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Closed ideals in certain Beurling algebras, and synthesis of hyperdistributions

by

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Abstract. We consider the ideal structure of two topological Beurling algebras which arise naturally in the study of closed ideals of A^+. Even in the case of closed ideals I such that $h(I) = E_{1/p}$, the perfect symmetric set of constant ratio $1/p$, some questions remain open, despite the fact that closed ideals J of A^+ such that $h(J) = E_{1/p}$ can be completely described in terms of inner functions. The ideal theory of the topological Beurling algebras considered in this paper is related to questions of synthesis for hyperdistributions such that $\limsup_{n\to-\infty} |\widehat{\varphi}(n)| < \infty$ and such that $\limsup_{n\to\infty} (\log^+ |\widehat{\varphi}(n)|)/\sqrt{n} < \infty$.

1. Introduction. Let $\mathcal{C}(\Gamma)$ be the algebra of all continuous, complex-valued functions on the unit circle Γ, and let

$$A(\Gamma) = \Big\{ f \in \mathcal{C}(\Gamma) \Bigm| \|f\|_1 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty \Big\}$$

be the usual Wiener algebra. By identifying continuous functions on the closed unit disc $\overline{D}$ which are analytic on D with their restrictions to Γ, we can interpret A^+, the algebra of absolutely convergent Taylor series, to be the algebra

$$\{f \in A(\Gamma) \mid \widehat{f}(n) = 0 \ (n < 0)\},$$

a closed subalgebra of $A(\Gamma)$.

There was some recent progress [8], [11], [12] in the theory of closed ideals of A^+. If I is a closed ideal of A^+, set $h(I) = \{z \in \overline{D} \mid f(z) = 0 \ (f \in I)\}$ and denote by $I^{A(\Gamma)}$ the set of elements of A^+ which belong to the closed ideal generated by I in $A(\Gamma)$.

Also, when $I \neq \{0\}$, denote by S_I the inner factor of I (i.e. the G.C.D. of the inner factors of all nonzero elements of I, see [15, p. 85]) and set $S_{\{0\}} = 1$. Bennett and Gilbert had conjectured in [3] (see also [17]) that all closed ideals I of A^+ satisfy

$$(1) \qquad I = I^{A(\Gamma)} \cap S_I \cdot H^\infty(D),$$

where $H^\infty(D)$ is the algebra of bounded analytic functions on D.

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Kahane and Bennett–Gilbert [3], [17] showed that (1) holds when $h(I)$ is finite or countable (then $I^{A(\Gamma)} = I^+(h(I)\cap\Gamma) = \{f \in A^+ \mid f_{|h(I)\cap\Gamma} = 0\}$).

The author [8] produced recently a counterexample to the conjecture, which is a closed ideal I of A^+ such that $S_I = 1$ and such that $h(I) \subset \Gamma$ is a Kronecker set (for the definition of Kronecker sets see for example [16, p. 89]).

In the other direction, E. Strouse, F. Zouakia and the author [12] showed that there exists a large class $\mathcal{C}$ of closed perfect subsets of Γ such that (1) holds whenever $h(I)\cap\Gamma$ is contained in some element of $\mathcal{C}$. The class $\mathcal{C}$ contains in particular $E_{1/p}$ for every integer $p \geq 3$, where

$$E_\zeta = \Big\{ \exp\Big(2i\pi\sum_{n=1}^{\infty}\varepsilon_n\zeta^{n-1}(1-\zeta)\Big) : \varepsilon_n = 0 \text{ or } 1\Big\}$$

for $\zeta \in (0,1/2)$.

During these recent works on the Bennett–Gilbert conjecture, some link between closed ideals of A^+ and closed ideals in some Beurling algebras on the unit circle was pointed out. Define a weight on $\mathbb{Z}$ to be a submultiplicative map $\omega : \mathbb{Z} \to [1,\infty[$ such that $\omega(0) = 1$. If ω is a weight on $\mathbb{Z}$, set

$$A_\omega(\Gamma) = \Big\{ f \in \mathcal{C}(\Gamma) \,\Big|\, \|f\|_\omega = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\omega(n) < \infty \Big\}.$$

Then $(A_\omega(\Gamma), \|\cdot\|_\omega)$ is a Banach algebra. If $\sum_{n\in\mathbb{Z}}(\log\omega(n))/(1+n^2) < \infty$, then $A_\omega(\Gamma)$ is a regular Banach algebra in the classical sense [19, p. 221]. For $z\in\Gamma$, denote by $\chi_z : \mathcal{C}(\Gamma) \to \mathbb{C}$ the map $f \to f(z)$. If the above condition is satisfied, then the map $z \to \chi_z$ is well known to be a homeomorphism from Γ into $\widehat{A_\omega(\Gamma)}$, the character space of $A_\omega(\Gamma)$. Weights ω such that $(\log\omega(n))/\sqrt{|n|} \to 0$ as $n \to -\infty$, and $\omega(n) = 1$ for $n \geq 0$, will be called Atzmon weights. Denote by $\mathcal{A}$ the set of all Atzmon weights on $\mathbb{Z}$, and set $B_0 = \bigcap_{\omega\in\mathcal{A}} A_\omega(\Gamma)$. Then $(B_0, (\|\cdot\|_\omega)_{\omega\in\mathcal{A}})$ is a locally multiplicatively convex complete algebra (see [29]).

Now for $p \geq 1$ set $\omega_p(n) = 1$ $(n \geq 0)$, $\omega_p(n) = e^{p\sqrt{|n|}}$ $(n < 0)$ and set $B_1 = \bigcap_{p\geq1} A_{\omega_p}(\Gamma)$. Then $(B_1, (\|\cdot\|_{\omega_p})_{p\geq1})$ is a Fréchet algebra.

Let μ be a positive measure on Γ which is singular with respect to the Lebesgue measure on Γ, and let

$$S_\mu : z \to \exp\Big[-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\Big]$$

be the singular inner function defined by μ. We will say that S_μ is nonatomic when the measure μ is nonatomic (i.e. $\mu(\{z_0\}) = 0$ for every $z_0 \in \Gamma$). The link between closed ideals of A^+ and closed ideals of B_0 and B_1 is given by the following theorem (the first assertion is implicitly contained in [12], and the second is a slight reformulation of a result of [29]).

THEOREM 1.1. *Let I be a closed ideal of A^+, and for $i = 0$ or 1 let $\overline{I}^{B_i}$ be the closure of I in B_i.*

(1) *If $h(I) \subset \Gamma$, then $I = \overline{I}^{B_1} \cap A^+$.*

(2) *If $h(I) \subset \Gamma$, and if the inner factor S_I of I is nonatomic, then $I = \overline{I}^{B_0} \cap A^+$.*

We are interested here in the study of closed ideals J of B_i such that $h(J) = \{z \in \Gamma \mid f(z) = 0\ (f \in J)\}$ is a set of Lebesgue measure zero. This theory is not contained in the theory of closed ideals of A^+. It follows from a result of Zarrabi [27] that for every uncountable, proper closed subset E of Γ there exists a closed ideal J of B_0 such that $h(J) = E$ and $J\cap A^+ = \{0\}$; we show by a similar method in Section 3 that for every nonempty, proper closed subset E of Γ there exists a closed ideal J of B_1 such that $h(J) = E$ and $J \cap A^+ = \{0\}$. Nevertheless the methods used in [8] and [12] to prove in some special cases the Bennett–Gilbert conjecture lead to some information on closed ideals J of B_i when $h(J)$ satisfies suitable geometric conditions. When $E \subset \Gamma$ is closed, denote by $J^+(E)$ the set of all $f \in A^+$ which satisfy synthesis with respect to E in the classical sense [16, p. 59]. Also let $A^\infty(D)$ be the algebra of C^∞-functions on Γ which have a continuous extension to $\overline{D}$ analytic on D, and set $J^+_\infty(E) = \{f \in A^\infty(D) \mid f^{(n)}|_E = 0\ (n \geq 0)\}$. We show in Section 2 that if J is a closed ideal of B_i, and if $J^+(h(J))$ is w^*-dense in A^+ with respect to the natural w^*-topology on A^+, then $J = J^A \cap J^0$, where J^A is the set of elements of B_i which belong to the closure of J in $(A(\Gamma), \|\cdot\|_1)$ and where J^0 is the intersection of the kernels of the elements ϕ of $J^\perp$ such that $\widehat{\phi}(n) \to 0$ as $n \to -\infty$ (see Section 2 for the interpretation of elements of B_i^* in terms of hyperdistributions).

Also we show in Section 4 that if $J^+_\infty(h(J))$ is w^*-dense in A^+ then $J = J^A \cap J^2$, where J^2 is the intersection of the kernels of the elements φ of $J^\perp$ such that $\sum_{n\leq0}|\widehat{\varphi}(n)|^2 < \infty$. In this situation, we always have $J^0 = J^2$, the notations being as above.

This result has an interpretation in terms of synthesis of hyperdistributions (see Remark 4.15). Elements of B_1^* (resp. B_0^*) can be interpreted as hyperdistributions on Γ such that

$$\sup_{n\leq0}|\widehat{\varphi}(n)| < \infty \quad\text{and}$$

$$\limsup_{n\to\infty}\frac{\log^+|\widehat{\varphi}(n)|}{\sqrt{n}} < \infty \quad \Big(\text{resp. } \frac{\log^+|\widehat{\varphi}(n)|}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0\Big).$$

These hyperdistributions can be considered as analytic functions on $\mathbb{C}_\infty \setminus \Gamma$ vanishing at infinity, where we denote by $\mathbb{C}_\infty$ the Riemann sphere. Denote by $\mathcal{J}_i(\varphi)$ the w^*-closure, with respect to the w^*-topology associated with B_i, of the linear span of translates of φ. The results of Section 4 show in particular that if $\varphi \in B_i^*$ satisfies $\widehat{\varphi}(n) \to 0$ as $n \to -\infty$, and if $J_\infty^+(\operatorname{supp}\varphi)$ is w^*-dense in A^+ then φ belongs to the w^*-closure of the set

$$\Big\{\psi \in \mathcal{J}_i(\varphi) \,\Big|\, \sum_{n\le 0} |\widehat{\psi}(n)|^2 < \infty\Big\}.$$

In other words, the above geometric condition on $\operatorname{supp}\varphi$ allows one to "synthetize c_0 by ℓ^2". These results hold in particular if φ is supported by the perfect symmetric set $E_{1/p}$ of constant ratio $1/p$ ($p \in \mathbb{N}$, $p \ge 3$).

