Posons

$$C_i = \left\{ z \in \mathbb{C} \;\middle|\; \frac{\varepsilon}{1+\alpha_i} < |z| < \frac{\varepsilon}{1+\alpha_{i-1}} \right\}$$

avec $\alpha_0 = 0$. Soit f_i la fonction definie par

$$f_i(z) = \begin{cases} 1 & \text{si } z \in C_i, \\ 0 & \text{si } z \notin C_i. \end{cases}$$

On a alors $f_i(t) = b_i$. En effet

$$\chi(f_i(t)) = f_i(\chi(t)) = \begin{cases} 0 & \text{si } \chi(b_i) = 0, \\ 1 & \text{si } \chi(b_i) = 1. \end{cases}$$

D'où $\chi(f_i(t)) = \chi(b_i)$ pour tout $\chi \in \Delta_B$. Comme $f_i(t)$ est un idempotent on a $b_i = f_i(t)$.

2^{ème} cas. Si a n'est pas inversible, on prend un $\lambda \notin \sigma(a)$ et on fait le raisonnement précédent avec $a - \lambda e$ au lieu de a.

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Exactness of skew products with expanding fibre maps

by

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To the memory of Wiesław Szlenk

Abstract. We give an elementary proof for the uniqueness of absolutely continuous invariant measures for expanding random dynamical systems and study their mixing properties.

Introduction. Let θ be a measure-preserving transformation of a Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\varphi := \{\varphi(\omega) : \omega \in \Omega\}$ be a family of nonsingular transformations of $(X, \mathcal{B}, m)$ such that $(\omega, x) \mapsto \varphi(\omega)x$ is $(\mathcal{F} \otimes \mathcal{B}, \mathcal{B})$-measurable. Then $(\omega, x) \mapsto (\theta\omega, \varphi(\omega)x) =: \Theta(\omega, x)$ defines a nonsingular transformation of the Lebesgue space $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}, \mathbb{P} \otimes m)$ which is called a *skew product* with base transformation θ and fibre maps $\varphi(\omega)$.

One also says that φ gives rise to a *random dynamical system* with state space X over the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ by defining

$$\varphi(n, \omega) := \varphi(\theta^{n-1}\omega) \circ \ldots \circ \varphi(\omega) \quad \text{for } n > 0.$$

Note that $\varphi(n, \omega)$ describes the action of Θ^n on X.

In this paper, we study the situation when the state space is a Riemannian manifold M and all fibre maps are expanding. In their classical paper [KS69] Krzyżewski and Szlenk proved that for each expanding map there is an invariant measure which is equivalent to the Riemannian volume. The fundamental problem in the random case is to find a Θ-invariant measure on $\Omega \times M$ whose disintegrations are equivalent to the Riemannian volume. This problem has been solved by Kifer [Kif92, Theorem B]

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using some statistical mechanics type of reasoning. We present a different approach using the classical "dynamics of densities" viewpoint (Lasota and Mackey [LM94]). This approach allows us to obtain some further properties of the measure under consideration. In particular, we prove that it is exact provided the base transformation is exact.

The paper is organized as follows. In Section 1 we lay down some preliminaries, while Section 2 is devoted to the concept of regularity. The latter plays a crucial role in the proof of our main theorem, which we present in Section 3.

1. Basic notations and preliminary results

The Frobenius–Perron operator for skew products. Let T be a measurable transformation of a Lebesgue space $(X,\mathcal{B},m)$. If for all $B\in\mathcal{B}$ with $m(B)=0$ we have $m(T^{-1}B)=0$ then T is said to be *nonsingular*. Associated with each nonsingular transformation T is the *Frobenius–Perron operator* $P_T:\mathbb{L}^1(m)\to\mathbb{L}^1(m)$ which is uniquely determined by the equation

$$\int_B P_T f\,dm=\int_{T^{-1}B} f\,dm\quad\text{for all } B\in\mathcal{B}. \tag{1}$$

For properties of P_T we generally refer to Lasota and Mackey [LM94]. In particular, we have that f is the density of a T-invariant measure which is absolutely continuous with respect to m if and only if f is a fixed point of P_T.

