On approach regions for the conjugate Poisson integral and singular integrals

by

S. FERRANDO (Mar del Plata), R. L. JONES (Chicago, Ill.) and K. REINHOLD (Albany, N.Y.)

Abstract. Let \( \tilde{u} \) denote the conjugate Poisson integral of a function \( f \in L^p(\mathbb{R}) \). We give conditions on a region \( \Omega \) so that

\[
\lim_{\{v,x\} \to \{0,0\}} \tilde{u}(x + v, x) = H f(x),
\]

the Hilbert transform of \( f \) at \( x \), for a.e. \( x \). We also consider more general Calderón-Zygmund singular integrals and give conditions on a set \( \Omega \) so that

\[
\sup_{\{v,x\} \in \Omega} \left| \int k(x + v - t) f(t) \, dt \right|
\]

is a bounded operator on \( L^p \), \( 1 < p < \infty \), and is weak \( (1,1) \).

Let \( f \in L^p(\mathbb{R}^d) \) and let \( u(x,y) \) denote the Poisson integral of \( f \). Then a classical theorem of Fatou [3] asserts that \( u \) has non-tangential limits a.e. on \( \mathbb{R}^d \). In 1984, Nagel and Stein [5] considered more general convergence than the classical non-tangential convergence and gave necessary and sufficient conditions for an approach region \( \Omega \) so that convergence occurs if \( u(x,y) \) approaches the boundary through the region \( \Omega \).

In this paper we consider the associated problem for the conjugate Poisson integral of a function \( f \) as well as for more general Calderón-Zygmund singular integrals.

Let \( k(x) \) be a Calderón-Zygmund kernel on \( \mathbb{R}^d \), that is, \( k(x) = w(x)/|x|^d \), where:

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In the first section we show that if $\Omega$ satisfies the cone condition then $H_{\Omega}^p f$ and $\sup_{(v,r)\in\Omega} |Q_v \ast f(x+v)|$, the maximal function associated with the conjugate Poisson kernel, are weak $(1,1)$ and strong $(p,p)$ operators, $1 < p < \infty$. The sufficiency of the cone condition in the one-dimensional case was already proved in S. Ferrando’s Ph.D. thesis [4]. Ferrando reduced the problem to the case in which $\Omega$ is a discrete set and proved the result using a covering argument plus a discrete version of the Hilbert transform. In the present work, we extend the result to $\mathbb{R}^d$ by using an argument involving atomic decompositions for functions in $L^p_{\text{loc}}$.

In Section 2, we show that the cone condition is also necessary for $H_{\Omega}^p f$ to be weak $(p,p)$, $1 < p < \infty$, when $k(x)$ is any of the Riesz kernels. In Section 3, we show the existence of the limit of $H_{\Omega} f(x+v)$ as $(v,r)$ approach $(0,0)$ on a region satisfying the cone condition. We apply this result to the convergence of $Q_v f(x+v)$, the conjugate Poisson integral of $f$, when $(v,y)$ tends to $(0,0)$ on an approach region $\Omega$ satisfying the cone condition. Lastly, in Section 4, we apply the results to the ergodic theory setting.

1. Maximal estimates. The proof that the maximal operator $H_{\Omega}^p f$ is weak $(1,1)$ and strong $(p,p)$, for $1 < p < \infty$, will make use of the atomic decomposition for operators in $L^p_{\text{loc}}$. This approach was suggested to us by E. M. Stein, greatly simplifying our original proof.

The atomic decomposition allows us to reduce the problem of showing that $H_{\Omega}^p f$ is weak $(1,1)$ and strong $(p,p)$, $1 < p < \infty$, to showing that a simpler operator is of the same type.

Let $\tilde{\Omega} = \{(x,y) : (x,\gamma y) \in \Omega \text{ for some } \gamma \leq y\}$. Then $H_{\tilde{\Omega}}^p f(x) \geq H_{\Omega}^p f(x)$, and if $\Omega$ satisfies the cone condition, so does $\tilde{\Omega}$ because $\tilde{\Omega}_\alpha = \tilde{\Omega}_\alpha$ (see Definition 1). Therefore, there is no harm in working with the extended set $\tilde{\Omega}$ instead, which simplifies the proof.