Such results are never true if we consider a Carleson set of multiplicity instead of $E_{1/p}$ (for example $E_\zeta = \{\exp(2i\pi \sum_{n=1}^\infty \varepsilon_n \zeta^n (1-\zeta)) : \varepsilon_n = 0 \text{ or } 1\}$, $\zeta \in (0, 1/2)$, when $1/\zeta$ is not a Pisot number). There are also Kronecker sets for which these results do not hold (see Remark 4.15).

We discuss in Section 5 the link between elements ψ of B_i^* such that $\sum_{n\le 0} |\widehat{\psi}(n)|^2 < \infty$ and inner functions. The condition $\sum_{n\le 0} |\widehat{\psi}(n)|^2 < \infty$ is equivalent to the fact that the function $\psi^- : z \to (1/z)\overline{\psi(1/\overline{z})}$ belongs to $H^2(D)$. If $\operatorname{supp}\psi$ is a Carleson set, and if $\psi^+ = \psi|_D$ belongs to the Nevanlinna class, then there exists a singular inner function S such that $\mathcal{T}_i(\varphi) = \mathcal{T}_i(S^*)$, where S^* is the element of B_i^* defined by S (see Section 3). But the fact that ψ^+ belongs to the Nevanlinna class, which is equivalent to the fact that ψ^- is a noncyclic vector for the adjoint of the shift operator on $H^2(D)$ by the Douglas–Shapiro–Shields theorem [6], is also equivalent to the fact that the restriction of ψ to A^+ is orthogonal to some nonzero ideal of A^+ if we assume that $\operatorname{supp}\psi$ is a Carleson set (Theorem 5.4). So the condition that ψ^+ belong to the Nevanlinna class seems to be a rather strong condition, and we show in Theorem 5.6 that if $E \subset \Gamma$ is any closed, infinite (resp. uncountable) set then there exists $\psi \in B_1^*$ (resp. B_0^*) such that $\operatorname{supp}\psi \subset E$, $\psi^- \in H^\infty(D)$ and such that ψ^+ does not belong to the Nevanlinna class (when $\psi \in B_1^*$ and when $\operatorname{supp}\psi$ is a finite set, ψ^+ belongs to the Nevanlinna class by a result of Atzmon [2]).

This leads to a question that seems interesting: given $\psi \in B_i^*$ such that $\psi^- \in H^2(D)$ and such that $\operatorname{supp}\psi$ is, say, a Carleson set, can we "synthetize ψ by inner functions", i.e. is it true that ψ belongs to the w^*-closure of the linear span of the set of elements of $\mathcal{T}_i(\varphi)$ which are defined by inner functions? We have only been able to answer this question for elements of B_1^* which are supported by a closed countable set (the problem is trivial for elements ψ of B_0^* with countable support because in this case $\limsup_{n\to-\infty} |\widehat{\psi}(n)| > 0$ if $\psi \ne 0$).

It would be interesting to consider similar questions for hyperdistributions ψ, supported by a single point, satisfying $\psi^- \in H^2(D)$ in the case where the sequence $(\widehat{\psi}(n))_{n\ge 0}$ satisfies growth conditions weaker than the condition $\limsup_{n\to\infty}(\log^+ |\widehat{\psi}(n)|)/\sqrt{n} < \infty$.

2. General description of closed ideals of B_0 and B_1. A *weight* on $\mathbb{Z}$ is a map $\omega : \mathbb{Z} \to [1,\infty)$ such that $\omega(n+m) \le \omega(n)\omega(m)$ ($n, m \in \mathbb{Z}$). We will say that a weight ω is *regular* if

$$\sum_{n\in\mathbb{Z}} \frac{\log \omega(n)}{1+n^2} < \infty. \tag{2.1}$$

In this case, set $A_\omega(\Gamma) = \{f \in \mathcal{C}(\Gamma) \mid \|f\|_\omega = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\omega(n) < \infty\}$. Then $A_\omega(\Gamma)$ is a regular algebra in the usual sense [19, Chapter 8] and the characters of $A_\omega(\Gamma)$ have the form $\chi_z : f \to f(z)$, where $z \in \Gamma$. For $f \in \mathcal{C}(\Gamma)$, denote by $\operatorname{supp} f$ the closed support of f, and for $M \subset \mathcal{C}(\Gamma)$ set $h(M) = \{z \in \Gamma \mid f(z) = 0 \ (f \in M)\}$. If $E \subset \Gamma$ is closed, set $I_\omega(E) = \{f \in A_\omega(\Gamma) \mid f_{|E} = 0\}$, and denote by $J_\omega(E)$ the closure in $A_\omega(\Gamma)$ of the set $\{f \in A_\omega(\Gamma) \mid \operatorname{supp} f \cap E = \emptyset\}$. Since $A_\omega(\Gamma)$ is regular, we have the following standard properties:

$$h(I_\omega(E)) = h(J_\omega(E)) = E, \tag{2.2}$$

(2.3) if I is a closed ideal of $A_\omega(\Gamma)$, then $J_\omega(h(I)) \subset I \subset I_\omega(h(I))$.

We will say that E satisfies *ω-synthesis*, or that E is a *set of ω-synthesis*, when $I_\omega(E) = J_\omega(E)$. Now consider the constant weight $\omega_0 \equiv 1$. We write $A(\Gamma)$ (resp. $I(E)$, resp. $J(E)$, resp. $\|\cdot\|_1$) instead of $A_{\omega_0}(\Gamma)$ (resp. $I_{\omega_0}(E)$, resp. $J_{\omega_0}(E)$, resp. $\|\cdot\|_{\omega_0}$). A *set of synthesis* is by definition a set of ω_0-synthesis (see for example [16, p. 59]).

An *Atzmon weight* on $\mathbb{Z}$ is a weight which satisfies the conditions

$$\omega(n) = 1 \quad (n \ge 0), \qquad \frac{\log \omega(-n)}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0. \tag{2.4}$$

We denote by $\mathcal{A}$ the set of all Atzmon weights. Also for $p \ge 1$ we denote by ω_p the weight defined by the conditions

$$\omega_p(n) = 1 \quad (n \ge 0), \qquad \omega_p(n) = e^{p\sqrt{|n|}} \quad (n < 0). \tag{2.5}$$

As in the introduction, we set

$$B_0 = \bigcap_{\omega\in\mathcal{A}} A_\omega(\Gamma), \tag{2.6}$$

$$B_1 = \bigcap_{p\ge 1} A_{\omega_p}(\Gamma). \tag{2.7}$$

We equip B_0 (resp. B_1) with the topology defined by the family $(\|\cdot\|_\omega)_{\omega\in\mathcal{A}}$ (resp. $(\|\cdot\|_{\omega_p})_{p\geq 1}$) so that B_0 is a locally multiplicatively convex complete algebra (resp. a Fréchet algebra).

If I is a closed ideal of B_0 (resp. B_1) and if $\omega \in \mathcal{A}$ (resp. $p \geq 1$) let I^ω (resp. I^{ω_p}) be the closure of I in $A_\omega(\Gamma)$ (resp. $A_{\omega_p}(\Gamma)$). It was observed in [29] that I^ω is a closed ideal of $A_\omega(\Gamma)$ ($\omega \in \mathcal{A}$) and that

$$I = \bigcap_{\omega\in\mathcal{A}} I^\omega \quad \text{for every closed ideal } I \text{ of } B_0. \tag{2.8}$$

By using the same easy arguments, we see that I^{ω_p} is a closed ideal of $A_{\omega_p}(\Gamma)$ ($p \geq 1$) and that

$$I = \bigcap_{p\geq 1} I^{\omega_p} \quad \text{for every closed ideal } I \text{ of } B_0. \tag{2.9}$$

Let $E \subset \Gamma$ be a closed set. For $i = 0, 1$ set $I_i(E) = \{f \in B_i \mid f|_E = 0\}$ and denote by $J_i(E)$ the closure in B_i of the set $\{f \in B_i \mid \operatorname{supp} f \cap E = \emptyset\}$.

Now set $\tau(n) = 1$ ($n \geq 0$) and $\tau(n) = e^{|n|^{2/3}}$ ($n < 0$).

For every closed set $E \subset \Gamma$, we have $J_\tau(E) \subset J_1(E) \subset J_0(E)$, and so $h(J_0(E)) = h(J_1(E)) = E$. Since $h(I^\omega) = h(I)$ for every closed ideal I of B_0, and since $J_0(E) \subset J_\omega(E)$, we have $J_0^\omega(E) = J_\omega(E)$ ($\omega \in \mathcal{A}$) and so

$$J_0(E) = \bigcap_{\omega\in\mathcal{A}} J_\omega(E) \quad \text{for every closed set } E \subset \Gamma. \tag{2.10}$$

Similarly $J_1^{\omega_p}(E) = J_{\omega_p}(E)$ ($p \geq 1$) and

$$J_1(E) = \bigcap_{p\geq 1} J_{\omega_p}(E) \quad \text{for every closed set } E \subset \Gamma. \tag{2.11}$$

It follows from the above properties that

$$J_i(h(I)) \subset I \subset I_i(h(I)) \quad \text{for every closed ideal } I \text{ of } B_i \ (i = 0 \text{ or } 1). \tag{2.12}$$

We will say that E is a *set of B_i-synthesis* when $J_i(E) = I_i(E)$. It was shown in [29], using results of [27], that every closed countable subset of Γ is a set of B_0-synthesis. Also, since closed arcs with nonempty interior are sets of synthesis in $A_\omega(\Gamma)$ for every regular weight ω [30], it follows from (2.10) and (2.11) that they are also sets of B_i-synthesis ($i = 0$ or 1). Now set $I^+(E) = \{f \in A^+ \mid f_{|E} = 0\}$. We will say that E is a ZA^+-*set* if $I^+(E) \neq \{0\}$. Clearly, a ZA^+-set has Lebesgue measure zero, but it follows from an old result of Carleson [4] that some closed sets E of Lebesgue measure zero are not ZA^+-sets. As we shall see in the next section, it follows from results and methods of Zarrabi [27] that a nonempty ZA^+-set is never a set of B_1-synthesis, and that an uncountable ZA^+-set is never a set of B_0-synthesis.