Now let θ be a measure-preserving transformation of a Lebesgue space $(\Omega,\mathcal{F},\mathbb{P})$, $\varphi:=\{\varphi(\omega):\omega\in\Omega\}$ a family of nonsingular transformations of $(X,\mathcal{B},m)$ such that $(\omega,x)\mapsto\varphi(\omega)x$ is $(\mathcal{F}\otimes\mathcal{B},\mathcal{B})$-measurable, and Θ the corresponding skew product on $(\Omega\times X,\mathcal{F}\otimes\mathcal{B},\mathbb{P}\otimes m)$.

1.1. Lemma. *For $f\in\mathbb{L}^1(\mathbb{P}\otimes m)$ put $(\widehat{P}_\theta f)(\omega,x):=(P_\theta f_x)(\omega)$ and $(\widehat{P}_\varphi f)(\omega,x):=(P_{\varphi(\omega)}f_\omega)(x)$, where $f_x:=f(\cdot,x)\in\mathbb{L}^1(\mathbb{P})$ and $f_\omega:=f(\omega,\cdot)\in\mathbb{L}^1(m)$. Then $\widehat{P}_\theta f$ and $\widehat{P}_\varphi f$ are measurable functions for each $f\in\mathbb{L}^1(\mathbb{P}\otimes m)$, and*

$$P_\Theta=\widehat{P}_\theta\circ\widehat{P}_\varphi. \tag{2}$$

Consequently,

$$P_\Theta^n=\widehat{P}_\theta^n\circ\widehat{P}_{\varphi(n,\cdot)}\quad\textit{for all } n\in\mathbb{N}. \tag{3}$$

Proof. Let $f\in\mathbb{L}^1(\mathbb{P}\otimes m)$. Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of finite partitions of X satisfying $\mathcal{P}_n\downarrow\varepsilon$. Then

$$\begin{aligned}(P_{\varphi(\omega)}f_\omega)(x)&=\Big(\frac{d}{dm}\int_{\varphi(\omega)^{-1}(\cdot)} f_\omega\,dm\Big)(x)\\&=\lim_{n\to\infty}\sum_{P\in\mathcal{P}_n}1_P(x)\frac{1}{m(P)}\int f_\omega 1_{\varphi(\omega)^{-1}P}\,dm.\end{aligned}$$

Obviously the measurability of $(\omega,x)\mapsto\varphi(\omega)x$ implies that of $(\omega,x)\mapsto 1_{\varphi(\omega)^{-1}P}(x)$, so $\widehat{P}_\varphi f$ is measurable. The same argument applies to $\widehat{P}_\theta f$.

To prove equation (2) it suffices to check equation (1) for product sets $F\times B$, $F\in\mathcal{F}$, $B\in\mathcal{B}$. For those sets we have

$$[\Theta^{-1}(F\times B)]_\omega=\begin{cases}\varphi(\omega)^{-1}B & \text{if } \omega\in\theta^{-1}F,\\ \emptyset & \text{otherwise,}\end{cases}$$

where $A_\omega:=\{x:(\omega,x)\in A\}$ for $A\in\mathcal{F}\otimes\mathcal{B}$. Hence

$$\begin{aligned}\int_{\Theta^{-1}(F\times B)} f\,d(\mathbb{P}\otimes m)&=\int_{\theta^{-1}F}\Big(\int_{\varphi(\omega)^{-1}B} f(\omega,x)\,dm(x)\Big)\,d\mathbb{P}(\omega)\\&=\int_B\Big(\int_{\theta^{-1}F}(\widehat{P}_\varphi f)(\omega,x)\,d\mathbb{P}(\omega)\Big)\,dm(x)\end{aligned}$$

by the definition of $\widehat{P}_\varphi$ and Fubini's theorem. By the definition of $\widehat{P}_\theta$ this gives the desired result. ∎