Let $\Gamma = \{(v,t) \in \mathbb{R}^{d+1} : |v| < t\}$. That is, $\Gamma$ is a single cone positioned at $(0,0)$. Then $H_{\Gamma}^p f(x) = \sup_{(v,r)\in\Gamma} |H_{\Omega} f(x+v)|$ is the standard non-tangential maximal function for the associated singular integral operator.

**Theorem 2.** If $\Omega$ satisfies the cone condition, then

(a) $\int_{\mathbb{R}^d} |H_{\Omega}^p f(x)|^p \ dx < c_p \int_{\mathbb{R}^d} |H_{\Gamma}^p f(x)|^p \ dx$, for $0 < p < \infty$,

(b) $|\{(x \in \mathbb{R}^d : H_{\Omega}^p f(x) > \lambda\}| \leq c \{x \in \mathbb{R}^d : H_{\Gamma}^p f(x) > \lambda\}$,

(c) $H_{\Omega}^p f(x) \leq H_{\Gamma}^p f(x) + C(|f(x)|)$, and

(d) $H_{\Omega}^p f$ is a weak $(1,1)$ and strong $(p,p)$ operator, for $1 < p < \infty$.

**Proof.** Parts (a) and (b) are an application of the results contained in Stein’s “Harmonic Analysis” [7], pages 68 and 69. For completeness, we include his argument.
(a) An atom associated to a ball $B \subset \mathbb{R}^d$ is a measurable function $a(x, t)$ supported in the tent $T(B) = \{(x, t) : |x| < r - t\} \subset \mathbb{R}^{d+1}$, such that $\|a\|_\infty \leq 1/|B|$.

If $H_0^\Phi f(x) \in L^p(\mathbb{R}^d)$ then we can apply the atomic decomposition to the function $H_\gamma f(x)$. Hence, to prove (a), it will be enough to consider the case where $p = 1$ and $H_\gamma f(x) = a(x, y)$ is an atom. Further, by translation, we can assume that the atom is supported in $T(B)$ for $B$ a ball of radius $r$ centered at the origin.

By the properties of the atom $a$, we clearly have $\sup_{(u, v) \in D} |a(x + u, v)| \leq 1/|B|$. If $\sup_{(u, v) \in D} |a(x + u, v)| \neq 0$ then there is a $(v, y) \in D$ such that $(x + u, v) \in T(B)$; that is, $|x + v| < r - y$. Since $(v, y) \in D$, it follows that $-x \in \Omega_1(r) - \Omega_1(r)$. Hence

\[ \|a(x + u, v)\| \neq 0 \leq \|\Omega_1(r)\| \leq c r^d. \]

and by assumption, $\|\Omega_1(r)\| \leq c r^d$. From this we get

\[ \int_{\mathbb{R}^d} \sup_{(v, y) \in D} |a(x + u, v)| \, dx \leq \frac{1}{|B|} \|\Omega_1(r)\| \leq c. \]

Since (1) holds for atoms, (a) holds in general (by Theorem 3.2.3 in [7]).

(b) To prove (b) we repeat the same proof, but replace the function $H_\gamma f(x)$ by the characteristic function of the set where $|H_\gamma f(x)| > \lambda$.

(c) It is easy to see that the operator $H_\gamma f$ can be compared with the maximal operator $H_\gamma^\Phi f$. Indeed,

\[ |H_\gamma f(x + v) - H_\gamma f(x)| \leq \int_{|t| > 2r} |f(x - t) - k(t - u)| \, dt + \int_{|t - u| > r} |f(x - t)| \, dt. \]

By property (k1), $|k(x)| \leq c/|x|^d$, thus the last two terms are majorized by

\[ c(d) \frac{1}{|B(0, 2r)|} \int_{B(0, 2r)} |f(x - t)| \, dt. \]

To handle the first term, recall that by (k3), $|k(t - u) - k(t)| \leq C|u|/|t|^{d+1}$ if $|t| > 2|u|$. Thus, if $|u| < r$, then

\[ |k(t - u) - k(t)| \leq C \frac{r}{|t|^{d+1}} = C \Phi_r(t), \quad \text{for } |t| > 2r, \]

where $\Phi_r(t) = r^{-d} \Phi_1(t/r)$, and $\Phi_1(t) = |t|^{-d-1}$ for $|t| > 2$. Thus

\[ \sup_{(u, r) \in D} |H_\gamma f(x + v) - H_\gamma f(x)| \leq C \sup_{r > 0} |f| * \Phi_r(x) + c(d) M f(x). \]

Since $\Phi_r$ is an integrable function on $\mathbb{R}^d$ which radially decreases at infinity with an appropriate rate, it follows that $\sup_{r > 0} |f| * \Phi_r(x)$ is also dominated by $M f(x)$. Hence

\[ \sup_{(u, r) \in D} |H_\gamma f(x + v)| \leq \sup_{r > 0} |H_\gamma f(x)| + C(d) M f(x), \]

finishing the proof of (c).