We now describe the dual spaces of B_0 and B_1. Recall that a *hyperdistribution* on Γ is an analytic function φ on $\mathbb{C} \setminus \Gamma$ such that $\varphi(z) \to 0$ as $|z| \to \infty$. Denote by $\mathrm{HD}(\Gamma)$ the set of all hyperdistributions on Γ. For $\varphi \in \mathrm{HD}(\Gamma)$ we define the Fourier coefficients $\widehat{\varphi}(n)$ ($n \in \mathbb{Z}$) of φ by the formulae

$$\varphi(z) = \sum_{n=1}^{\infty} \widehat{\varphi}(n) z^{n-1} \qquad (z \in D), \tag{2.13}$$

$$\varphi(z) = -\sum_{n\leq 0} \widehat{\varphi}(n) z^{n-1} \qquad (z \in \mathbb{C} \setminus \overline{D}). \tag{2.14}$$

If ω is a regular weight, set

$$\mathrm{HD}_\omega(\Gamma) = \left\{ \varphi \in \mathrm{HD}(\Gamma) \;\middle|\; \|\varphi\|_\omega^* = \sup_{n\in\mathbb{Z}} \frac{|\widehat{\varphi}(-n)|}{\omega(n)} < \infty \right\}.$$

Similarly set

$$\mathrm{HD}_0(\Gamma) = \left\{ \varphi \in \mathrm{HD}(\Gamma) \;\middle|\; \sup_{n\leq 0} |\widehat{\varphi}(n)| < \infty,\ \frac{\log^+ |\widehat{\varphi}(n)|}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \right\}$$

and

$$\mathrm{HD}_1(\Gamma) = \left\{ \varphi \in \mathrm{HD}(\Gamma) \;\middle|\; \sup_{n\leq 0} |\widehat{\varphi}(n)| < \infty,\ \limsup_{n\to\infty} \frac{\log^+ |\widehat{\varphi}(n)|}{\sqrt{n}} < \infty \right\}.$$

We can identify the dual of $A_\omega(\Gamma)$ with $\mathrm{HD}_\omega(\Gamma)$ by using the formula

$$\langle f, \varphi \rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \widehat{\varphi}(-n), \tag{2.15}$$

and it is well known (see for example [8]) that

$$\langle f, \varphi \rangle = \lim_{r\to 1^-} \frac{1}{2i\pi} \int_\Gamma [\varphi(r\zeta) - \varphi(\zeta/r)] f(\zeta)\, d\zeta. \tag{2.16}$$

For $f \in A_\omega(\Gamma)$ and $\varphi \in \mathrm{HD}_\omega(\Gamma)$, define $f.\varphi$ by the formula

$$\langle g, f.\varphi \rangle = \langle f.g, \varphi \rangle \qquad (g \in A_\omega(\Gamma)). \tag{2.17}$$

Then routine computations show that

$$\widehat{f.\varphi}(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) \widehat{\varphi}(n-m) \qquad (n \in \mathbb{Z}). \tag{2.18}$$

It follows immediately from the definition of the topology of B_0 that we can identify the dual of B_0 with $\bigcup_{\omega\in\mathcal{A}} \mathrm{HD}_\omega(\Gamma)$, by using (2.15). It follows from an elementary technical result of Zarrabi [27] that for every sequence $(\lambda_n)_{n\geq 1}$ such that $(\log^+ |\lambda_n|)/\sqrt{n} \to 0$ there exists a submultiplicative map $\tau : \mathbb{N} \to [1, \infty)$ such that $(\log \tau(n))/\sqrt{n} \to 0$ and $|\lambda_n| = O(\tau(n))$ as $n \to \infty$. Hence we have the following result (which follows directly from the definition of the topology of B_1 for $i = 1$).

PROPOSITION 2.19. *Formula* (2.15) *implements an isomorphism between* $\mathrm{HD}_i(\Gamma)$ *and the dual of* B_i, *and* (2.16)–(2.18) *hold for* $f \in B_i$ *and* $\varphi \in \mathrm{HD}_i(\Gamma)$ $(i = 0$ *or* $1)$.

We now introduce two important subsets of $\mathrm{HD}(\Gamma)$.

DEFINITION 2.20. $\mathrm{HD}^2(\Gamma) = \{\varphi \in \mathrm{HD}(\Gamma) \mid \sum_{n\le 0} |\widehat{\varphi}(n)|^2 < \infty\}$ and $\mathrm{HD}^0(\Gamma) = \{\varphi \in \mathrm{HD}(\Gamma) \mid \widehat{\varphi}(n) \to 0$ as $n \to -\infty\}$. Similarly $\mathrm{HD}^0_\omega(\Gamma) = \mathrm{HD}^0(\Gamma) \cap \mathrm{HD}_\omega(\Gamma)$, etc.

Denote by $\alpha : z \to z$ the identity map on Γ. Clearly, $\alpha^p.\varphi \in \mathrm{HD}^0_\omega(\Gamma)$ $(p \in \mathbb{Z})$ if ω is a regular weight and $\varphi \in \mathrm{HD}^0_\omega(\Gamma)$. Hence if $\omega(n) = 1$ $(n \ge 0)$, then $f.\varphi \in \mathrm{HD}^0_\omega(\Gamma)$ for $\varphi \in \mathrm{HD}^0_\omega(\Gamma)$ and $f \in A_\omega(\Gamma)$. Since $\mathrm{HD}^0_0(\Gamma) = \bigcup_{\omega\in\mathcal{A}} \mathrm{HD}^0_\omega(\Gamma)$ and $\mathrm{HD}^0_1(\Gamma) = \bigcup_{p\ge 1} \mathrm{HD}^0_{\omega_p}(\Gamma)$, we obtain

$$f.\varphi \in \mathrm{HD}^0_i(\Gamma) \quad (f \in B_i,\ \varphi \in \mathrm{HD}^0_i(\Gamma)). \tag{2.21}$$

Now for $\varphi \in \mathrm{HD}(\Gamma)$, denote by $\operatorname{supp}\varphi$ the smallest closed subset E of Γ such that φ has an analytic extension to $\mathbb{C} \setminus E$, and set $\mathrm{HD}(E) = \{\varphi \in \mathrm{HD}(\Gamma) \mid \operatorname{supp}\varphi \subset E\}$, $\mathrm{HD}_\omega(E) = \mathrm{HD}_\omega(\Gamma) \cap \mathrm{HD}(E)$, etc. It follows from (2.16) and Gelfand theory (see for example [5]) that for every regular weight ω and every closed set $E \subset \Gamma$ we have

$$\mathrm{HD}_\omega(E) = [J_\omega(E)]^\perp. \tag{2.22}$$

We now extend this result to the algebras B_i.

PROPOSITION 2.23. *For every closed set* $E \subset \Gamma$, $\mathrm{HD}_i(E) = [J_i(E)]^\perp$ $(i = 0$ *or* $1)$.

Proof. For $i = 0$ let $\varphi \in [J_0(E)]^\perp$, and let $\omega \in \mathcal{A}$ be such that $\varphi \in \mathrm{HD}_\omega(\Gamma)$. Then $\varphi \perp J^\omega_0(E) = J_\omega(E)$, and so $\operatorname{supp}\varphi \subset E$ and $\varphi \in \mathrm{HD}_0(E)$. Conversely, let $\varphi \in \mathrm{HD}_0(E)$ and let $\omega \in \mathcal{A}$ be such that $\varphi \in \mathrm{HD}_\omega(\Gamma)$. Then $\varphi \perp J_\omega(E) \supset J_0(E)$. A similar argument works for $i = 1$.

If I is a closed ideal in the locally multiplicatively convex complete algebra B_0, let $\pi_I : B_0 \to B_0/I$ be the canonical surjection. Then the quotient algebra B_0/I is a locally multiplicatively convex complete algebra with respect to the family $(\|\cdot\|_\omega)_{\omega\in\mathcal{A}}$, where

$$\|\pi_I(f)\|_\omega = \inf_{g\in I} \|f + g\|_\omega \quad (f \in B_0,\ \omega \in \mathcal{A}).$$

Similarly if I is a closed ideal of B_1 then B_1/I is a Fréchet algebra with respect to the family $(\|\cdot\|_{\omega_p})_{p\ge1}$, where $\|\pi_I(f)\|_{\omega_p} = \inf_{g\in I} \|f + g\|_{\omega_p}$ $(f \in B_1,\ p \ge 1)$. According to the previous notations, we will set in both cases $\|\pi_I(f)\|_1 = \inf_{g\in I} \|f + g\|_1$.

The following elementary lemma extends to the algebras B_i a property pointed out in [8, Lemma 2-6] for the algebras $A_\omega(\Gamma)$, when $\omega(n) = 1$ $(n \ge 0)$.

LEMMA 2.24. *Let I be a closed ideal of B_i $(i = 0$ or $1)$ and denote by I^A the set of elements of B_i which belong to the closure of I in $A(\Gamma)$. Then* $I^A = \{f \in B_i \mid \pi_I(f\alpha^p) \to 0$ *as* $p \to \infty\}$.

Proof. We observe as in [8] that if $\omega(n) = 1$ $(n \ge 0)$ and if $f \in A_\omega(\Gamma)$, then $\lim_{p\to\infty} \|f\alpha^p\|_\omega = \|f\|_1$. Now let I be a closed ideal of B_0. If $\pi_I(f\alpha^p) \to 0$ as $p \to \infty$ then in particular $\|\pi_I(f\alpha^p)\|_1 \to 0$ as $p \to \infty$. Since $\|\pi_I(f\alpha^n)\|_1 = \|\pi_I(f)\|_1$ for every $n \in \mathbb{Z}$ we obtain $\|\pi_I(f)\|_1 = 0$, and $f \in I^A$. Conversely, if $f \in I^A$ let $\omega \in \mathcal{A}$ and $g \in I$. We have

$$\begin{aligned}\limsup_{p\to\infty} \|\pi_I(f\alpha^p)\|_\omega &\le \limsup_{p\to\infty} \|f\alpha^p - g\alpha^p\|_\omega \\ &\le \limsup_{p\to\infty} \|f\alpha^p - g\alpha^p\|_1 = \|f - g\|_1.\end{aligned}$$

Hence $\limsup_{p\to\infty} \|\pi_I(f\alpha^p)\|_\omega \le \|\pi_I(f)\|_1 = 0$ and so $\pi_I(f\alpha^p) \to 0$ as $p \to \infty$. A similar argument holds for the algebra B_1.