1.2. Remark. We did not use the assumption that θ is measure-preserving, so (2) holds whenever θ is nonsingular.

1.3. Remark. If θ is an *automorphism*, i.e. a measure-preserving bijection with measurable inverse, then $P_\theta g=g\circ\theta^{-1}$ for all $g\in\mathbb{L}^1(\mathbb{P})$. In this case, thus, (3) reads

$$(P_\Theta^n f)(\omega,x)=(P_{\varphi(n,\theta^{-n}\omega)}f_{\theta^{-n}\omega})(x)\quad\text{for all } f\in\mathbb{L}^1(\mathbb{P}\otimes m),\ n\in\mathbb{N}.$$

Absolutely continuous measures on product spaces. The ergodic theory of random dynamical systems is concerned with Θ-invariant measures μ which have marginal $\mathbb{P}$ on Ω. The following says that this is not too much of a restriction.

1.4. Lemma. (i) *If θ is ergodic then any Θ-invariant measure μ on $\Omega\times X$ which is absolutely continuous with respect to $\mathbb{P}\otimes m$ has marginal $\mathbb{P}$ on Ω.*

(ii) *Vice versa: If there exists an ergodic Θ-invariant measure μ on $\Omega\times X$ which is equivalent to $\mathbb{P}\otimes m$ then θ is necessarily ergodic.*

Proof. Write $\widetilde{\mathbb{P}}:=\pi_\Omega\mu$. If μ is Θ-invariant then $\widetilde{\mathbb{P}}$ is θ-invariant.

(i) $\mu\ll\mathbb{P}\otimes m$ implies $\widetilde{\mathbb{P}}\ll\mathbb{P}$, hence $\widetilde{\mathbb{P}}=\mathbb{P}$ by ergodicity (cf. Corollary A.2).

(ii) In this case $\widetilde{\mathbb{P}}$ is ergodic and equivalent to $\mathbb{P}$. Hence, since $\mathbb{P}$ is θ-invariant, again $\widetilde{\mathbb{P}}=\mathbb{P}$. ∎

If X is a Polish space then any measure μ on $\Omega \times X$ with marginal $\mathbb{P}$ on Ω is determined by an essentially unique family of conditional probabilities $\{\mu_\omega\}$ satisfying $d\mu(\omega, x) = d\mu_\omega(x)\, d\mathbb{P}(\omega)$.

1.5. LEMMA. *Let m be a Borel measure on a Polish space X and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose μ is a measure on $\Omega \times X$ with marginal $\mathbb{P}$ on Ω. Then*

$$\mu \ll \mathbb{P} \otimes m \;\Leftrightarrow\; \mu_\omega \ll m \text{ } \mathbb{P}\text{-a.s.}$$

Proof. $\Leftarrow$: Let $A \in \mathcal{F} \otimes \mathcal{B}$ with $\mathbb{P} \otimes m(A) = 0$. Then $m(A_\omega) = 0$ $\mathbb{P}$-a.s., hence by the assumption $\mu_\omega(A_\omega) = 0$ $\mathbb{P}$-a.s. Thus $\mu(A) = \int \mu_\omega(A_\omega)\, d\mathbb{P}(\omega) = 0$.

$\Rightarrow$: Let g be the density of μ with respect to $\mathbb{P} \otimes m$. For each $F \in \mathcal{F}$ and each $B \in \mathcal{B}$ we have

$$\mu(F \times B) = \int_F \int_B g(\omega, x)\, dm(x)\, d\mathbb{P}(\omega) = \int_F \mu_\omega(B)\, d\mathbb{P}(\omega).$$

