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On approach regions for the conjugate Poisson integral and singular integrals

by

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Abstract. Let $\tilde{u}$ denote the conjugate Poisson integral of a function $f \in L^p(\mathbb{R})$. We give conditions on a region Ω so that

$$\lim_{\substack{(v,\varepsilon)\to(0,0)\\(v,\varepsilon)\in\Omega}} \tilde{u}(x+v,\varepsilon) = Hf(x),$$

the Hilbert transform of f at x, for a.e. x. We also consider more general Calderón–Zygmund singular integrals and give conditions on a set Ω so that

$$\sup_{(v,r)\in\Omega} \Big| \int_{|t|>r} k(x+v-t)f(t)\,dt \Big|$$

is a bounded operator on L^p, $1 < p < \infty$, and is weak $(1,1)$.

Let $f \in L^p(\mathbb{R}^d)$ and let $u(x,y)$ denote the Poisson integral of f. Then a classical theorem of Fatou [3] asserts that u has non-tangential limits a.e. on $\mathbb{R}^d$. In 1984, Nagel and Stein [5] considered more general convergence than the classical non-tangential convergence and gave necessary and sufficient conditions for an approach region Ω so that convergence occurs if $u(x,y)$ approaches the boundary through the region Ω.

In this paper we consider the associated problem for the conjugate Poisson integral of a function f, as well as for more general Calderón–Zygmund singular integrals.

Let $k(x)$ be a Calderón–Zygmund kernel on $\mathbb{R}^d$, that is, $k(x) = w(x)/|x|^d$, where:

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(k1) w is homogeneous of degree 0 and $w \in L^\infty(S^{d-1})$,
(k2) its integral over the sphere S^{d-1} vanishes, and
(k3) $|k(x+y) - k(x)| \le C|y|/|x|^{d+1}$ if $|x| > 2|y|$.

Let $k_1(x) = k(x)$ if $|x| > 1$ and 0 otherwise, and define $k_r(x) = r^{-d}k_1(x/r)$. Consider the d-dimensional singular integral defined by this kernel, i.e.

$$H_r f(x) = \int_{|x-t|>r} f(t)k(x-t)\,dt = f * k_r(x).$$

Given a set $\Omega \subset \mathbb{R}^d \times \mathbb{R}^+$, consider the maximal transform

$$H^{\#}_{\Omega} f(x) = \sup_{(v,r)\in\Omega} |H_r f(x+v)|.$$

We will also use the notation

$$H^{\#} f(x) = \sup_{r>0} |H_r f(x)|, \qquad Hf(x) = \lim_{r\to 0} H_r f(x),$$

and the standard Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x+t)|\,dt.$$

In this paper we find necessary and sufficient conditions on the sets Ω for which $H^{\#}_{\Omega} f$ is a weak $(1,1)$ and strong (p,p) operator, $1 < p < \infty$. It turns out that such sets coincide with those Ω's for which the moved Hardy–Littlewood maximal operator

$$M_\Omega f(x) = \sup_{(v,r)\in\Omega} \frac{1}{|B(v,r)|} \int_{B(v,r)} |f(x+t)|\,dt$$

is a weak $(1,1)$ and strong (p,p) operator, $1 < p < \infty$. Nagel and Stein [5] showed that a necessary and sufficient condition for $M_\Omega f$ to be weak $(1,1)$ and strong (p,p), $1 < p < \infty$, is that the set Ω satisfies the following condition, known as the cone condition.

DEFINITION 1. We say that a set $\Omega \subset \mathbb{R}^d \times \mathbb{R}^+$ satisfies the *cone condition* if for any α, the set

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^+ : \exists (v,r) \in \Omega \text{ such that } |x-v| < \alpha(y-r)\}$$

has the property that there exists a constant $C = C(\alpha)$ such that the cross-section set

$$\Omega_\alpha(\lambda) = \{x \in \mathbb{R}^d : (x,\lambda) \in \Omega_\alpha\}$$

satisfies

$$|\Omega_\alpha(\lambda)| \le C\lambda^d,$$

for all $\lambda > 0$.