(d) The proof of (d) is a straightforward application of (a), (b) and (c). \( \blacksquare \)

Let

\[ Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2} \]

denote the conjugate Poisson kernel in $\mathbb{R}^d$. For a set $\Omega \subset \mathbb{R}^d$, let $Q_y^\Phi f(x) = \sup_{(u, v) \in D} |Q_x f(x + v)|$. With this notation, the corresponding version of Theorem 2 also holds for this maximal operator.

**Theorem 3.** If $\Omega$ satisfies the cone condition, then

(a) $\int_{\mathbb{R}^d} |Q_y^\Phi f(x)|^p \, dx \leq c_p \int_{\mathbb{R}^d} |Q_x f(x)|^p \, dx$, for $0 < p < \infty$,

(b) $|\{x \in \mathbb{R}^d : Q_y^\Phi f(x) > \lambda\}| \leq c |\{x \in \mathbb{R}^d : Q_x^\Phi f(x) > \lambda\}|$,

(c) $Q_y^\Phi f(x) \leq \pi^{-1} [H_y^\Phi f(x) + c(d) M f(x)]$, and

(d) $H_y^\Phi f$ is a weak $(1, 1)$ and strong $(p, p)$ operator, for $1 < p < \infty$.

**Proof.** The proof is exactly the same as the proof of Theorem 2. \( \blacksquare \)

2. Necessity of the cone condition. The Riesz kernels in $\mathbb{R}^d$ are defined by the $j$th coordinate in the following way:

\[ k_j(x) = w_j(x)/|x|^d, \quad \text{where} \quad w_j(x) = x_j/|x|. \]

**Proposition 4.** Let $k$ be a Riesz kernel in $\mathbb{R}^d$. If $H_y^\Phi f$ is weak $(p, p)$ for some $1 \leq p < \infty$ then $\Omega$ satisfies the cone condition.

**Proof.** Recall that

\[ \Omega_\alpha = \{(x, t) : \exists (v, r) \in \Omega \text{ such that } |x - v| < \alpha (t - r)\}. \]
Without loss of generality we can assume \( k(x) = k_1(x) \). For a fixed \( \alpha \), we need to estimate the measure of \( \Omega_\alpha(\lambda) = \{ x : (x, \lambda) \in \Omega_\alpha \} \) for any \( \lambda > 0 \).

Let \( b \geq 2\alpha \lambda \) to be determined and

\[
f(x) = \begin{cases} 1 & \text{if } 0 \leq x_i \leq b \text{ and } |x_i| \leq \alpha \lambda \text{ for all } 2 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( x \in \Omega_\alpha(\lambda) \) and \( (v, r) \in \Omega \) such that \( |x - v| < \alpha(\lambda - r) \). Then

\[
|H_r f(v - x)| \leq \int_{|t| > r} f(t - (v - x)) \frac{|u_k(t)|}{|t|^d} dt
\]

\[
= \int_{\{t : (v_1 - x_1, \ldots, v_d - x_d) \leq \alpha \lambda, t \neq 0 \}} \frac{1}{|t|^d} dt
\]

by the symmetry of the kernel.

**Case 1:** \( r < \alpha \lambda \).

\[
(v_2 - x_2) + \alpha \lambda
\]

\[
(v_2 - x_2) - \alpha \lambda
\]

\[
r
\]

In this case, since \( |x - v| < \alpha \lambda \),

\[
|H_r f(v - x)| \geq \int_{\{t : t \neq 0, t_1 < \alpha \lambda, t \neq 0 \}} \frac{1}{|t|^d} dt
\]

\[
\geq c(d) \frac{(b - 2\alpha \lambda)(\alpha \lambda)^{d-1}}{(b + d\alpha \lambda)^d}
\]

if \( b = 3\alpha \lambda \).