For each $B \in \mathcal{B}$, thus, we have

$$\mu_\omega(B) = \int_B g(\omega, x)\, dm(x) \quad \mathbb{P}\text{-a.s.}$$

Since $\mathcal{B}$ is countably generated we can construct from this a set $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$ such that $g(\omega, \cdot)$ is an m-density of μ_ω for all $\omega \notin N$. ■

Expanding random dynamical systems. Let $X = M$ be a compact connected smooth manifold equipped with a Riemannian metric $\|\cdot\|$. A random dynamical system φ on M is called *expanding* if each $\varphi(\omega)$ is a $\mathcal{C}^2$-mapping and if there exists a constant $\gamma > 1$ such that for all $\omega \in \Omega$ the differential $D_x\varphi(\omega)$ of $\varphi(\omega)$ at $x \in M$ satisfies

$$\|(D_x\varphi(\omega))(v)\| \geq \gamma\|v\| \quad \text{for all } v \in T_xM.$$

The following result has been proved by Kifer [Kif92, Theorem B].

THEOREM. *Assume that φ is an expanding random dynamical system on M over an automorphism θ. Then there exists a Θ-invariant probability μ with marginal $\mathbb{P}$ on Ω such that μ_ω is $\mathbb{P}$-a.s. equivalent to the Riemannian volume m on M.*

In the proof μ is constructed as the relative equilibrium state of the function $-\log|\det D_x\varphi(\omega)|$. For ergodic θ this yields automatically ergodicity of μ. So in view of Lemma 1.5, μ is in fact the unique Θ-invariant measure with $\mu_\omega \ll m$ $\mathbb{P}$-a.s. (cf. Corollary A.2).

Our aim is to give an alternative proof of this result which also allows us to deduce further mixing properties of μ. Recently, Khanin and Kifer [KK94, Theorem 3.2] have obtained results in this direction even for random dynamical systems which are only expanding in average.

2. Regularity. Let M be a compact connected smooth Riemannian manifold and m the normalized Lebesgue measure on M. Further, let $\mathrm{Lip}(M)$ denote the set of all nonnegative Lipschitz functions on M.

2.1. DEFINITION. The *regularity* of $f \in \mathrm{Lip}(M)$ is given by

$$\mathrm{Reg}(f) := \sup\{|f'(x)|/f(x) : x \in M \text{ with } f'(x) \text{ defined and } f(x) > 0\}.$$

Here $f'(x)$ denotes the length of the gradient of f at x.

f is said to be *regular* if $\mathrm{Reg}(f) < \infty$.

We will apply the following result due to Lasota [Las80, Proposition 2].

2.2. PROPOSITION. *If $f \in \mathrm{Lip}(M)$ is strictly positive with $\int f\, dm = 1$ and $\mathrm{Reg}(f) \leq \alpha$ then*

$$e^{-\alpha r} \leq f(x) \leq e^{\alpha r} \quad \textit{and} \quad |f'(x)| \leq e^{\alpha r} \quad \textit{for all } x \in M,$$

where $r = \operatorname{diam} M$.

2.3. Remark. Since obviously $\mathrm{Reg}(cf) = \mathrm{Reg}(f)$ for any $c > 0$ we obtain

$$e^{-\alpha r}\int f\, dm \leq f(x) \leq e^{\alpha r}\int f\, dm \quad \text{for all } x \in M \tag{4}$$

for any strictly positive Lipschitz function with $\mathrm{Reg}(f) \leq \alpha$ by simply applying the above to the normalized function $f/\int f\, dm$.

Let $D := \{f \in \mathbb{L}^1(\mathbb{P} \times m) : f \geq 0 \text{ and } \int f\, d\mathbb{P} \otimes m = 1\}$ be the set of all densities in $\mathbb{L}^1(\mathbb{P} \times m)$. Further, let

$$D_R := \{f \in D : f_\omega \in \mathrm{Lip}(M) \text{ } \mathbb{P}\text{-a.s. and } \omega \mapsto \mathrm{Reg}(f_\omega) \in \mathbb{L}^\infty(\mathbb{P})\}$$

be the set of densities with essentially bounded regularity.