In the first section we show that if Ω satisfies the cone condition then $H^{\#}_{\Omega} f$ and $\sup_{(v,r)\in\Omega} |Q_r * f(x+v)|$, the maximal function associated with the conjugate Poisson kernel, are weak $(1,1)$ and strong (p,p) operators, $1 < p < \infty$. The sufficiency of the cone condition in the one-dimensional case was already proved in S. Ferrando's Ph.D. thesis [4]. Ferrando reduced the problem to the case in which Ω is a discrete set and proved the result using a covering argument plus a discrete version of the Hilbert transform. In the present work, we extend the result to $\mathbb{R}^d$ by using an argument involving atomic decompositions for functions in $\mathbb{R}^{d+1}_+$.

In Section 2, we show that the cone condition is also necessary for $H^{\#}_{\Omega}$ to be weak (p,p), $1 \le p < \infty$, when $k(x)$ is any of the Riesz kernels. In Section 3, we show the existence of the limit of $H_r f(x+v)$ as (v,r) approach $(0,0)$ on a region satisfying the cone condition. We apply this result to the convergence of $Q_y f(x+v)$, the conjugate Poisson integral of f, when (v,y) tends to $(0,0)$ on an approach region Ω satisfying the cone condition. Lastly, in Section 4, we apply the results to the ergodic theory setting.

1. Maximal estimates.

The proof that the maximal operator $H^{\#}_{\Omega}$ is weak $(1,1)$ and strong (p,p), for $1 < p < \infty$, will make use of the atomic decomposition for operators in $\mathbb{R}^{d+1}_+$. This approach was suggested to us by E. M. Stein, greatly simplifying our original proof.

The atomic decomposition allows us to reduce the problem of showing that $H^{\#}_{\Omega} f$ is weak $(1,1)$ and strong (p,p), $1 < p < \infty$, to showing that a simpler operator is of the same type.

Let $\widetilde{\Omega} = \{(x,y) : (x,y_0) \in \Omega \text{ for some } y_0 \le y\}$. Then $H^{\#}_{\widetilde{\Omega}} f(x) \ge H^{\#}_{\Omega} f(x)$, and if Ω satisfies the cone condition, so does $\widetilde{\Omega}$ because $\widetilde{\Omega}_\alpha = \Omega_\alpha$ (see Definition 1). Therefore, there is no harm in working with the extended set $\widetilde{\Omega}$ instead, which simplifies the proof.

Let $\Gamma = \{(v,t) \in \mathbb{R}^{d+1}_+ : |v| < t\}$. That is, Γ is a single cone positioned at $(0,0)$. Then $H^{\#}_{\Gamma} f(x) = \sup_{(v,r)\in\Gamma} |H_r f(x+v)|$ is the standard non-tangential maximal function for the associated singular integral operator.

THEOREM 2. *If Ω satisfies the cone condition, then*

(a) $\int_{\mathbb{R}^d} |H^{\#}_{\Omega} f(x)|^p\,dx < c_p \int_{\mathbb{R}^d} |H^{\#}_{\Gamma} f(x)|^p\,dx$, *for* $0 < p < \infty$,
(b) $|\{x \in \mathbb{R}^d : H^{\#}_{\Omega} f(x) > \lambda\}| \le c|\{x \in \mathbb{R}^d : H^{\#}_{\Gamma} f(x) > \lambda\}|$,
(c) $H^{\#}_{\Gamma} f(x) \le H^{\#} f(x) + C(d)Mf(x)$, *and*
(d) $H^{\#}_{\Omega} f$ *is a weak* $(1,1)$ *and strong* (p,p) *operator, for* $1 < p < \infty$.

Proof. Parts (a) and (b) are an application of the results contained in Stein's "Harmonic Analysis" [7], pages 68 and 69. For completeness, we include his argument.

(a) An atom associated to a ball $B \subset \mathbb{R}^d$ is a measurable function $a(x,t)$ supported in the tent $T(B) = \{(x,t) : |x| < r - t\} \subset \mathbb{R}^{d+1}_+$, such that $\|a\|_\infty \le 1/|B|$.

If $H^\#_\Gamma f(x) \in L^p(\mathbb{R}^d)$ then we can apply the atomic decomposition to the function $|H_y f(x)|^p$. Hence, to prove (a), it will be enough to consider the case where $p = 1$ and $H_y f(x) = a(x,y)$ is an atom. Further, by translation, we can assume that the atom is supported in $T(B)$ for B a ball of radius r centered at the origin.