DEFINITION 2.25. Let I be a closed ideal of B_i $(i = 0$ or $1)$. For $j = 0$ or 2, set $(I^\perp)_j = \{\varphi \in \mathrm{HD}^j_i(\Gamma) \mid I \subset \operatorname{Ker}\varphi\}$ and $I^j = \bigcap_{\varphi\in(I^\perp)_j} \operatorname{Ker}\varphi$.

THEOREM 2.26. *For every closed ideal of* B_i $(i = 0$ *or* $1)$ *we have*

$$I^A \cdot I^0 \subset I \subset I^A \cap I^0.$$

Proof. Let $\varphi \in I^\perp$ and let $\widetilde{\varphi}$ be the linear functional on B_i/I defined by the formula

$$\langle \pi_I(f), \widetilde{\varphi}\rangle = \langle f, \varphi\rangle \quad (f \in B_i). \tag{2.27}$$

Let $g \in I^A$ and $h \in I^0$. We have $\widehat{g.\varphi}(-n) = \langle \alpha^n, g.\varphi\rangle = \langle \pi_I(\alpha^n g), \widetilde{\varphi}\rangle$ and so it follows from Lemma 2.24 that $g.\varphi \in (I^\perp)^0$. Hence $\langle g.h, \varphi\rangle = \langle h, g.\varphi\rangle = 0$, and $g.h \in I$. The other inclusion is obvious.

As a Banach space, A^+ is isomorphic to ℓ^1. So we can identify A^+ with the dual of $c_0(\mathbb{Z}^-)$ by using the formula

$$\langle u, f\rangle = \sum_{n=0}^{\infty} \widehat{f}(n) u_{-n} \quad \text{for } u = (u_m)_{m\le 0} \in c_0(\mathbb{Z}^-).$$

This duality induces a w^*-topology on A^+, and every $\varphi \in \mathrm{HD}^0_1(\Gamma)$ is w^*-continuous when restricted to A^+.

COROLLARY 2.27. *Let $E \subset \Gamma$ be a closed set. Let $J^+(E) = J(E) \cap A^+$. If $J^+(E)$ is w^*-dense in A^+, then $I = I^A \cap I^0$ for every closed ideal I of B_i such that $h(I) \subset E$ $(i = 0$ or $1)$.*

Proof. Let $f \in I^A \cap I^0$. Since $h(I) \subset E$, the closure of I in $A(\Gamma)$ contains $J(E)$, and so $J^+(E) \subset I^A$.

Let $\varphi \in I^\perp$, and let $g \in J^+(E)$. It follows from the theorem that $fg \in I$, and so $f \cdot \varphi \perp J^+(E)$. But since $f \in I^A$ and $\varphi \in I^\perp$, we see as above that

$f \cdot \varphi \in \mathrm{HD}^0(\Gamma)$. Hence $\widehat{f \cdot \varphi}(-n) = 0$ $(n \geq 0)$, since $J^+(E)$ is w^*-dense in A^+, and so $f \cdot \varphi$ vanishes on $\mathbb{C} \setminus \overline{D}$. Since $J_i(E) \subset I$, it follows from Proposition 2.23 that $\operatorname{supp} f \cdot \varphi \subset E$, and so $f \cdot \varphi = 0$. In particular, $\langle f, \varphi \rangle = \langle 1, f \cdot \varphi \rangle = 0$ for every $\varphi \in I^\perp$, and so $f \in I$.

There are many examples of closed sets E such that $J^+(E)$ is w^*-dense in A^+ (of course, such sets must be ZA^+-sets and sets of uniqueness). For example, every "strong AA^+-set" (i.e. every closed set $E \subset \Gamma$ such that $J^+(E) + A^+ = A(\Gamma)$) enjoys this property. So Corollary 2.27 is valid for Dirichlet sets, for Helson sets of synthesis, for the sets

$$E_{1/p} = \left\{ \exp\left(2i\pi \sum_{n=1}^{\infty} \varepsilon_n p^{1-n}(1 - 1/p)\right) : \varepsilon_n = 0 \text{ or } 1 \right\}$$

when $p \in \mathbb{N}$, $p \geq 3$, and for closed countable sets.

It is known that $\mathrm{HD}_0^0(E) = \{0\}$ when E is countable [27] (more precisely, the support of every element of $\mathrm{HD}_0^0(\Gamma)$ is a perfect set, see [9]). Thus, $(J_0^+(E))^0 = B_0$ when E is countable. Since closed countable sets are sets of synthesis, we also have in this case $(J_0^+(E))^A = I(E) \cap B_0 = I_0(E)$, and we obtain, by a slightly different method, a result of Zerouali:

Corollary 2.28 [29]. *Every closed countable subset of Γ is a set of B_0-synthesis.*

3. Singular inner functions, and closed ideals of B_i that A^+ cannot see. We now describe elements of $\mathrm{HD}_i^2(\Gamma) \subset \mathrm{HD}_i^0(\Gamma)$ associated with singular inner functions. Recall that a *singular inner function* is a function

$$S : z \to \exp\left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right) \quad (z \in D),$$

where μ is a positive measure on Γ which is *singular*, i.e. concentrated on a set of Lebesgue measure zero (the notation $d\mu(t)$ has the same meaning as in [24, Chapter 11]).

We can consider S as a function on $\mathbb{C} \setminus \Gamma$, by using the same formula for $|z| > 1$, and we will say that S is *nonatomic* when μ is nonatomic (i.e. $\mu(\{z_0\}) = 0$ for every $z_0 \in \Gamma$). We denote by $\mathcal{S}_1$ (resp. $\mathcal{S}_0$) the set of all singular (resp. nonatomic singular) inner functions.

Notice that S is analytic at infinity. In other words, if we set

$$S(\infty) = \exp\left[\frac{1}{2\pi}\|\mu\|\right] = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\mu(t)\right]$$

then S is analytic on $\mathbb{C}_\infty \setminus \Gamma$, where we denote by $\mathbb{C}_\infty$ the Riemann sphere. Now, using a notation slightly different from the notations in [27], [28], set

$$(3.1) \qquad S^*(z) = -\frac{1}{S(z)} + \frac{1}{S(\infty)} \quad (z \in \mathbb{C} \setminus \Gamma).$$

Clearly, $S^* \in \mathrm{HD}(\Gamma)$, and if the singular measure μ which defines S is concentrated on a closed set $E \subset \Gamma$, then $S^* \in \mathrm{HD}(E)$. Set $\psi(z) = \overline{S}^*(1/\overline{z})$ for $|z| < 1$. Then $\psi(z) = \sum_{n \leq 0} \overline{\widehat{S}^*(n)} z^{1-n}$. Also $\psi(z) = S(0) - S(z)$, by an easy calculation, and so $\psi \in H^\infty(D) \subset H^2(D)$. In particular, $\sum_{n \leq 0} |\widehat{S}^*(n)|^2 < \infty$, and $S^* \in \mathrm{HD}^2(\Gamma)$.

We have $\limsup_{|z| \to 1^-} (1 - |z|) \log^+ |S^*(z)| < \infty$ by an immediate calculation on the Poisson kernel. More precise standard estimates (see for example [1, Lemma 5]) show that $(1 - |z|) \log^+ |S^*(z)| \to 0$ as $|z| \to 1^-$ if S is nonatomic.

Conversely, assume that $\mu(\{z_0\}) > 0$ for some $z_0 \in \Gamma$, and set $s = 2\pi\mu(\{z_0\})$. Denote by δ_{z_0} the Dirac measure at z_0. Let S_1 be the singular inner function defined by the positive, singular measure $\mu - s\delta_{z_0}$. Then $S(z) = S_1(z) e^{s(z_0 + z)/(z_0 - z)}$ so that

$$\left| S^*(z) - \frac{1}{S(\infty)} \right| \geq |e^{s(z + z_0)/(z - z_0)}| \quad (z \in D).$$

It follows immediately that $\limsup_{|z| \to 1^-} (1 - |z|) \log^+ |S^*(z)| > 0$.

Hence $(1 - |z|) \log^+ |S^*(z)| \to 0$ as $|z| \to 1^-$ if and only if S is nonatomic. By using Cauchy's inequalities in a standard way (see for example [1]) we obtain the following result, implicitly contained in [1], [27].

Proposition 3.1. *For $i = 0, 1$ set $\mathcal{S}_i^* = \{S^*\}_{S \in \mathcal{S}_i}$. Then $\mathcal{S}_1^* \subset \mathrm{HD}_1^2(\Gamma)$ and $\mathcal{S}_1^* \cap \mathrm{HD}_0(\Gamma) = \mathcal{S}_0^*$.*

Let $S \in \mathcal{S}_i$, and let S_* be the radial limit of S, defined almost everywhere by the formula

$$(3.3) \qquad S_*(e^{it}) = \lim_{r \to 1^-} S(re^{it}).$$

We have $\overline{S}(1/\overline{z}) = 1/S(z)$ for $z \neq 0$, $|z| \neq 1$ and so, since $\zeta\overline{\zeta} = 1$ for $\zeta \in \Gamma$, we obtain by (2.16), for $f \in B_i$,

$$\langle f, S^* \rangle = \lim_{r \to 1^-} \frac{1}{2i\pi} \int_{\Gamma} f(\zeta) \left(\overline{S(r\zeta)} - \frac{1}{S(r\zeta)} \right) d\zeta.$$

It follows from the dominated convergence theorem that

$$\int_{\Gamma} f(\zeta) \overline{S(r\zeta)}\, d\zeta \underset{r \to 1^-}{\longrightarrow} \int_{\Gamma} f(\zeta) \overline{S_*(\zeta)}\, d\zeta.$$

We obtain, for $S \in \mathcal{S}_i$ and $f \in B_i$, $i = 0$ or 1,

$$(3.4)\qquad \langle f, S^*\rangle = \frac{1}{2i\pi}\int_\Gamma f(\zeta)\overline{S_*(\zeta)}\,d\zeta - \lim_{r\to 1^-}\frac{1}{2i\pi}\int_\Gamma \frac{f(\zeta)}{S(r\zeta)}\,d\zeta.$$

Now if $f \in A^+$, then the last integral is zero for $r < 1$, by Cauchy's theorem, and we have

$$(3.5)\qquad \langle f, S^*\rangle = \frac{1}{2i\pi}\int_\Gamma f(\zeta)\overline{S_*(\zeta)}\,d\zeta \quad (f \in A^+,\ S \in \mathcal{S}_i).$$

We obtain the following result (established in [27] in a slightly less general situation):

PROPOSITION 3.6. *Let $S \in \mathcal{S}_1$, and let $f \in A^+$. Then $f.S^* = 0$ if and only if $f \in S.H^\infty(D)$.*

Proof. A standard characterization of the boundary values of elements of $H^\infty(D)$ [24, p. 235] shows that $f \in S.H^\infty(D)$ if and only if $\int_{-\pi}^{\pi} f(e^{it})\overline{S_*(e^{it})}e^{i(n+1)t}\,dt = 0$ for every $n \ge 0$, which is equivalent to the condition $\widehat{f.S^*}(n) = 0$ $(n \le 0)$. So if $f.S^* = 0$ then $f \in S.H^\infty(D)$.

