),

$$Tw(0) = \sum_{k=1}^{\infty} w(k) = \sum_{k=1}^{\infty} a_k = \beta_1 < \gamma = w(0)$$

and

$$Tw(k) = a_k w(0) + w(k+1) = a_k\gamma + a_{k+1}$$
$$= \Big(\gamma + \frac{a_{k+1}}{a_k}\Big)a_k \le \Big(\gamma + \frac{1-\beta_1}{2}\Big)a_k \quad \text{for } k \ge 1.$$

Thus, letting $\delta = \max\{\beta_1\gamma^{-1}, \gamma + (1-\beta_1)2^{-1}\}$, we have

(12) $$0 < Tw \le \delta w \quad \text{on } X, \quad \text{and} \quad 0 < \delta < 1.$$

It follows that $\lim_n \|T^n w\|_1 \le \lim_n \delta^n \|w\|_1 = 0$. Since T is power bounded, (ii) follows from an approximation argument.

To prove (iii), let LIM denote any Banach limit on L_∞ $(= \ell_\infty)$, and define a bounded linear functional ν on L_∞ by the relation

(13) $$\nu(f) = f(0) + \sum_{k=1}^{\infty} \beta_k f(k) + (1-\alpha)\,\mathrm{LIM}(f)$$

for all $f \in L_\infty$.

By (5) we see that $0 \le \nu \in L_\infty^*$ with $\|\nu\| = \nu(1) = 2$; further (8) gives

$$\nu(T^*f) = T^*f(0) + \sum_{k=1}^{\infty} \beta_k T^*f(k) + (1-\alpha)\,\mathrm{LIM}(T^*f)$$
$$= \sum_{k=1}^{\infty} a_k f(k) + \Big(\sum_{k=1}^{\infty}\beta_k\Big) f(0) + \sum_{k=2}^{\infty}\beta_k f(k-1)$$
$$+ (1-\alpha)(f(0) + \mathrm{LIM}(f))$$
$$= \sum_{k=1}^{\infty}(a_k + \beta_{k+1}) f(k) + \alpha f(0) + (1-\alpha) f(0) + (1-\alpha))\,\mathrm{LIM}(f)$$
$$= \sum_{k=1}^{\infty}\beta_k f(k) + f(0) + (1-\alpha)\,\mathrm{LIM}(f) = \nu(f).$$

Hence $\nu = T^{**}\nu$, and clearly $\nu(f) = \nu(T^*f) > 0$ for all $0 \ne f \in L_\infty^+$. This completes the proof.

Remark. By a slight modification of the construction of T we may sharpen (i) in the theorem as

(i)′ $1 < \|T\| < \|T^2\| < \ldots < 1 + \varepsilon$,

where ε is an arbitrary positive number.

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Department of Mathematics
Faculty of Science
Okayama University
Okayama, 700 Japan

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