By the properties of the atom a, we clearly have $\sup_{(v,y)\in\Omega} |a(x+v,y)| \le 1/|B|$. If $\sup_{(v,y)\in\Omega} |a(x+v,y)| \ne 0$ then there is a $(v,y) \in \widetilde{\Omega}$ such that $(x+v,y) \in T(B)$; that is, $|x+v| < r - y$. Since $(v,y) \in \widetilde{\Omega}$, it follows that $-x \in \widetilde{\Omega}_1(r) = \Omega_1(r)$. Hence

$$|\{x : \sup_{(v,y)\in\Omega} |a(x+v,y)| \ne 0\}| \le |\Omega_1(r)|,$$

and by assumption, $|\Omega_1(r)| \le cr^d$. From this we get

$$\int_{\mathbb{R}^d} \sup_{(v,y)\in\Omega} |a(x+v,y)|\,dx \le \frac{1}{|B|}|\Omega_1(r)| \le c. \tag{1}$$

Since (1) holds for atoms, (a) holds in general (by Theorem 3.2.3 in [7]).

(b) To prove (b) we repeat the same proof, but replace the function $H_y f(x)$ by the characteristic function of the set where $|H_y f(x)| > \lambda$.

(c) It is easy to see that the operator $H^\#_\Gamma f$ can be compared with the maximal operator $H^\# f$. Indeed,

$$\begin{aligned} |H_r f(x+v) - H_r f(x)| \le & \int_{|t|>2r} |f(x-t)|\cdot|k(t-v)-k(t)|\,dt \\ & + \int_{\substack{|t-v|>r\\ |t|\le 2r}} |f(x-t)|\cdot|k(t-v)|\,dt \\ & + \int_{r<|t|\le 2r} |f(x-t)|\cdot|k(t)|\,dt. \end{aligned}$$

By property (k1), $|k(x)| \le c/|x|^d$, thus the last two terms are majorized by

$$c(d)\,\frac{1}{|B(0,2r)|}\int_{B(0,2r)} |f(x-t)|\,dt.$$

To handle the first term, recall that by (k3), $|k(t-v) - k(t)| \le C|v|/|t|^{d+1}$ if $|t| > 2|v|$. Thus, if $|v| < r$, then

$$|k(t-v) - k(t)| \le C\frac{r}{|t|^{d+1}} = C\Phi_r(t), \qquad \text{for } |t| > 2r,$$

where $\Phi_r(t) = r^{-d}\Phi_1(t/r)$, and $\Phi_1(t) = |t|^{-d-1}$ for $|t| > 2$. Thus

$$\sup_{(v,r)\in\Gamma} |H_r f(x+v) - H_r f(x)| \le C\sup_{r>0} |f| * \Phi_r(x) + c(d)Mf(x).$$

Since Φ_1 is an integrable function on $\mathbb{R}^d$ which radially decreases at infinity with an appropriate rate, it follows that $\sup_{r>0} |f| * \Phi_r(x)$ is also dominated by $Mf(x)$. Hence

$$\sup_{(v,r)\in\Gamma} |H_r f(x+v)| \le \sup_{r>0} |H_r f(x)| + C(d)Mf(x),$$

finishing the proof of (c).

(d) The proof of (d) is a straightforward application of (a), (b) and (c). ■

Let

$$Q_y(x) = \frac{1}{\pi}\,\frac{x}{x^2+y^2}$$

denote the conjugate Poisson kernel in $\mathbb{R}^2_+$. For a set $\Omega \subset \mathbb{R}^2_+$, let $Q^\#_\Omega f(x) = \sup_{(v,\varepsilon)\in\Omega} |Q_\varepsilon * f(x+v)|$. With this notation, the corresponding version of Theorem 2 also holds for this maximal operator.

THEOREM 3. *If Ω satisfies the cone condition, then*

(a) $\int_{\mathbb{R}^d} |Q^\#_\Omega f(x)|^p\,dx < c_p \int_{\mathbb{R}^d} |Q^\#_\Gamma f(x)|^p\,dx$, *for* $0 < p < \infty$,

(b) $|\{x \in \mathbb{R}^d : Q^\#_\Omega f(x) > \lambda\}| \le c|\{x \in \mathbb{R}^d : Q^\#_\Gamma f(x) > \lambda\}|$,

(c) $Q^\#_\Gamma f(x) \le \pi^{-1}[H^\#_\Gamma f(x) + c(d)Mf(x)]$, *and*

(d) $H^\#_\Omega f$ *is a weak* $(1,1)$ *and strong* (p,p) *operator, for* $1 < p < \infty$.