**Case 2:** \( r \geq \alpha \lambda \).

\[
(v_2 - x_2) + r
\]

\[
(v_2 - x_2) - r
\]

\[
\text{Fig. 2}
\]

Now we will use also the fact that \( |x - v| \leq \alpha(\lambda - r) \), so in particular, \( r \leq \lambda \) and \( \alpha \leq 1 \). We have

\[
|H_r f(v - x)| \geq \int_{r < t_1 < \alpha \lambda} \frac{1}{|t|^d} dt
\]

\[
\geq c(d) \frac{(b - 2\lambda)(\alpha \lambda)^{d-1}}{(b + \alpha \lambda)^d}
\]

if \( b = 3\lambda \).

Let

\[
A(\alpha) = \begin{cases} c(d)(3 + d)^d & \text{if } \alpha \geq 1, \\ c(d)x^{d-1} / (3 + d)^d & \text{if } 0 < \alpha < 1. \end{cases}
\]

Then, if \( H_{\Omega}^p f \) is a weak \((p,p)\) operator, we have

\[
|\Omega_\alpha(\lambda)| = \{ x : (x, \lambda) \in \Omega \text{ such that } |x - v| < \alpha(\lambda - r) \}
\]

\[
\leq \| x : \sup_{(v, r) \in \Omega, r \leq \lambda} |H_r f(v - x) > A(\alpha) \|
\]

\[
\leq |\{ x : H_{\Omega}^p f(x) > A(\alpha) \}||f||_p = C(\alpha, \lambda)^d.
\]

Hence \( \Omega \) satisfies the cone condition. ■
3. Almost everywhere convergence along $\Omega$. Let $\Omega$ satisfy the cone condition. In this section we prove pointwise convergence of

$$
\lim_{(v,r) \to (0,0)} H_r f(x + v) = \limsup_{(v,r) \in \Omega} H_r f(x + v)
$$

for any $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

**Theorem 5.** Let $\Omega \subset \mathbb{R}^d \times \mathbb{R}^+$ satisfy the cone condition, such that $(0,0) \in \Omega$. Then, for any $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, we have

$$
\lim_{(v,r) \to (0,0)} H_r f(x + v) = H f(x) \quad \text{a.e.}
$$

**Proof.** Let $C^1_c(\mathbb{R}^d)$ be the set of functions with compact support and continuous partial derivatives. Let $f \in C^1_c(\mathbb{R}^d)$. Then

$$
H_r f(x + v) = f \ast k_1(x + v) + \int_{(r-x-y)<1} f(y) k(x-y) \, dy
$$

$$
= I(x,v,r) + II(x,v,r).
$$

By continuity of $f$ and compactness of its support, $I(x,v,r) \to f \ast k_1(x)$ as $(v,r) \to (0,0)$. For the second term, notice that by (k2),

$$
\int_{(r-x-y)<1} k(x-y) \, dy = 0,
$$

thus

$$
II(x,v,r) = \int [f(y+v) - f(x+v)] k(x-y) \chi_{(r-x)<1} (x-y) \, dy.
$$

Since the differential of $f$ is continuous of compact support, the integrand is majorized by

$$
c|x-y|^{-d+1} \chi_{(r-x)<1} (x-y),
$$

which is integrable. And, as $(v,r) \to (0,0)$, the integrand converges to

$$
[f(y) - f(x)] k(x-y) \chi_{(0-x)<1} (x-y).
$$

From these two estimations,

$$
\lim_{(v,r) \to (0,0)} H_r f(x + v) = H f(x) \quad \text{for all } x.
$$

Now let $f \in L^p(\mathbb{R}^d)$. Given $\varepsilon > 0$ choose $g \in C^1_c(\mathbb{R}^d)$ such that

$$
\|f - g\|_p < \varepsilon.
$$

Let

$$
Af(x) := \limsup_{(v,r) \to (0,0)} H_r f(x + v) - \liminf_{(v,r) \in \Omega} H_r f(x + v).
$$

Then, $Af = \Lambda(f - g)$ and, by Theorem 2,

$$
|\{x : Af(x) > \alpha\}| = |\{x : \Lambda(f - g)(x) > \alpha\}| \leq \frac{C(d)\|f - g\|_p}{\alpha^p} \leq \frac{C(d)}{\alpha^p} \varepsilon^p.
$$

Since $\varepsilon$ is arbitrary, the limit

$$
\lim_{(v,r) \to (0,0)} H_r f(x + v) = H f(x) \quad \text{a.e.}
$$

exists for almost every $x$.