Conversely, assume that $f \in S.H^\infty(D)$. If $f = 0$, then $f.S^* = 0$. If $f \ne 0$, let μ be the positive singular measure on Γ which defines S. Since f/S is bounded on D, it follows from [15, p. 69] that if $f(e^{it_0}) \ne 0$ then $\mu(]t_0 - s, t_0 + s[) = 0$ for some $s \ne 0$. Hence μ is concentrated on $F = \{z \in \Gamma \mid f(z) = 0\}$. Since $\widehat{f.S^*}(n) = 0$ for $n \le 0$, $f.S^*$ vanishes on $\mathbb{C}\setminus\overline{D}$. But $\operatorname{supp} f.S^* \subset \operatorname{supp} S^* \subset F$, and so $f.S^* = 0$, which concludes the proof of the proposition.

Notice that it follows from the above argument that if $f \in A^+$ and $S \in \mathcal{S}_1$, then $f.S^* = 0$ if and only if $\langle \alpha^m.S, f\rangle = 0$ $(m \ge 0)$.

COROLLARY 3.7. *Let $S \in \mathcal{S}_i$ $(i = 0$ or $1)$ and set $K^n(S) = \{f \in \mathcal{S}_i \mid f.S^{*n} = 0\}$ and $K^\infty(S) = \bigcap_{n\ge1} K^n(S)$. Then $K^\infty(S)\cap A^+ = 0$ for every $S \in \mathcal{S}_i$.*

Proof. We have $K^\infty(S)\cap A^+ = (\bigcap_{n\ge1} S^n.H^\infty(D))\cap A^+$ and it is well known that $\bigcap_{n\ge1} S^n.H^\infty(D) = \{0\}$ ([27, Prop. 2.2], but there should be much earlier references).

The following result is essentially contained in [27] in the case where $i = 0$.

COROLLARY 3.8. (i) *For every closed nonempty set $E \subset \Gamma$, $J_1(E)\cap A^+ = \{0\}$.*

(ii) *For every closed uncountable set $E \subset \Gamma$, $J_0(E)\cap A^+ = \{0\}$.*

Proof. Let $z_0 \in E$, and set $S(z) = e^{(z_0+z)/(z_0-z)}$. Then $(S^*)^n \in \mathrm{HD}_1(E)$ $(n \ge 1)$ and so $J_1(E) \subset K^\infty(S)$, by Proposition 2.23. If E is uncountable, then E contains a perfect closed set F, and it is well known that there exists a singular nonatomic positive measure μ concentrated on F. Let S be the singular inner function defined by μ. Then $S^n \in \mathcal{S}_0$, and so $(S^*)^n \in \mathrm{HD}_0(E)$ $(n \ge 1)$. It follows then again from Proposition 2.23 that $J_0(E) \subset K^\infty(S)$.

Zarrabi's paper [27] actually contains information more precise than Corollary 3.8(ii): if S is the singular inner function associated with μ as above, Zarrabi shows that there exists an Atzmon weight ω such that $(S^*)^n \in \mathrm{HD}_\omega(\Gamma)$ $(n \ge 1)$, and so $J_\omega(E)\cap A^+ = \{0\}$.

Of course, as pointed out in [29], it follows from Corollary 3.8(ii) that if E is a ZA^+-set (which means that $I^+(E) \ne \{0\}$) and if E is uncountable, then E is not a set of B_0-synthesis (see [27]). Similarly it follows from Corollary 3.8(i) that no ZA^+-set is a set of B_1-synthesis. In the other direction, closed arcs not reduced to a single point are sets of ω-synthesis for every regular weight ω [30] and so, by (2.10) and (2.11), they are sets of B_i-synthesis $(i = 0$ or $1)$.

4. $\mathrm{HD}_i^0(\Gamma)$ and $\mathrm{HD}_i^2(\Gamma)$. We showed in Section 2 that if $J^+(E)$ is w^*-dense in A^+, then $I = I^A \cap I^0$ for every closed ideal I of B_i such that $h(I) \subset E$. We now wish to prove that, for a significant class of closed sets E, we have in fact $I = I^A \cap I^2$.

We state as a lemma standard estimates due to Domar and Taylor–Williams, which play a basic role in the theory of ideals of algebras of analytic functions on D.

LEMMA 4.1. *Let $E \subset \Gamma$ be a closed set, and let $\varphi \in \mathrm{HD}_1(E)$. Then there exists $M > 0$ such that*

(1) $\log^+|\varphi(z)| \le M\operatorname{dist}(z,E)^{-1}$ $(z \in D)$,

(2) $|\varphi(z)| \le M\operatorname{dist}(z,E)^{-2}$ $(1 \le |z| \le 2,\ z \notin E)$.

Proof. Since $\log^+|\widehat{\varphi}(n)| = O(\sqrt{n})$ as $n \to \infty$, a standard computation shows that $\log^+|\varphi(z)| = O(1/(1-|z|))$ as $|z| \to 1^-$. Clearly, we have $\sup_{|z|>1}(|z|-1)|\varphi(z)| < \infty$ and the result follows then directly from [25, Lemmas 5.8 and 5.9].

DEFINITION 4.2. A *Carleson set* is a closed set $E \subset \Gamma$ such that

$$\int_{-\pi}^{\pi}\log\frac{2}{\operatorname{dist}(e^{it},E)}\,dt < \infty.$$

Recall that the usual *Nevanlinna class* $\mathcal{N}$ consists of those functions f holomorphic on D which satisfy the condition

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ |f(re^{it})|\, dt < \infty.$$

If $f \in \mathcal{N}$, the radial limit f_* is defined almost everywhere by the formula

$$f_*(e^{it}) = \lim_{r\to 1^-} f(re^{it}). \tag{4.3}$$

Also, if $f \neq 0$, then $\log|f_*| \in L^1(\Gamma)$, and there exists a unique $c \in \Gamma$, a unique Blaschke product B and a unique real singular measure μ_f such that the following decomposition holds:

$$f(z) = cB(z)\exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_f(t)\right] \times \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f_*(e^{it})|\, dt\right] \quad (z \in D). \tag{4.4}$$

Recall that if μ is a real measure on Γ, the Jordan decomposition of μ gives a unique pair (μ^+, μ^-) of mutually singular positive measures on Γ such that $\mu = \mu^+ - \mu^-$.

DEFINITION 4.5. Let $f \in \mathcal{N}$ with $f \neq 0$. The *denominator* of f, denoted by $S(f)$, is the singular inner function associated with μ_f^-.

It follows from the definition of $S(f)$ that for $f \in \mathcal{N}$ with $f \neq 0$ we have

$$\log|f(z)| + \log|S(f)(z)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log|f_*(e^{it})|\, dt \tag{4.6}$$

for $z = re^{i\theta} \in D$, $r \in [0,1)$, $\theta \in \mathbb{R}$ (we denote by $P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}$ the usual Poisson kernel).

Let $A^\infty(D) = \{f \in H^\infty(D) \mid f^{(n)} \in H^\infty(D)\ (n \ge 1)\}$. A standard application of Cauchy's inequalities shows that elements of $A^\infty(D)$ are characterized in $\mathcal{H}(D)$ by the condition $n^p|f^{(n)}(0)/n!| \to 0$ as $n \to \infty$ for every $p \ge 1$; in particular, $A^\infty(D) = \{f \in A^+ \mid n^p|\widehat{f}(n)| \to 0 \text{ as } n \to \infty\ (p \ge 1)\}$. Also, by identifying elements of $A^\infty(D)$ with their restrictions to Γ, we can interpret $A^\infty(D)$ to be the algebra $\{f \in \mathcal{C}^\infty(\Gamma) \mid \widehat{f}(n) = 0\ (n < 0)\}$.

If f is analytic on D, continuous on $\overline{D}$, and satisfies a Lipschitz condition, then $Z(f) \cap \Gamma$ is a Carleson set, where $Z(f) = \{z \in \overline{D} \mid f(z) = 0\}$ [4]. Conversely, an improvement of a construction of [4] shows that if E is a Carleson set there exists an outer function $f \in A^\infty(D)$ such that $E = Z(f)$, and $E \subset Z(f^{(n)})$ for every $n \ge 1$ (see [25]).

Now for $\varphi \in \mathrm{HD}(\Gamma)$, we define φ^+ and φ^- on D by the formulae

$$\varphi^+ = \varphi|_D, \tag{4.7}$$

$$\varphi^-(z) = \frac{1}{z}\overline{\varphi\left(\frac{1}{\overline{z}}\right)} \quad (0 < |z| < 1). \tag{4.8}$$

Of course, since $\varphi(z) \to 0$ as $|z| \to \infty$, φ^- has a removable singularity at the origin, and we can consider φ^- as an analytic function on D.

LEMMA 4.9. *Let $E \subset \Gamma$ be a Carleson set, and let $f \in A^\infty(D)$ be such that $E \subset Z(f) \cap Z(f')$. Then $f.\varphi^- \in H^\infty(D)$ for every $\varphi \in \mathrm{HD}_1(E)$. In particular, $\varphi^- \in \mathcal{N}$ and $S(\varphi^-) = 1$ for every nonzero $\varphi \in \mathrm{HD}_1(E)$.*

Proof. A routine computation shows that for $r \in [0,1)$ and $t \in \mathbb{R}$ we have

$$\frac{|1-e^{it}|^2}{|r-e^{it}|^2} \le \frac{4}{(1+r)^2}.$$

Hence for $z_0, z_1 \in \Gamma$ and $r \in [0,1)$ we have $|z_0 - z_1| \le 2|rz_0 - z_1|$ and $\mathrm{dist}(re^{it}, E) \ge \frac{1}{2}\mathrm{dist}(e^{it}, E)$. It follows then from Lemma 4.1 that there exists $M > 0$ such that $|\varphi^-(z)| \le M\,\mathrm{dist}(z,E)^{-2}$ $(z \in D)$.