Proof. The proof is exactly the same as the proof of Theorem 2. ■

2. Necessity of the cone condition. The Riesz kernels in $\mathbb{R}^d$ are defined by the jth coordinate in the following way:

$$k_j(x) = w_j(x)/|x|^d, \quad \text{where} \quad w_j(x) = x_j/|x_j|.$$

PROPOSITION 4. *Let k be a Riesz kernel in $\mathbb{R}^d$. If $H^\#_\Omega f$ is weak (p,p) for some $1 \le p < \infty$ then Ω satisfies the cone condition.*

Proof. Recall that

$$\Omega_\alpha = \{(x,t) : \exists (v,r) \in \Omega \text{ such that } |x-v| < \alpha(t-r)\}.$$

Without loss of generality we can assume $k(x) = k_1(x)$. For a fixed α, we need to estimate the measure of $\Omega_\alpha(\lambda) = \{x : (x,\lambda) \in \Omega_\alpha\}$ for any $\lambda > 0$.

Let $b \ge 2\alpha\lambda$ to be determined and

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x_1 \le b \text{ and } |x_i| \le \alpha\lambda \text{ for all } 2 \le i \le d,\\ 0 & \text{otherwise.}\end{cases}$$

Let $x \in \Omega_\alpha(\lambda)$ and $(v,r) \in \Omega$ such that $|x - v| < \alpha(\lambda - r)$. Then

$$\begin{aligned}|H_r f(v-x)| &= \Big|\int\limits_{|t|>r} f(t-(v-x))\frac{w_1(t)}{|t|^d}\,dt\Big|\\ &= \int\limits_{\substack{|t|>r\\ |v_1-x_1|<t_1<b+(v_1-x_1)\\ |t_i-(v_i-x_i)|<\alpha\lambda,\, i\ne 1}} \frac{1}{|t|^d}\,dt\end{aligned}$$

by the symmetry of the kernel.

Case 1: $r < \alpha\lambda$.

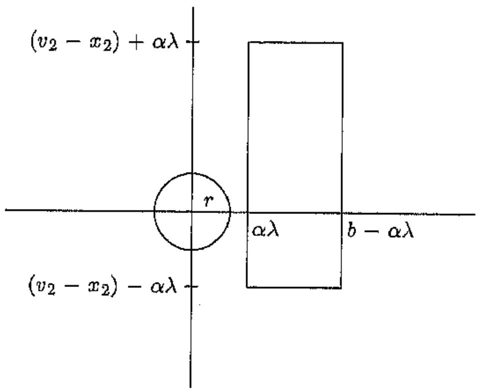

Fig. 1

In this case, since $|x - v| < \alpha\lambda$,

$$\begin{aligned}H_r f(v-x)| &\ge \int\limits_{\substack{\alpha\lambda<t_1<b-\alpha\lambda\\ |t_i-(v_i-x_i)|<\alpha\lambda,\, i\ne 1}} \frac{1}{|t|^d}\,dt\\ &\ge c(d)\frac{(b-2\alpha\lambda)(\alpha\lambda)^{d-1}}{(b+d\alpha\lambda)^d} = c(d)\frac{1}{(3+d)^d}\end{aligned}$$

if $b = 3\alpha\lambda$.

Case 2: $r \ge \alpha\lambda$.

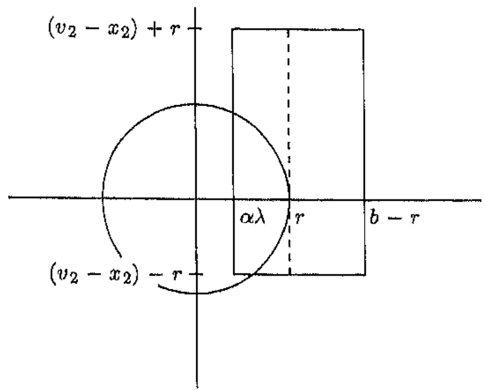

Fig. 2

Now we will use also the fact that $|x - v| \le \alpha(\lambda - r)$, so in particular, $r \le \lambda$ and $\alpha \le 1$. We have

$$\begin{aligned}H_r f(v-x)| &\ge \int\limits_{\substack{r<t_1<b-\alpha\lambda\\ |t_i-(v_i-x_i)|<\alpha\lambda,\, i\ne 1}} \frac{1}{|t|^d}\,dt\\ &\ge c(d)\frac{(b-\alpha\lambda-r)(\alpha\lambda)^{d-1}}{(b+d\alpha\lambda)^d}\\ &\ge c(d)\frac{(b-2\lambda)(\alpha\lambda)^{d-1}}{(b+d\lambda)^d} = c(d)\alpha^{d-1}\frac{1}{(3+d)^d}\end{aligned}$$

if $b = 3\lambda$.