Similar arguments show that

$$
\lim_{(v,r) \to (0,0)} H_r f(x + v) = H f(x) \quad \text{a.e.}
$$

**Theorem 6.** Recall that

$$
Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2}
$$

denotes the conjugate Poisson kernel in $\mathbb{R}^d$. If $\Omega$ satisfies the cone condition, then

$$
\lim_{(v,r) \to (0,0)} Q_y \ast f(x + v) \quad \text{exists for a.e. } x,
$$

and is equal to $H f(x)$.

**Proof.** This follows from Theorem 3 and the fact that

$$
\lim_{\varepsilon \to 0} Q_{\varepsilon} \ast f(x + v) = H f(x)
$$

(by arguments similar to those in Theorem 5).

4. Hilbert transform for measurable flows. Let $(X,\beta,\mu)$ be a $\sigma$-finite measure space and $\{\tau_t\}_{t \in \mathbb{R}^d}$ a measure preserving action of $\mathbb{R}^d$ acting on $X$, which is jointly measurable from $\mathbb{R}^d \times X$ to $X$. We will now consider the truncated ergodic singular integrals

$$
H_r f(x) = \int \, f(\tau_t x) k(t) \, dt, \quad f \in L^p(X),
$$

and the related moving maximal operator

$$
H_r^\# = \sup_{(v,r) \in \Omega} |H_r f(\tau_v x)|.
The singular integral results obtained in Section 1 can be translated to this setting by means of a Calderón transfer principle. However, we first need to establish a modified version of the results in Section 1, for the truncated singular integrals.

Since we are interested in the limit as \((v, r) \to (0, 0)\), in this section we will assume that for all \((v, r) \in \Omega\), we have \(r \leq 1\).

**Corollary 7.** Let \(\Omega \subset \mathbb{R} \times \mathbb{R}^+\) satisfy the cone condition. Then

\[
\sup_{(v, r) \in \Omega} \left| \int_{r < |t| < 1/r} f(x + v + t)k(t) \, dt \right|
\]

is a weak \((1, 1)\) and strong \((p, p)\) operator for \(1 < p < \infty\).

**Proof.** The result follows from Theorem 2 because

\[
\left| \int_{r < |t| < 1/r} f(x + v + t)k(t) \, dt \right| \leq |H_r f(x + v)| + |H_{1/r} f(x + v)|,
\]

and \((v, 1/r) : (v, r) \in \Omega\) satisfies the cone condition if \(r \leq 1\). \(\blacksquare\)

**Proposition 8.** (Transfer principle). Let \(\Omega \subset \mathbb{R}^d \times \mathbb{R}^+\) and \(1 < p < \infty\).

If

\[
\sup_{(v, r) \in \Omega} \left| \int_{r < |t| < 1/r} \varphi(x + v + t)k(t) \, dt \right|
\]

is a weak \((p, p)\) operator in \(L^p(\mathbb{R})\), then \(H_{\Omega}^{\varphi} f\) is a weak \((p, p)\) operator in \(L^p(\mathbb{R}^d)\).

**Proof.** Fix \(M > 0\) and let \(N = 3M\). Given \(f \in L^p(\mathbb{R}^d)\) define

\[
\varphi_x(t) = \begin{cases} f(\tau_r x) & \text{if } |t| \leq N, \\ 0 & \text{otherwise.} \end{cases}
\]

Then, for almost every \(x, v\), we \(\varphi_x \in L^p(\mathbb{R}^d)\). Indeed,

\[
\int_{X} \int_{|t| \leq N} |\varphi_x(t)|^p \, dx \, dt = \int_{X} \int_{|t| \leq N} |f(\tau_r x)|^p \, dx \, dt = c(d)N^d \|f\|^p_p,
\]

because the flow is measure preserving.