2.4. LEMMA. *D_R is dense in D.*

Proof. Let γ be a partition of M given by an open cover of M by local charts. Let E denote the set of all members f of D which can be written in the form

$$f(\omega, x) = \sum_{A \in \alpha,\, B \in \beta} c_{AB} 1_A(\omega) 1_B(x)$$

where α is a finite partition of Ω, $\beta > \gamma$ is a finite partition of M on closed sets such that $m(\partial B) = 0$ for all $B \in \beta$, and c_{AB} are nonnegative coefficients varying with the pairs $(A, B) \in \alpha \times \beta$.

Since E is dense in D the proof will be complete if we can approximate each member of E by members of D_R. So let $f \in E$ and $\varepsilon > 0$ be given.

For each $B \in \beta$ we can choose a closed set C and an open set U with $C \subset U \subset B$ and $m(B \setminus C) < \varepsilon$. Further, we can choose functions f_B which are $\mathcal{C}^\infty$ in local coordinates and satisfy $\varepsilon \le f_B \le 1$ and

$$f_B(x) = \begin{cases} \varepsilon & \text{for } x \notin U, \\ 1 & \text{for } x \in C. \end{cases}$$

Define

$$g(\omega, x) = \sum_{A\in\alpha, B\in\beta} c_{AB} 1_A(\omega) f_B(x).$$

Obviously, $g/\int g\, d\mathbb{P}\otimes m \in D_R$ and

$$\int |g - f|\, d\mathbb{P}\otimes m \le 2\varepsilon \sum_{A\in\alpha, B\in\beta} c_{AB}\mathbb{P}(A),$$

which finishes the proof. ∎

3. The main result. From now on, we assume that φ is an expanding random dynamical system with state space M over the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, and let Θ denote the corresponding skew product. We write $J_i(\omega)(x) := |\det d\psi_i(\omega)(x)|$, where $\psi_i(\omega)$ are the inverse branches of $\varphi(\omega)$, $1 \le i \le n(\omega)$.

3.1. THEOREM. *Let φ be an expanding random dynamical system on M over an endomorphism θ. Assume there exists $K > 0$ such that*

$$K(\omega) := \sup_{i,x} \frac{|J_i'(\omega)(x)|}{J_i(\omega)(x)} \le K \quad \mathbb{P}\text{-a.s.} \tag{5}$$

(i) *There exists a Θ-invariant probability μ which is absolutely continuous with respect to $\mathbb{P}\otimes m$.*

(ii) *If θ is ergodic this μ is uniquely determined, equivalent to $\mathbb{P}\otimes m$, and ergodic. Moreover, for the density $f_0 = d\mu/d\mathbb{P}\otimes m$ we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} P_\Theta^i f = f_0 \quad \text{for all } f \in D.$$

(iii) *If θ is exact then the system $(\Omega\times M, \mathcal{F}\otimes\mathcal{B}, \mu, \Theta)$ is exact, and we have*

$$\lim_{n\to\infty} P_\Theta^n f = f_0 \quad \text{for all } f \in D.$$

3.2. Remark. Proofs of statement (i) and the first part of (ii) already exist under weaker assumptions than (5) (cf. Kifer [Kif92], Khanin and Kifer [KK94]) using methods borrowed from statistical mechanics. Our proof, which has been inspired by Lasota [Las80], is more elementary.

We also remark that Khanin and Kifer [KK94, Theorem 3.2] proved that if θ is a mixing automorphism then μ is mixing. Our proof of statement (iii) has been inspired by Morita [Mor85].