Let

$$A(\alpha) = \begin{cases} c(d)/(3+d)^d & \text{if } \alpha \ge 1,\\ c(d)\alpha^{d-1}/(3+d)^d & \text{if } 0 < \alpha < 1.\end{cases}$$

Then, if $H^{\#}_\Omega f$ is a weak (p,p) operator, we have

$$\begin{aligned}|\Omega_\alpha(\lambda)| &= |\{x : \exists (v,r) \in \Omega \text{ such that } |x-v| < \alpha(\lambda - r)\}|\\ &\le |\{x : \sup_{\substack{(v,r)\in\Omega\\ r\le\lambda}} H_r f(v-x) > A(\alpha)\}|\\ &\le |\{x : H^{\#}_\Omega f(-x) > A(\alpha)\}| \le \frac{C}{A(\alpha)^p}\|f\|_p^p = C(d,\alpha)\lambda^d.\end{aligned}$$

Hence Ω satisfies the cone condition. ∎

3. Almost everywhere convergence along Ω. Let Ω satisfy the cone condition. In this section we prove pointwise convergence of

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)$$

for any $f\in L^p(\mathbb{R}^d)$, $1\le p<\infty$.

THEOREM 5. *Let $\Omega\subset\mathbb{R}^d\times\mathbb{R}^+$ satisfy the cone condition, such that $(0,0)\in\overline{\Omega}$. Then, for any $f\in L^p(\mathbb{R}^d)$, $1\le p<\infty$, we have*

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v) = Hf(x) \quad \textit{a.e.}$$

Proof. Let $C_c^1(\mathbb{R}^d)$ be the set of functions with compact support and continuous partial derivatives. Let $f\in C_c^1(\mathbb{R}^d)$. Then

$$\begin{aligned} H_r f(x+v) &= f*k_1(x+v) + \int\limits_{\{r<|x-y|<1\}} f(y+v)k(x-y)\,dy \\ &= I(x,v,r) + II(x,v,r). \end{aligned}$$

By continuity of f and compactness of its support, $I(x,v,r)\to f*k_1(x)$ as $(v,r)\to(0,0)$. For the second term, notice that by (k2),

$$\int\limits_{\{r<|x-y|<1\}} k(x-y)\,dy = 0,$$

thus

$$II(x,v,r) = \int [f(y+v)-f(x+v)]k(x-y)\chi_{\{r<|u|<1\}}(x-y)\,dy.$$

Since the differential of f is continuous of compact support, the integrand is majorized by

$$c|x-y|^{-d+1}\chi_{\{0<|u|<1\}}(x-y),$$

which is integrable. And, as $(v,r)\to(0,0)$, the integrand converges to

$$[f(y)-f(x)]k(x-y)\chi_{\{0<|u|<1\}}(x-y).$$

From these two estimations,

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v) = Hf(x) \quad \text{for all } x.$$

Now let $f\in L^p(\mathbb{R}^d)$. Given $\varepsilon>0$ choose $g\in C_c^1(\mathbb{R}^d)$ such that $\|f-g\|_p<\varepsilon$. Let

$$\Lambda f(x) := \Big|\limsup_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v) - \liminf_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)\Big|.$$

Then, $\Lambda f = \Lambda(f-g)$ and, by Theorem 2,

$$|\{x:\Lambda f(x)>\alpha\}| = |\{x:\Lambda(f-g)(x)>\alpha\}| \le \frac{C(d)}{\alpha^p}\|f-g\|_p^p \le \frac{C(d)}{\alpha^p}\varepsilon^p.$$

Since ε is arbitrary, the limit

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)$$

exists for almost every x.