Let \(\Omega_M = \{(v, r) \in \Omega : |v| \leq M, 1/M \leq r \leq M\}\). Then

\[
\int_{X} \left| \int_{r < |s| < 1/r} \varphi_x(t)k(s + v + t) \, dt \right| \, dx \geq \lambda
\]

\[
\leq \frac{C}{\lambda^p} \int_{X} \|\varphi_x\|^p_p \leq c(d)N^d \frac{C}{\lambda^p} \|f\|^p_p.
\]

Let

\[
A = \{(x, s) \in X \times \mathbb{R}^d : \sup_{(v, r) \in \Omega_M} \left| \int_{r < |s + v + t| < 1/r} \varphi_x(t)k(s + v + t) \, dt \right| \geq \lambda \}.
\]

Notice that if \((v, r) \in \Omega_M, |v| \leq M\) and \(|t| < 1/r\), then

\[
f(\tau_{v+s+t} x) = \varphi_x(v + s + t)\] because \(3M = N\). Thus,

\[
\int_{X} \left| \int_{r < |s + v + t| < 1/r} \varphi_x(t)k(s + v + t) \, dt \right| \geq \lambda
\]

\[
\geq \int_{X} \int_{|x| \leq 2M} |H_{v+s+t}(\mathbb{R}^d)(s)| \, dx \, ds
\]

\[
\geq \int_{X} \int_{|x| \leq 2M} m(x, v, r) |H_{v+s+t}(\mathbb{R}^d)(x)| \, dx \, ds
\]

\[
= c(d)M^d \int_{X} \sup_{(v, r) \in \Omega_M} \left| H_{v+s+t}(\mathbb{R}^d)(x) \right| \, dx
\]

Since \(N = 3M\), we obtain

\[
m(x, v, r) \geq \lambda \leq \frac{3dC}{\lambda^p} \|f\|^p_p.
\]

The proposition follows by letting \(M \to \infty\). \(\blacksquare\)

**Corollary 9.** If \(\Omega\) satisfies the cone condition, then \(H_{\Omega}^{\varphi} f\) is a weak \((1, 1)\) and strong \((p, p)\) operator for \(1 < p < \infty\).

**Proof.** This follows from Corollary 7 and Proposition 8. \(\blacksquare\)

**Theorem 10.** Let \(\Omega \subset \mathbb{R}^d \times \mathbb{R}^+\) satisfy the cone condition and \((0, 0) \in \overline{\Omega}\). Then

\[
\lim_{(v, r) \to (0, 0)} H_{\Omega}^{\varphi} f(\tau_v x)
\]

exists a.e. for all \(f \in L^p(\mathbb{R}^d), 1 < p < \infty\).

**Proof.** It suffices to prove that

\[
k_{\Omega, \varphi}(u) := \int_{r < |t| < 1/r} k(t) \phi(u - v - t) \, dt
\]

converges in \(L^1(\mathbb{R}^d)\) as \((v, r) \to (0, 0), (v, r) \in \Omega\), for any \(\phi \in C_0^1(\mathbb{R}^d)\) satisfying \(\int_{\mathbb{R}^d} \phi \, ds = 0\). Indeed, let

\[
O \{ h \in L^1(\mathbb{R}^d) : h(x) = \int g(\tau_x) \phi(t) \, dt, g \in L^1(\mathbb{R}^d), \phi \in C_0^1(\mathbb{R}^d) \}.
\]

Then

\[
H_{\Omega}^{\varphi} h(\tau_v x) = \int g(\tau_x) k_{\Omega, \varphi}(\phi(s)) \, ds.
\]
The orthogonal complement of $O \cap L^2(X)$ consists of the invariant functions under the action (see [2]). Thus the theorem would hold for a dense class of functions and then the result would follow for all functions by an application of Corollary 9.

Let us introduce some notation:

$$K_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if } r \leq |s-v|, \\ 0 & \text{otherwise}, \end{cases}$$

$$k_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if } r \leq |s-v| \leq 1/r, \\ 0 & \text{otherwise}. \end{cases}$$

Hence

$$\left| \int_{r \leq |s| \leq 1/r} f(u-v-s)k(s) \, ds \right| \leq \int_{r \leq |u-v-s| \leq 1/r} f(s)k(u-v-s) \, ds = \left| k_{(v,r)}(u-s)f(s) \, ds \right| = |k_{(v,r)} * f(u)|.$$

The $L^1$-convergence of $k_{(v,r)} * \phi$ follows from the following two properties:

(A) $K_{(v,r)} * \phi$ converges in $L^1$, and

(B) $\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0$ as $r \to 0$.

Property (A) follows from Lebesgue's Dominated Convergence Theorem. By Theorem 8, $K_{(v,r)} * \phi$ converges a.e. Assume that supp$(\phi) \subseteq \{y \leq L\}$.