Proof. Following the argument in the proof of Theorem 3 of [Las80] for each $\omega \in \Omega$ (outside a set of measure zero) separately we obtain $\mathrm{Reg}(P_{\varphi(\omega)} f_\omega) \le \frac{1}{\gamma}\mathrm{Reg}(f_\omega) + K(\omega)$. By induction, and in view of assumption (5), this yields

$$\mathrm{Reg}(P_{\varphi(n,\omega)} f_\omega) \le \frac{1}{\gamma^n}\mathrm{Reg}(f_\omega) + \frac{K}{\gamma - 1}.$$

Let $\alpha > K/(\gamma-1)$. For $f \in D_R$, thus, there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Reg}(P_{\varphi(n,\omega)} f_\omega) < \alpha$ for all $n \ge n_0$ and $\mathbb{P}$-a.a. ω (because $\mathrm{Reg}(f_\omega)$ is essentially bounded).

Applying (4) we see that $\mathbb{P}$-a.s.

$$e^{-\alpha r}\int f_\omega\, dm \le (P_{\varphi(n,\omega)} f_\omega)(x) \le e^{\alpha r}\int f_\omega\, dm \quad \text{for all } n \ge n_0,\ x \in M.$$

With the notation $\widehat{f}(\omega) := \int f_\omega\, dm$ this yields that $\mathbb{P}$-a.s.

$$e^{-\alpha r}(P_\theta^n \widehat{f})(\omega) \le (P_\Theta^n f)(\omega, x) \le e^{\alpha r}(P_\theta^n \widehat{f})(\omega) \quad \text{for all } n \ge n_0,\ x \in M \tag{6}$$

by formula (3).

The Dunford–Pettis Theorem (cf. Diestel [Die84, p. 93]) allows us to conclude from this that $\{P_\Theta^n f : n \in \mathbb{N}\}$ is relatively weakly compact in $\mathbb{L}^1(\mathbb{P}\otimes m)$, hence $\{n^{-1}\sum_{i=0}^{n-1} P_\Theta^i f : n \in \mathbb{N}\}$ has a weak cluster point, say $P_\Theta^* f$. The mean ergodic theorem (cf. Krengel [Kre85, p. 72]) implies that in fact

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} P_\Theta^i f = P_\Theta^* f \quad \text{in the strong } \mathbb{L}^1(\mathbb{P}\otimes m)\text{-norm.} \tag{7}$$

Obviously, $P_\Theta^* f$ is a fixed point of P_Θ, hence the density of a Θ-invariant measure absolutely continuous with respect to $\mathbb{P}\otimes m$.

Now assume that θ is ergodic. Write $C(\omega, x) := \sum_{i=0}^{n_0-1}(P_\Theta^i f)(\omega, x)$. By (6) we obtain

$$\begin{aligned} e^{-\alpha r}\Big(\frac{1}{n}\sum_{i=n_0}^{n-1}(P_\theta^i \widehat{f})(\omega) + \frac{1}{n}C(\omega,x)\Big) &\le \frac{1}{n}\sum_{i=0}^{n-1} P_\Theta^i f(\omega, x) \\ &\le e^{\alpha r}\Big(\frac{1}{n}\sum_{i=n_0}^{n-1}(P_\theta^i \widehat{f})(\omega) + \frac{1}{n}C(\omega,x)\Big) \end{aligned}$$

for all $n > n_0$. Since $\widehat{f}$ is a density in $\mathbb{L}^1(\Omega, \mathbb{P})$ and θ is ergodic the sequence $\{P_\theta^n \widehat{f}\}$ is Cesàro convergent to 1 (cf. Lasota and Mackey [LM94, Theorem 4.4.1]); so letting n tend to infinity the above yields $e^{-\alpha r} \le P_\Theta^* f \le e^{\alpha r}$ for all $f \in D_R$. Since D_R is dense in D (Lemma 2.4) the limit (7) exists for all $f \in D$ and we obtain

$$e^{-\alpha r} \le P_\Theta^* f \le e^{\alpha r} \quad \text{for all } f \in D.$$

This means that every Θ-invariant measure absolutely continuous with respect to $\mathbb{P}\otimes m$ is equivalent to $\mathbb{P}\otimes m$, proving the assertions of (ii) (cf. Corollary A.3).