Similar arguments show that

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v) = Hf(x) \quad \text{a.e.} \ ■$$

THEOREM 6. *Recall that*

$$Q_y(x) = \frac{1}{\pi}\,\frac{x}{x^2+y^2}$$

denotes the conjugate Poisson kernel in $\mathbb{R}_+^2$. If Ω satisfies the cone condition, then

$$\lim_{\substack{(v,\varepsilon)\to(0,0)\\(v,\varepsilon)\in\Omega}} Q_\varepsilon * f(x+v) \quad \textit{exists for a.e. } x,$$

and is equal to $Hf(x)$.

Proof. This follows from Theorem 3 and the fact that

$$\lim_{\varepsilon\to 0} Q_\varepsilon * f(x) = Hf(x)$$

(by arguments similar to those in Theorem 5). ■

4. Hilbert transform for measurable flows. Let (X,β,m) be a σ-finite measure space and $\{\tau_t\}_{t\in\mathbb{R}^d}$ a measure preserving action of $\mathbb{R}^d$ acting on X, which is jointly measurable from $\mathbb{R}^d\times X$ to X. We will now consider the truncated ergodic singular integrals

$$H'_r f(x) = \int\limits_{r<|t|<1/r} f(\tau_t x)k(t)\,dt, \quad f\in L^p(X),$$

and the related moving maximal operator

$$H'^{\#}_{\Omega} = \sup_{(v,r)\in\Omega} |H'_r f(\tau_v x)|.$$

The singular integral results obtained in Section 1 can be translated to this setting by means of a Calderón transfer principle. However, we first need to establish a modified version of the results in Section 1, for the truncated singular integrals.

Since we are interested in the limit as $(v,r)\to(0,0)$, in this section we will assume that for all $(v,r)\in\Omega$, we have $r\le 1$.

COROLLARY 7. *Let $\Omega\subset\mathbb{R}\times\mathbb{R}^+$ satisfy the cone condition. Then*

$$\sup_{(v,r)\in\Omega}\Big|\int_{r<|t|<1/r} f(x+v+t)k(t)\,dt\Big|$$

is a weak $(1,1)$ and strong (p,p) operator for $1<p<\infty$.

Proof. The result follows from Theorem 2 because

$$\Big|\int_{r<|t|<1/r} f(x+v+t)k(t)\,dt\Big|\le|H_rf(x+v)|+|H_{1/r}f(x+v)|,$$

and $\{(v,1/r):(v,r)\in\Omega\}$ satisfies the cone condition if $r\le1$. ■

PROPOSITION 8 (Transfer principle). *Let $\Omega\subset\mathbb{R}^d\times\mathbb{R}^+$ and $1\le p<\infty$. If*

$$\sup_{(v,r)\in\Omega}\Big|\int_{r<|t|<1/r}\varphi(x+v+t)k(t)\,dt\Big|$$

is a weak (p,p) operator in $L^p(\mathbb{R})$, then $H'^{\#}_{\Omega}f$ is a weak (p,p) operator in $L^p(X)$.

Proof. Fix $M>0$ and let $N=3M$. Given $f\in L^p(X)$ define

$$\varphi_x(t)=\begin{cases} f(\tau_tx) & \text{if } |t|\le N,\\ 0 & \text{otherwise.}\end{cases}$$

Then, for almost every x, we $\varphi_x\in L^p(\mathbb{R}^d)$. Indeed,

$$\int_X\int_{\mathbb{R}^d}|\varphi_x(t)|^p\,dt\,dx=\int_{|t|\le N}\int_X|f(\tau_tx)|^p\,dx\,dt=c(d)N^d\|f\|_p^p,$$

because the flow is measure preserving.

Let $\Omega_M=\{(v,r)\in\Omega:|v|\le M,\ 1/M\le r\le M\}$. Then

$$\int_X\Big|\Big\{|s|\le M:\sup_{(v,r)\in\Omega_M}\Big|\int_{r<|s+v-t|<1/r}\varphi_x(t)k(s+v-t)\,dt\Big|\ge\lambda\Big\}\Big|\,dx \le\frac{C}{\lambda^p}\int_X\|\varphi_x\|_p^p\le c(d)N^d\frac{C}{\lambda^p}\|f\|_p^p.$$

Let

$$A=\{(x,s)\in X\times\mathbb{R}^d:\sup_{(v,r)\in\Omega_M}\Big|\int_{r<|s+v-t|<1/r}\varphi_x(t)k(s+v-t)\,dt\Big|\ge\lambda\}.$$