Now if $f \in A^\infty(D)$ satisfies the conditions of the lemma, it follows easily from Taylor's formula that $|f(z)| \le \|f^{(2)}\|_\infty \mathrm{dist}(z,E)^2$. Hence $f.\varphi^- \in H^\infty(D)$. Since there exists an outer function f which satisfies the above condition, $\varphi^- = (f.\varphi^-)/f \in \mathcal{N}$, and $S(\varphi^-) = 1$.

Some versions of Lemma 4.9 exist in classical papers about ideals of algebras of analytic functions in the disk. The following lemma, which we prove in a very simple way, is also related to classical tools [12], [21], [22], [25].

LEMMA 4.10. *Let $f \in A^+$ and let $\varphi \in \mathrm{HD}(\Gamma)$. If $\sum_{n=1}^{\infty} n|\widehat{f}(n)| < \infty$, and if $\limsup_{n\to-\infty} |\widehat{\varphi}(n)| < \infty$, then $(f.\varphi)^+ - f.\varphi^+ \in A^+$. If, further, $f \in A^\infty(D)$, then $(f.\varphi)^+ - f.\varphi^+ \in A^\infty(D)$.*

Proof. Let a_n be the nth Taylor coefficient of $(f.\varphi)^+ - f.\varphi^+$ at the origin. For $n \ge 0$ we have

$$\begin{aligned} a_n &= \widehat{f.\varphi}(n+1) - \sum_{p=0}^{n} \widehat{f}(p)\widehat{\varphi}(n-p+1) \\ &= \sum_{p=0}^{\infty} \widehat{f}(p)\widehat{\varphi}(n-p+1) - \sum_{p=0}^{n} \widehat{f}(p)\widehat{\varphi}(n-p+1) \\ &= \sum_{p=n+1}^{\infty} \widehat{f}(p)\widehat{\varphi}(n-p+1). \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}|a_n| \le \sum_{n=0}^{\infty}\Big(\sum_{p=n+1}^{\infty}|\widehat{f}(p)|\cdot|\widehat{\varphi}(n-p+1)|\Big)$$
$$=\sum_{p=1}^{\infty}\Big(\sum_{n=0}^{p-1}|\widehat{f}(p)|\cdot|\widehat{\varphi}(n-p+1)|\Big)$$
$$\le \Big(\sum_{p=1}^{\infty}p|\widehat{f}(p)|\Big)\sup_{n\le 0}|\widehat{\varphi}(n)|<\infty.$$

Assume that $f\in A^\infty(D)$ and let $k\ge 1$. There exists $m\ge 0$ such that $|\widehat{f}(p)|\le m/(p+1)^{k+1}$ $(p\ge 0)$ and there exists $M>0$ such that $\sum_{p=n+1}^{\infty}1/(p+1)^{k+1}\le M/(n+1)^k$. We obtain

$$(n+1)^k|a_n|\le mM\sup_{n\ge 0}|\widehat{\varphi}(n)|,$$

and so $(f.\varphi)^+ - f.\varphi^+\in A^\infty(D)$.

LEMMA 4.11. *Let E be a Carleson set, and let $\varphi\in \mathrm{HD}_1(E)$. If $f\in A^\infty(D)$, and if $E\subset Z(f)\cap Z(f')$, then $(f.\varphi)^-\in H^\infty(D)$. In particular, $f.\varphi\in \mathrm{HD}_1^2(E)$. If, further, $E\subset\bigcap_{n\ge 0}Z(f^{(n)})$, then $(f.\varphi)^-\in A^\infty(D)$.*

Proof. It follows from Lemma 4.10 that $(f.\varphi)^+ - f.\varphi^+\in A^+$. Since $\operatorname{supp}\varphi\cup\operatorname{supp} f\varphi\subset E$, the functions $(f.\varphi)^-$, $f.\varphi^-$, $(f.\varphi)^+$ and $f.\varphi^+$ extend continuously to $\overline{D}\setminus E$. But $f.\varphi^-$ is bounded on D (Lemma 4.9) so $f.\varphi^+$ and $(f.\varphi)^+$ are bounded on $\Gamma\setminus E$. Hence $(f.\varphi)^-$ is bounded on $\Gamma\setminus E$. But $(f.\varphi)^-\in\mathcal{N}$, and $S[(f.\varphi)^-]=1$. It follows then from (4.6) that $(f.\varphi)^-\in H^\infty(D)$.

Now assume that $E\subset\bigcap_{n\ge0}Z(f^{(n)})$. It follows from Lemma 4.10 that $(f.\varphi)^+ - f.\varphi^+\in A^\infty(D)$. But $f(\zeta)\varphi_*^+(\zeta)=f(\zeta)\varphi(\zeta)=O(\operatorname{dist}(\zeta,E)^k)$ on $\Gamma\setminus E$ for every $k\ge 0$ and so $f.\varphi_*^+\in\mathcal{C}^\infty(\Gamma)$.

So $(f.\varphi)_*^+\in\mathcal{C}^\infty(\Gamma)$ and $(f.\varphi)_*^-\in\mathcal{C}^\infty(\Gamma)$. Since $(f.\varphi)^-\in H^\infty(D)$, this shows that $(f.\varphi)^-\in A^\infty(D)$.

DEFINITION 4.12. Let E be a Carleson set. We set $J_\infty^+(E)=\{f\in A^\infty(D)\mid E\subset Z(f^{(n)})$ for all $n\ge 0\}$.

THEOREM 4.13. *Let E be a Carleson set such that $J_\infty^+(E)$ is w^*-dense in A^+, and let I be a closed ideal of B_i ($i=0$ or 1) such that $h(I)\subset E$. Then $I^2=I^0$, and $I=I^A\cap I^2$.*

Proof. Let $f\in I^2$, and let $\varphi\in(I^\perp)_0=I^\perp\cap\mathrm{HD}_i^0(\Gamma)$. Then $\varphi\perp J_1(E)$, and so $\operatorname{supp}\varphi\subset E$. So if $g\in J_\infty^+(E)$, then $g.\varphi\in\mathrm{HD}_i^2(\Gamma)$, and so $g.\varphi\in(I^\perp)_2$. So if we set $H=\{g\in A^+\mid\langle g.f,\varphi\rangle=0\}$ we have $J_\infty^+(E)\subset H$.

Since $\varphi\in\mathrm{HD}_i^0(\Gamma)$, $f.\varphi\in\mathrm{HD}_i^0(\Gamma)$, and so the restriction of $f.\varphi$ to A^+ is w^*-continuous. So $H=\{g\in A^+\mid\langle g,f.\varphi\rangle=0\}$ is w^*-closed in $A^\perp$.

Hence $H=A^+$, and $\langle f,\varphi\rangle=\langle 1,f.\varphi\rangle=0$. So $I^2\subset I^0$, and the other inclusion is obvious. The other assertion follows then from Corollary 2.27.

COROLLARY 4.14. *Let $p\ge 3$ be an integer. Then $I^2=I^0$, and $I=I^A\cap I^2$ for every closed ideal of B_i ($i=0$ or 1) such that $h(I)\subset E_{1/p}$. In particular, $I=I_i(E_{1/p})\cap I^2$ if $h(I)=E_{1/p}$.*

Proof. It was shown in [12] that $J_\infty^+(E_{1/p})$ is indeed w^*-dense in A^+. If $h(I)=E_{1/p}$, then $I^A=I(E_{1/p})\cap B_i=I_i(E_{1/p})$, since $E_{1/p}$ satisfies synthesis, by a well known theorem of Herz [16, p. 60].

Remark 4.15. 1) Theorem 4.13 and Corollary 4.14 hold, with the same proof, for closed ideals in the Beurling algebra $A_\omega(\Gamma)$, where ω is a weight such that $\omega(n)=1$ $(n\ge 0)$ and $\limsup_{n\to\infty}(\log\omega(n))/\sqrt{n}<\infty$.

2) We can interpret Theorem 4.13 in terms of spectral synthesis in $\mathrm{HD}_i(\Gamma)$. We briefly recall the classical notion of synthesis of pseudomeasures. Define $\mathrm{PM}(\Gamma)=\{\varphi\in\mathrm{HD}(\Gamma)\mid\|\varphi\|_{\mathrm{PM}}=\sup_{n\in\mathbb{Z}}|\widehat{\varphi}(n)|<\infty\}$. A *character* is a pseudomeasure φ_z with Fourier transform of the form $\{z^{-n}\}_{n\in\mathbb{Z}}$, where $z\in\Gamma$, so that, according to formula (2.15), we have $\langle f,\varphi_z\rangle=f(z)$ $(f\in A(\Gamma))$. Denote by $\mathcal{C}$ the set of all characters. Also for $M\subset\mathrm{PM}(\Gamma)$ denote by $\mathcal{T}(M)$ the w^*-closure (according to the duality between $A(\Gamma)$ and $\mathrm{PM}(\Gamma)$) of the linear span of the set $\{\alpha^n\varphi\}_{n\in\mathbb{Z},\varphi\in M}$. For $\varphi\in\mathrm{PM}(\Gamma)$ set $\mathcal{T}(\varphi)=\mathcal{T}(\{\varphi\})$, and $I_\varphi=\mathcal{T}(\varphi)^\perp=\{f\in A(\Gamma)\mid\langle f,\psi\rangle=0\ (\psi\in\mathcal{T}(\varphi))\}$. Clearly, $I_\varphi=\{f\in A(\Gamma)\mid\langle f,\alpha^n\varphi\rangle=0\ (n\in\mathbb{Z})\}=\{f\in A(\Gamma)\mid\widehat{f.\varphi}(n)=0\ (n\in\mathbb{Z})\}=\{f\in A(\Gamma)\mid f.\varphi=0\}$. Since $\mathcal{T}(\varphi)$ is w^*-closed, $\mathcal{T}(\varphi)=[I_\varphi]^\perp$.

Let $E=\operatorname{supp}\varphi$. Then $E=\operatorname{supp}(\alpha^n\varphi)$ $(n\in\mathbb{Z})$ and so $J(E)\subset I_\varphi$ and $E=h(J(E))\supset h(I_\varphi)$. Conversely, since $J(h(I_\varphi))\subset I_\varphi$, $\varphi\in J(h(I_\varphi))^\perp=\mathrm{PM}(h(I_\varphi))$, and so $E=h(I_\varphi)$.