Then

$$|K_{(v,r)} * \phi(u)| \leq \left( c \chi_{\{y \leq 2L\}}(u) + \frac{c(d,L)}{|u|^{d+1}} \chi_{\{|u| \leq 2L\}}(u) \right) \in L^1(\mathbb{R}^d).$$

First consider $|u| \geq 2L$. Then, using the basic properties of $\phi$ and $K_{(v,r)}$ (recall (k2)), we can compute (for $\phi(v) * \phi(u)$ small enough)

$$|K_{(v,r)} * \phi(u)| = \left| \int [K_{(v,r)}(s) - K_{(v,r)}(u)] \phi(s) \, ds \right| \leq \int_{|s| \leq K} |K_{(v,r)}(s) - K_{(v,r)}(u)| \cdot |\phi(s)| \, ds \leq \int_{|s| \leq K} \frac{|s|}{|u-v|^d} \cdot |\phi(s)| \, ds \leq \frac{c}{|u|^{d+1}} \cdot c|d| \frac{1}{|u|^{d+1}},$$

by (k3). Here $c = c(\phi)$.

Consider now $|u| \leq 2L$. Taking $(v,r)$ small enough we get

$$|K_{(v,r)} * \phi(u)| = \left| \int [K_{(v,r)}(s) - K_{(v,r)}(u)] \phi(s) \, ds \right| \leq \int_{4L \leq |s| \geq 2r} \frac{u(s)}{|s|^d} \phi(s) \, ds \leq \frac{c}{4L |v|^{d+1}} \left| \phi(u) \right| \leq c |\phi(u)|.$$

(where $c = c(\phi)$) because the differential of $\phi$ is continuous of compact support. This ends the proof of (A).

To prove (B), assume supp$(\phi) \subseteq \{|y| \leq K\}$. By definition of $K_{(v,r)}$ and $k_{(v,r)}$, we have

$$k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u) = k_{1/r} * \phi(u) - \phi(u).$$

Now $K_{1/r} * \phi(u) = 0$ if $u \notin S_{(v,r)} := \mathbb{R}^d \setminus \{u : |u| < 1/r - v - L\}$. We can choose $(u,v)$ small enough such that $u \in S_{(v,r)}$ implies $|u| \geq 2L$. Then a similar computation to that in (A) gives $|k_{1/r} * \phi(u)| \leq c/|u|^{d+1}$. In summary,

$$|k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u)| \leq \chi_{S_{(v,r)}}(u) k_{1/r} * \phi(u) \leq \chi_{S_{(v,r)}}(u) \frac{c}{|u|^{d+1}}.$$

Hence

$$\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0.$$

References

On invariant measures for power bounded positive operators

by

RYOTARO SATO (Okayama)

To the memory of Hisao Tominaga

Abstract. We give a counterexample showing that \((I-T^*)L_\infty \cap L_\infty^+ = \{0\}\) does not imply the existence of a strictly positive function \(u \in L_1\) with \(Tu = u\), where \(T\) is a power bounded positive linear operator on \(L_1\) of a \(\sigma\)-finite measure space. This settles a conjecture by Brunel, Horowitz, and Lin.

1. Introduction. Let \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space and \(T\) a positive linear operator in \(L_1 = L_1(X, \Sigma, m)\). \(T\) is called a contraction if \(\|T\| \leq 1\), power bounded if \(\sup_n \|T^n\| < \infty\), and Cesàro bounded if \(\sup_n \|n^{-1} \sum_{k=0}^n T^k\| < \infty\). Many ergodic theorems for positive \(L_1\) contractions require the existence of a finite invariant measure equivalent to the original one, i.e., a strictly positive \(u \in L_1\) with \(Tu = u\). This problem has attracted many top researchers, and one of the conditions equivalent to the existence of such a \(u \in L_1\), obtained by Brunel [1], is that

\[(I-T^*)L_\infty \cap L_\infty^+ = \{0\}\]

For any \(T\) positive and Cesàro bounded, condition (1) is seen, by using the known fact that \(n^{-1}||T^n|| \to 0\) as \(n \to \infty\), to be equivalent to the following condition:

\[\limsup_n \frac{1}{n} \sum_{k=1}^n T^{*k} 1_A \to 0\]

Stuchiel [7] started a systematic study of power bounded positive linear operators in \(L_1\), and Pong [4] studied the problem of existence of strictly positive fixed points under an additional assumption of a null disappearing part. The problem in general was studied by Derriennic and Lin [3] (see also Sato [6]), who proved that for any \(T\) positive and Cesàro bounded, an

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