It remains to prove (iii), so assume θ is exact. This means, in particular, that θ is totally ergodic, i.e. θ^n is ergodic for all $n\in\mathbb{N}$. Write $\varphi^{(n)}(\omega):=\varphi(n,\omega)$. Since this gives an expanding random dynamical system over θ^n we may apply to $\varphi^{(n)}$ what we have shown so far. As a result, we conclude that the system (Θ,μ) is totally ergodic.

Assume that Θ is not exact, i.e. the σ-algebra

$$(\mathcal{F}\otimes\mathcal{B})_\infty=\bigcap_{n=0}^{\infty}\Theta^{-n}(\mathcal{F}\otimes\mathcal{B})$$

is nontrivial. Then, by total ergodicity, it is atomless, so there is a sequence $(B_k)_{k\in\mathbb{N}}\subset(\mathcal{F}\otimes\mathcal{B})_\infty$ with $\mathbb{P}\otimes m(B_k)>0$ and $\lim_{n\to\infty}\mathbb{P}\otimes m(B_k)=0$.

Define

$$f_k:=\frac{1}{\mathbb{P}\otimes m(B_k)}1_{B_k}.$$

Clearly $\int f_k\,d\mathbb{P}\otimes m=1$ for all $k\in\mathbb{N}$. Also $B_k=\Theta^{-n}\Theta^nB_k$ for all $k,n\in\mathbb{N}$ (since $B_k\in(\mathcal{F}\otimes\mathcal{B})_\infty$; cf. Rokhlin [Rho64]), hence $f_k=f_k\cdot 1_{\Theta^nB_k}\circ\Theta^n$. Therefore

$$\begin{aligned}1&=\int f_k\,d\mathbb{P}\otimes m=\int f_k\cdot 1_{\Theta^nB_k}\circ\Theta^n\,d\mathbb{P}\otimes m\\&=\int_{\Theta^nB_k}P_\Theta^nf_k\,d\mathbb{P}\otimes m\quad\text{ for all } k,n\in\mathbb{N}.\end{aligned}$$

Applying (6) to an approximation of f_k by an element of D_R (cf. Lemma 2.4) yields

$$\limsup_{n\to\infty}\int_{\Theta^nB_k}P_\Theta^nf_k\,d\mathbb{P}\otimes m\le e^{\alpha r}\limsup_{n\to\infty}\int_{\Theta^nB_k}P_\theta^n\widehat{f}_k\,d\mathbb{P}\otimes m\quad\text{ for all } k\in\mathbb{N}.$$

By exactness of θ, $P_\theta^n\widehat{f}_k$ is strongly convergent to 1 (cf. Lasota and Mackey [LM94, Theorem 4.4.1]), so the above yields

$$\limsup_{n\to\infty}\mathbb{P}\otimes m(\Theta^nB_k)\ge e^{-\alpha r}\quad\text{ for all } k\in\mathbb{N}.$$

But $\lim_{k\to\infty}\mathbb{P}\otimes m(\Theta^nB_k)=0$ for all $n\in\mathbb{N}$, i.e. we reached a contradiction. ■

3.3. Remark. If θ is positive nonsingular and ergodic, one can show that $f_0\in D_R$. In this case, one can even prove that $\mathbb{P}$-a.s.

$$|f_0(\omega,x)-f_0(\omega,y)|\le e^{\alpha r}\varrho(x,y),$$

where ϱ is the usual distance on M. This can be done by applying the second part of Proposition 2.2 in the same way as we did with the first part to obtain (6) in the above proof.

Appendix. The following result is well known. We add the proof for the sake of completeness.