Hence $\mathcal{C}\cap\mathcal{T}(\varphi)=\mathcal{C}\cap[I_\varphi]^\perp=\{\varphi_z\}_{z\in\operatorname{supp}\varphi}$.

By definition (see for example [13], p. 69]) the pseudomeasure φ satisfies synthesis if φ belongs to the w^*-closure of the linear span of $\mathcal{C}\cap\mathcal{T}(\varphi)$, which is equivalent to the condition $\langle f,\varphi\rangle=0$ for every $f\in[\mathcal{C}\cap\mathcal{T}(\varphi)]^\perp=I(\operatorname{supp}\varphi)$. In particular, if $\operatorname{supp}\varphi$ is a set of synthesis, then $\varphi\in[J(\operatorname{supp}\varphi)]^\perp=[I(\operatorname{supp}\varphi)]^\perp$, and φ satisfies synthesis. We can extend these notions to $\mathrm{HD}_i(\Gamma)$ ($i=0$ or 1). Since $\mathrm{HD}_i(\Gamma)$ can be identified with the dual of B_i, the linear forms $\varphi\to\langle f,\varphi\rangle$ $(f\in B_i)$ define the w^*-topology on $\mathrm{HD}_i(\Gamma)$. Denote by $\mathcal{T}_i(\varphi)$ the w^*-closure of the linear span of $(\alpha^n\varphi)_{n\in\mathbb{Z}}$ and set $I_{i,\varphi}=\{f\in B_i\mid\langle f,\psi\rangle=0\ (\psi\in\mathcal{T}_i(\varphi))\}$. We see as above that $I_{i,\varphi}=\{f\in B_i\mid f.\varphi=0\}$, that $\mathcal{T}_i(\varphi)=[I_{i,\varphi}]^\perp$ and that $h(I_{i,\varphi})=\operatorname{supp}\varphi$. Notice (this is also true for pseudomeasures) that $\mathcal{T}_i(\varphi)$ is for every $k\ge 0$ the w^*-closure of the linear span of either of the sets $(\alpha^n\varphi)_{n\ge k}$ and $(\alpha^{-n}\varphi)_{n\ge k}$, provided that $\operatorname{supp}\varphi\subsetneq\Gamma$: in this case if $\widehat{f.\varphi}(n)=0$ for $n\ge k$, then $f.\varphi$ agrees with a polynomial on D, hence on $\mathbb{C}\setminus\operatorname{supp}\varphi$, so that $f.\varphi=0$, since

$f.\varphi$ vanishes at infinity, and if $\widehat{f.\varphi}(n) = 0$ for $n \leq -k$, then $f.\varphi$ agrees with a rational function, with only possible pole at the origin, on $\mathbb{C} \setminus \overline{D}$, hence on $\mathbb{C} \setminus \operatorname{supp} \varphi$, which also implies that $f \in I_{i,\varphi}$.

We can then state Theorem 4.13 in the following way: if $\varphi \in \mathrm{HD}_i^0(\Gamma)$, if $\operatorname{supp}\varphi$ is a Carleson set and if $J_\infty^+(\operatorname{supp}\varphi)$ is w^*-dense in A^+, then φ belongs to the w^*-closure of $\mathrm{HD}_i^2(\Gamma) \cap \mathcal{T}(\varphi)$ (this set is clearly a linear space stable by multiplication by α^n, $n \in \mathbb{Z}$). In other words, elements φ of $\mathrm{HD}_i^0(\Gamma)$ such that $J_\infty^+(\operatorname{supp}\varphi)$ is w^*-dense in A^+ can be "synthetized" by elements of $\mathrm{HD}_i^2(\Gamma)$. Similar results hold also for elements of $\mathrm{HD}_\omega^0(\Gamma)$ if ω is a weight such that $\omega(n) = 1$ $(n \geq 0)$ and $\limsup_{n\to\infty}(\log\omega(-n))/\sqrt{n} < \infty$ (in fact if $\varphi \in \mathrm{HD}_\omega^0(\Gamma)$, and if $J_\infty^+(\operatorname{supp}\varphi)$ is w^*-dense in A^+, then the elements of $\mathcal{T}_\omega(\varphi) \cap \mathrm{HD}_\omega^2(\Gamma)$ used to "synthetize" φ belong to the norm-closure of $(\alpha^n\varphi)_{n\geq 0}$ in $\mathrm{HD}_\omega^2(\Gamma)$).

3) Theorem 4.13 and Remark 4.15(2) do not extend to general closed ideals I of B_i (resp. to general elements $\varphi \in \mathrm{HD}_i^0(\Gamma)$) for which $E = h(I)$ (resp. $E = \operatorname{supp}\varphi$) is a Carleson set, even in the case where $J^+(E)$ is w^*-dense in A^+. First notice that if $f \in A^+$ is outer, and if $f.\varphi = 0$ for some $\varphi \in \mathrm{HD}_i^2(\Gamma)$, then $\varphi = 0$. To see this, set $\widetilde{f} = (\widehat{f}(n))_{n\geq 0}$ and $\widetilde{\varphi} = (\widehat{\varphi}(-n))_{n\geq 0}$, so that $\widetilde{f}$ and $\widetilde{\varphi}$ belong to ℓ^2, and denote by T the standard shift operator on ℓ^2. Since $f.\varphi = 0$, we have $\widetilde{\varphi} \perp T^n(\widetilde{f})$ $(n > 0)$. By Beurling's theorem [24, Chapter 17], $\widetilde{f}$ is cyclic for T, since f is outer, and so $\widetilde{\varphi} = 0$. Since $f.\varphi = 0$, we have $\operatorname{supp}\varphi \subset Z(f) \subsetneq \Gamma$, and so $\varphi = 0$.

Now consider a set of multiplicity which is also a Carleson set (for example $E = E_\zeta$ when $1/\zeta$ is not a Pisot number), and set $I = J(E) \cap B_i$. Then $J_\infty^+(E) \subset I$, and so I contains outer functions, $(I^\perp)_2 = 0$ and $I^2 = B_i$. But there exists $\varphi \in \mathrm{HD}(E)$ such that $\widehat{\varphi}(n) \to 0$ as $|n| \to \infty$ and so $(I^\perp)^0 \neq \{0\}$ and $I^0 \subsetneq B_i$. We also see in this case that $\mathcal{T}_i(\varphi) \cap \mathrm{HD}_i^2(\Gamma) = \{0\}$. We can also obtain a similar example where $J^+(E)$ is w^*-dense in A^+. The author's counterexample to the Bennett–Gilbert conjecture gives a nonzero distribution φ supported by a Kronecker set $E \subset E_\zeta$ such that $\widehat{\varphi}(n) \to 0$ as $n \to -\infty$. Then $J_\infty^+(E) \subset I_{i,\varphi}$ and we see as above that $(I_{i,\varphi})^2 = B_i$, $(I_{i,\varphi})^0 \subsetneq B_i$, and $\mathcal{T}_i(\varphi) \cap \mathrm{HD}_i^2(\Gamma) = \{0\}$. Similar remarks are valid for the algebras $A_\omega(\Gamma)$ and the spaces $\mathrm{HD}_\omega^0(\Gamma)$ for all weights ω such that $\omega(n) = 1$ $(n \geq 0)$ and $\limsup_{n\to\infty}(\log\omega(-n))/\sqrt{n} < \infty$.

5. Closed ideals of B_0 and B_1, and inner functions. We saw in the previous section that when $J_\infty^+(E)$ is w^*-dense in A^+, then every closed ideal I of B_i such that $h(I) \subset E$ has the form $I = I^A \cap I^2$, where I^2 is the closure of I with respect to the weak topology $\sigma(B_i, \mathrm{HD}_i^2(\Gamma))$. For closed ideals J of A^+ such that $h(J) \subset E$, a more precise result holds: in this case $J = J^A \cap S_J.H^\infty(D)$, where J^A is the closed ideal of $A(\Gamma)$ generated by J, and where S_J is the G.C.D. of the inner factors of all nonzero elements of J. Since $(S_J.H^\infty(D)) \cap A^+ = \{f \in A^+ \mid f.S_J^* = 0\}$, the notations being as in Section 3, it is natural to consider whether $I^2 = \bigcap_{S\in k_i(I)} \operatorname{Ker} S^*$, where $k_i(I) = \{S \in \mathcal{S}_i \mid I \subset \operatorname{Ker} S^*\}$ for I as above. In other words, it is natural, given $\varphi \in \mathrm{HD}_i^2(\Gamma)$, to consider whether φ belongs to the w^*-closure of the linear span of $\mathcal{S}_i^* \cap \mathcal{T}_i(\varphi)$, the notations being as in Section 4. We have only been able to give a positive answer to this question for elements of $\mathrm{HD}_1^2(\Gamma)$ whose hull is countable (Corollary 5.18). We first discuss elements of $\mathrm{HD}_i^2(\Gamma)$ such that $\mathcal{T}_i(\varphi) = \mathcal{T}_i(S^*)$ for some $S \in \mathcal{S}_i$. This implies that $\operatorname{supp}\varphi = h(I_\varphi) = h(I_{S^*}) = \operatorname{supp} S^*$, and $\operatorname{supp} S^*$ is well known to be the closed support (i.e. the support as a distribution) of the singular positive measure μ which defines S. So if $\operatorname{supp}\varphi$ is a Carleson set and if $f \in J_\infty^+(\operatorname{supp}\varphi)$, this implies that $(f.S).\varphi = 0$ (the function $f.S$ belongs in this case to $A^\infty(D) \subset A^+$, see [25]).

The following lemma uses the well known characterization in terms of analytic quasiextensions of elements of the Hardy space H^2 which are not cyclic for the adjoint T^* of the shift operator T.

Lemma 5.1. *Let $\varphi \in \mathrm{HD}_1(\Gamma)$, and assume that $f.\varphi = 0$ for some nonzero $f \in A^+$. Then*

$$\limsup_{|z|\to 1^-}(1-|z|)|f(z)| \cdot |\varphi^+(z)| < \infty.$$

If, further, $\varphi \in \mathrm{HD}_1^2(\Gamma)$, or if $\operatorname{supp}\varphi$ is a Carleson set, then $\varphi^+ \in \mathcal{N}$.