Notice that if $(v,r)\in\Omega_M$, $|s|\le M$ and $|t|<1/r$, then $f(\tau_{v+s+t}x)=\varphi_x(v+s+t)$ because $3M=N$. Thus,

$$\begin{aligned}\int_X\Big|\Big\{s:\sup_{(v,r)\in\Omega_M}\Big|\int_{r<|s+v-t|<1/r}\varphi_x(t)k(s+v-t)\,dt\Big|\ge\lambda\Big\}\Big|\,dx &\ge\int_{\mathbb{R}}\int_X\chi_A(x,s)\chi_{\{|u|<M\}}(s)\,dx\,ds\\ &\ge\int_{|s|<M}m(x:\sup_{(v,r)\in\Omega_M}|H'_rf(\tau_{v+s}x)|\ge\lambda)\,ds\\ &=c(d)M^dm(x:\sup_{(v,r)\in\Omega_M}|H'_rf(\tau_vx)|\ge\lambda).\end{aligned}$$

Since $N=3M$, we obtain

$$m(x:\sup_{(v,r)\in\Omega_M}|H'_rf(\tau_vx)|\ge\lambda)\le\frac{3^dC}{\lambda^p}\|f\|_p^p.$$

The proposition follows by letting $M\to\infty$. ■

COROLLARY 9. *If Ω satisfies the cone condition, then $H'^{\#}_{\Omega}f$ is a weak $(1,1)$ and strong (p,p) operator for $1<p<\infty$.*

Proof. This follows from Corollary 7 and Proposition 8. ■

THEOREM 10. *Let $\Omega\subset\mathbb{R}^d\times\mathbb{R}^+$ satisfy the cone condition and $(0,0)\in\overline{\Omega}$. Then*

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}}H'_rf(\tau_vx)$$

exists a.e. for all $f\in L^p(X)$, $1\le p<\infty$.

Proof. It suffices to prove that

$$k_{v,r}\phi(u):=\int_{r<|t|<1/r}k(t)\phi(u-v-t)\,dt$$

converges in $L^1(\mathbb{R}^d)$ as $(v,r)\to(0,0)$, $(v,r)\in\Omega$, for any $\phi\in C^1_c(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}}\phi\,ds=0$. Indeed, let

$$O=\Big\{h\in L^1(X):h(x)=\int g(\tau_tx)\phi(t)\,dt,\ g\in L^1(X),\ \phi\in C^1_c(\mathbb{R}^d)\Big\}.$$

Then

$$H'_rh(\tau_vx)=\int g(\tau_sx)k_{v,r}\phi(s)\,ds.$$

The orthogonal complement of $O \cap L^2(X)$ consists of the invariant functions under the action (see [2]). Thus the theorem would hold for a dense class of functions and then the result would follow for all functions by an application of Corollary 9.

Let us introduce some notation:

$$K_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if } r \le |s-v|, \\ 0 & \text{otherwise,} \end{cases}$$

$$k_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if } r \le |s-v| \le 1/r, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\Big| \int\limits_{r \le |s| \le 1/r} f(u-v-s)k(s)\,ds \Big| = \Big| \int\limits_{r \le |u-v-s| \le 1/r} f(s)k(u-v-s)\,ds \Big| = \Big| \int k_{(v,r)}(u-s)f(s)\,ds \Big| = |k_{(v,r)} * f(u)|.$$

The L^1-convergence of $k_{(v,r)} * \phi$ follows from the following two properties:

(A) $K_{(v,r)} * \phi$ converges in L^1, and

(B) $\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0$ as $r \to 0$.

Property (A) follows from Lebesgue's Dominated Convergence Theorem. By Theorem 8, $K_{(v,r)} * \phi$ converges a.e. Assume that $\operatorname{supp}(\phi) \subseteq \{|y| \le L\}$. Then

$$|K_{(v,r)} * \phi(u)| \le \Big(c\chi_{\{|y|<2L\}}(u) + \frac{c(d,L)}{|u|^{d+1}} \chi_{\mathbb{R}^d \setminus \{|y|<2L\}}(u) \Big) \in L^1(\mathbb{R}^d).$$

First consider $|u| \ge 2L$. Then, using the basic properties of ϕ and $K_{(v,r)}$ (recall (k2)), we can compute (for (v,r) small enough)

$$\begin{aligned} |K_{(v,r)} * \phi(u)| &= \Big| \int [K_{(v,r)}(u-s) - K_{(v,r)}(u)]\phi(s)\,ds \Big| \\ &\le \int\limits_{|s| \le K} |K_{(v,r)}(u-s) - K_{(v,r)}(u)| \cdot |\phi(s)|\,ds \\ &\le \int\limits_{|s| \le K} |k(u-v-s) - k(u-v)| \cdot |\phi(s)|\,ds \\ &\le c \int\limits_{|s| \le K} \frac{|s|}{|u-v|^{d+1}} |\phi(s)|\,ds \le cc(d) \frac{L^{d+1}}{|u|^{d+1}}, \end{aligned}$$

by (k3). Here $c = c(\phi)$.