A.1. THEOREM. *Let T be a measurable transformation of a Lebesgue space $(X,\mathcal{B},m)$. If there are two different T-invariant measures which are absolutely continuous with respect to m, then there exist two singular T-invariant measures which are absolutely continuous with respect to m.*

Proof. Let μ_1 and μ_2 be two different T-invariant measures with $\mu_1\ll m$ and $\mu_2\ll m$. First, assume that both are ergodic. Since $\mathcal{B}$ is countably generated, Birkhoff's ergodic theorem yields a μ_1-nullset N_1 and a μ_2-nullset N_2 such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}1_B(T^ix)=\begin{cases}\mu_1(B)&\text{for all } x\notin N_1,\\ \mu_2(B)&\text{for all } x\notin N_2\end{cases}$$

for all $B\in\mathcal{B}$. Since $\mu_1(B)\ne\mu_2(B)$ for some $B\in\mathcal{B}$ we must have $N_1^c\cap N_2^c=\emptyset$. Hence $\mu_1(N_2)=1$, i.e. $\mu_1\perp\mu_2$.

Now assume μ_1 is not ergodic. Then there is a set $A\in\mathcal{B}$ with $T^{-1}A=A$ and $\mu_1(A)\notin\{0,1\}$. Put

$$\mu_A(\cdot):=\frac{1}{\mu_1(A)}\mu_1(A\cap\cdot).$$

Clearly μ_A is T-invariant and $\mu_A\ll m$. If $\mu_2(A)=0$ we have $\mu_A\perp\mu_2$, and if $\mu_2(A)=1$ we have $\mu_{A^c}\perp\mu_2$. If $\mu_2(A)\notin\{0,1\}$ we put

$$\widetilde{\mu}_A(\cdot):=\frac{1}{\mu_2(A)}\mu_2(A\cap\cdot)$$

and find $\mu_{A^c}\perp\widetilde{\mu}_A$. ■

Here are some easy consequences which we used in the main body of our paper.

A.2. COROLLARY. *Let T be a measurable transformation of a Lebesgue space $(X,\mathcal{B},m)$. If there exists an ergodic T-invariant measure μ which is equivalent to m, then μ is the unique T-invariant measure which is absolutely continuous with respect to m. Moreover,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}1_B(T^ix)=\mu(B)\quad\textit{for } m\textit{-a.a. } x$$

for all $B\in\mathcal{B}$.

A.3. COROLLARY. *Let T be a measurable transformation of a Lebesgue space $(X,\mathcal{B},m)$. If all T-invariant measures which are absolutely continuous with respect to m are equivalent to m, then there exists at most one such measure. This measure is necessarily ergodic.*

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On approach regions for the conjugate Poisson integral and singular integrals

by

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Abstract. Let $\tilde{u}$ denote the conjugate Poisson integral of a function $f \in L^p(\mathbb{R})$. We give conditions on a region Ω so that

$$\lim_{\substack{(v,\varepsilon)\to(0,0)\\(v,\varepsilon)\in\Omega}} \tilde{u}(x+v,\varepsilon) = Hf(x),$$

the Hilbert transform of f at x, for a.e. x. We also consider more general Calderón–Zygmund singular integrals and give conditions on a set Ω so that

$$\sup_{(v,r)\in\Omega}\Big|\int_{|t|>r} k(x+v-t)f(t)\,dt\Big|$$

is a bounded operator on L^p, $1 < p < \infty$, and is weak $(1,1)$.

Let $f \in L^p(\mathbb{R}^d)$ and let $u(x,y)$ denote the Poisson integral of f. Then a classical theorem of Fatou [3] asserts that u has non-tangential limits a.e. on $\mathbb{R}^d$. In 1984, Nagel and Stein [5] considered more general convergence than the classical non-tangential convergence and gave necessary and sufficient conditions for an approach region Ω so that convergence occurs if $u(x,y)$ approaches the boundary through the region Ω.

In this paper we consider the associated problem for the conjugate Poisson integral of a function f, as well as for more general Calderón–Zygmund singular integrals.

Let $k(x)$ be a Calderón–Zygmund kernel on $\mathbb{R}^d$, that is, $k(x) = w(x)/|x|^d$, where:

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