Consider now $|u| \le 2L$. Taking (v,r) small enough we get

$$\begin{aligned} |K_{(v,r)} * \phi(u)| &= \Big| \int K_{(v,r)}(s)\phi(u-s)\,ds \Big| \\ &= \Big| \int K_{(v,r)}(s-v)\phi(u+v-s)\,ds \Big| \\ &= \Big| \int\limits_{4L \ge |s| \ge r} \frac{w(s)}{|s|^d} \phi(u-v-s)\,ds \Big| \\ &\le \int\limits_{4L \ge |s| \ge r} \frac{1}{|s|^d} |\phi(u-v-s) - \phi(u-v)|\,ds \le c \end{aligned}$$

(where $c = c(\phi)$) because the differential of ϕ is continuous of compact support. This ends the proof of (A).

To prove (B), assume $\operatorname{supp}(\phi) \subseteq \{|y| \le K\}$. By definition of $K_{(v,r)}$ and $k_{(v,r)}$ we have

$$k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u) = k_{1/r} * \phi(u-v).$$

Now $K_{1/r} * \phi(u-v) = 0$ if $u \notin S_{(v,r)} := \mathbb{R}^d \setminus \{u : |u| < 1/r - v - L\}$. We can choose (u,v) small enough such that $u \in S_{(v,r)}$ implies $|u| \ge 2L$. Then a similar computation to that in (A) gives $|k_{1/r} * \phi(u)| \le c/|u|^{d+1}$. In summary,

$$|k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u)| \le \chi_{S_{(v,r)}}(u)|k_{1/r} * \phi(u-v)| \le \chi_{S_{(v,r)}}(u) \frac{c}{|u|^{d+1}}.$$

Hence

$$\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0. \ \blacksquare$$

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On invariant measures for power bounded positive operators

by

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To the memory of Hisao Tominaga

Abstract. We give a counterexample showing that $\overline{(I-T^*)L_\infty} \cap L_\infty^+ = \{0\}$ does not imply the existence of a strictly positive function u in L_1 with $Tu = u$, where T is a power bounded positive linear operator on L_1 of a σ-finite measure space. This settles a conjecture by Brunel, Horowitz, and Lin.

1. Introduction. Let (X, Σ, m) be a σ-finite measure space and T a positive linear operator in $L_1 = L_1(X, \Sigma, m)$. T is called a *contraction* if $\|T\| \le 1$, *power bounded* if $\sup_n \|T^n\| < \infty$, and *Cesàro bounded* if $\sup_n \|n^{-1}\sum_{k=1}^n T^k\| < \infty$. Many ergodic theorems for positive L_1 contractions require the existence of a finite invariant measure equivalent to the original one, i.e., a strictly positive $u \in L_1$ with $Tu = u$. This problem has attracted many top researchers, and one of the conditions equivalent to the existence of such a $u \in L_1$, obtained by Brunel [1], is that

$$(1) \qquad \overline{(I-T^*)L_\infty} \cap L_\infty^+ = \{0\}.$$

For any T positive and Cesàro bounded, condition (1) is seen, by using the known fact that $n^{-1}\|T^n\| \to 0$ as $n \to \infty$, to be equivalent to the following condition:

$$(2) \qquad \limsup_n \Big\| \frac{1}{n} \sum_{k=1}^n T^{*k} 1_A \Big\|_\infty > 0 \quad \text{for any } A \in \Sigma \text{ with } m(A) > 0.$$

Sucheston [7] started a systematic study of power bounded positive linear operators in L_1, and Fong [4] studied the problem of existence of strictly positive fixed points under an additional assumption of a null disappearing part. The problem in general was studied by Derriennic and Lin [3] (see also Sato [6]), who proved that for any T positive and Cesàro bounded, an

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