Proof. Denote by a_n the nth Taylor coefficient of $f.\varphi^+$ at the origin. We have

$$a_n = \sum_{p=0}^{n} \widehat{f}(p)\widehat{\varphi}(n+1-p).$$

Since $f.\varphi = 0$, we have

$$\sum_{p=0}^{\infty} \widehat{f}(p)\widehat{\varphi}(n+1-p) = 0,$$

and so

$$a_n = -\sum_{p=n+1}^{\infty} \widehat{f}(p)\widehat{\varphi}(n+1-p).$$

For $z \in D$, we obtain

$$|f(z)| \cdot |\varphi^+(z)| \leq \sum_{n=0}^{\infty}\sum_{p=n+1}^{\infty} |\widehat{f}(p)| \cdot |\widehat{\varphi}(n+1-p)| \cdot |z|^n$$

$$= \sum_{p=1}^{\infty} \sum_{n=0}^{p-1} |\widehat{f}(p)| \cdot |\widehat{\varphi}(n+1-p)| \cdot |z|^n$$

$$\le \sup_{m\le 0} |\widehat{\varphi}(m)| \sum_{p=1}^{\infty} |\widehat{f}(p)| \frac{1-|z|^p}{1-|z|} \le \sup_{m\le 0} |\widehat{\varphi}(m)| \cdot \|f\|_1 \frac{1}{1-|z|},$$

which proves the first assertion.

Now assume that $\varphi \in \mathrm{HD}_1^2(\Gamma)$, and set $\widetilde{\varphi} = (\widehat{\varphi}(-n))_{n\ge 0}$ and $\widetilde{f} = (\widehat{f}(n))_{n\ge 0}$. Then $\widetilde{f}$ and $\widetilde{\varphi}$ belong to ℓ^2, and if we denote by T the usual unilateral shift we have

$$\langle \widetilde{f}, T^{*p}(\widetilde{\varphi})\rangle = \sum_{n=0}^{\infty} \widehat{f}(n)\widehat{\varphi}(-n-p) = \widehat{f.\varphi}(-p) = 0 \quad (p \ge 1).$$

Hence $\widetilde{\varphi}$ is not a cyclic vector for T^*, and it follows from the Douglas–Shapiro–Shields theorem [6] that there exists a meromorphic function ψ on D such that

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ |\psi(re^{it})|\, dt < \infty$$

and such that $\overline{\psi}_*(e^{it}) = \lim_{r\to 1^-} \overline{\psi}(re^{it})$ agrees almost everywhere on Γ with $u_*(e^{it}) = \lim_{r\to 1^-} u(re^{it})$, where $u : z \to \sum_{n=0}^{\infty} \widetilde{\varphi}(-n) z^n$ is the element of the Hardy space $H^2(D)$ defined by $\widetilde{\varphi}$.

We have

$$u(z) = \frac{1}{z}\overline{\varphi}\left(\frac{1}{\overline{z}}\right) = \varphi^-(z) \quad (z \in D)$$

and so

$$u_*(\zeta) = \varphi^-(\zeta) = \overline{\zeta}\,\overline{\varphi}(\zeta) \quad \text{for } \zeta \in \Gamma \setminus \operatorname{supp}\varphi.$$

Since $f \in I_\varphi$, we have $\operatorname{supp}\varphi \subset \{z \in \Gamma \mid f(z) = 0\}$ and so $\operatorname{supp}\varphi$ has Lebesgue measure zero.

The radial limit $(\varphi^+)_*(\zeta) = \lim_{r\to 1^-} \varphi^+(r\zeta) = \varphi(\zeta)$ exists for every $\zeta \in \Gamma \setminus \operatorname{supp}\varphi$, hence for almost every $\zeta \in \Gamma$, and we obtain $\psi_*(\zeta) = \overline{u}_*(\zeta) = \zeta(\varphi^+)_*(\zeta)$ for almost every $\zeta \in \Gamma$. It follows then from Privalov's theorem [23] that $\psi(z) = z\varphi^+(z)$ $(z \in D)$ and so $\varphi^+ \in \mathcal{N}$.

Now assume that $f.\varphi = 0$ for some nonzero $f \in A^+$, and that $\operatorname{supp}\varphi$ is a Carleson set. Let g be any nonzero element of $J_\infty^+(E)$. It follows from Lemmas 4.10 and 4.11 that $g.\varphi^+ - (g.\varphi)^+ \in H^\infty(D)$ and that $g.\varphi \in \mathrm{HD}_1^2(\Gamma)$. Since $f.(g.\varphi) = g.(f.\varphi) = 0$, we have $(g.\varphi)^+ \in \mathcal{N}$ and so $g.\varphi^+ \in \mathcal{N}$. Hence $\varphi^+ \in \mathcal{N}$, since g is bounded on D.

Notice that we can apply Lemma 5.1 to any $\varphi \in \mathrm{HD}(\Gamma)$ such that $\sup_{n\le 0} |\widehat{\varphi}(n)| < \infty$, since in this case we can define $f.\varphi$ for every $f \in A^+$.

Then

$$\limsup_{|z|\to 1^-} (1-|z|)|f(z)| \cdot |\varphi^+(z)| < \infty \quad \text{if } f.\varphi = 0.$$

But if $f \ne 0$, then [1, Lemma 5.1] gives $\limsup_{|z|\to 1^-}(1-|z|)\log^+|\varphi(z)| < \infty$, and by Cauchy's inequalities, $\limsup_{n\to\infty}(\log^+|\widehat{\varphi}(n)|)/\sqrt{n} < \infty$, so that $\varphi \in \mathrm{HD}_1(\Gamma)$.

Recall that if $f \in \mathcal{N} \ne 0$ we have the decomposition

$$f(z) = cB(z)\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f_*(e^{it})|\, dt\right] \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_f(t)\right],$$

where μ_f is a singular, real measure on Γ, $|c| = 1$ and $B(z)$ is the Blaschke product defined by the zeroes of f in the disc.

LEMMA 5.2. *Let $\varphi \in \mathrm{HD}(\Gamma)$, and assume that $\varphi^+ \in \mathcal{N}$. Then the singular measure μ_{φ^+} is concentrated on* $\operatorname{supp}\varphi$. *Also the denominator $S(\varphi^+)$ belongs to $\mathcal{S}_0$ if $\varphi \in \mathrm{HD}_0(\Gamma)$.*

Proof. Let $f \in \mathcal{C}(\Gamma)$ be such that $\operatorname{supp} f \cap \operatorname{supp}\varphi = \emptyset$. Since φ is analytic on a neighboorhood of $\operatorname{supp} f$, the set of zeroes of φ on $\operatorname{supp} f$, repeated according to multiplicities, is a finite set $\{z_1, \ldots, z_k\}$. Set

$$D(z) = B(z)\prod_{j\le k}(z - z_j) \quad \text{and} \quad g(z) = \varphi(z)/D(z) \quad (z \in D).$$

Denote by Ω the set of poles of B in $\mathbb{C}\setminus\overline{D}$. Then $\overline{\Omega}\cap\Gamma \subset \operatorname{supp}\varphi$, and so g extends analytically to $\mathbb{C}\setminus(\Omega\cup\operatorname{supp}\varphi)$. Since the functions $z \to z - z_j$ are outer on D, it follows that $g \in \mathcal{N}$ and $\mu_g = \mu_{\varphi^+}$. Set $V = \bigcup_{0\le r\le 1} r\operatorname{supp} f$. Then g is continuous on V, and $g(z) \ne 0$ for every $z \in D$. Hence $\log|g(r\zeta)| \to \log|g_*(\zeta)|$ as $r \to 1^-$ uniformly for $\zeta \in \operatorname{supp} f$. We obtain

$$\int_{-\pi}^{\pi} \log|g_*(e^{it})| f(e^{it})\, dt + \int_{-\pi}^{\pi} f(e^{it})\, d\mu_g(t) = \lim_{r\to 1^-}\int_{-\pi}^{\pi} \log|g(re^{it})| f(e^{it})\, dt$$
$$= \int \log|g_*(e^{it})| f(e^{it})\, dt.$$

Hence

$$\int_{-\pi}^{\pi} f(e^{it})\, d\mu_{\varphi^+}(t) = \int_{-\pi}^{\pi} f(e^{it})\, d\mu_g(t) = 0.$$

Since this holds for every $f \in \mathcal{C}(\Gamma)$ such that $\operatorname{supp} f \cap \operatorname{supp}\varphi = \emptyset$, μ_{φ^+} is concentrated on the closed set $\operatorname{supp}\varphi$.

Now assume that $\varphi \in \mathrm{HD}_0(\Gamma)$, and let $z_0 = e^{it_0} \in \operatorname{supp}\varphi$. Let $\mu_{\varphi^+} = \mu_+ - \mu_-$ be the Jordan decomposition of μ_{φ^+}, so that $S(\varphi^+)$ is the singular inner function defined by μ_-, according to the notations of Section 4.

If $d = 2\pi\mu_-(\{t_0\}) > 0$, set

$$U(z) = \varphi^+(z)S(\varphi^+)(z), \quad V(z) = U(z)\exp\left[d\frac{z_0+z}{z_0-z}\right], \quad W(z) = \frac{U(z)}{cB(z)},$$

where c and B are the constant and the Blaschke product in the decomposition of φ^+. Then $|V(z)| \le |\varphi^+(z)|$ $(z \in D)$ and so $\limsup_{r\to 1^-}(1-r)m(r) = 0$, where $m(r) = \sup_{t\in\mathbb{R}} \log|V(re^{it})|$ for $r \in [0,1)$.

Since μ^+ and μ_- are mutually singular, we have $\mu^+(\{t_0\}) = 0$, and $\nu(\{t_0\}) = 0$ if we set $d\nu(t) = \log|\varphi^+_*(e^{it})|dt - d\mu_+(t)$. We have

$$W(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\,d\nu(t)\right],$$

so that $\log|W(z)|$ is harmonic on D. Set

$$h(z) = d\,\mathrm{Re}\left(\frac{z_0+z}{z_0-z}\right) \quad (z \in D).$$

Then $h + \log|W| = \log|V/(cB)|$ is harmonic on D.

Fix $z \in D$ and set $\varrho = (1+|z|)/2$, so that $1-\varrho = \varrho - |z| = (1-|z|)/2$. It follows from Poisson's formula that

$$h(z) + \log|W(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varrho^2-|z|^2}{|\varrho - ze^{-it}|^2}\log|V(\varrho e^{it})|\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varrho^2-|z|^2}{|\varrho-ze^{-it}|^2}\log\left|\frac{1}{B(\varrho e^{it})}\right|dt.$$

We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varrho^2-|z|^2}{|\varrho-ze^{-it}|^2}\,dt = 1,$$

by standard properties of Poisson kernels. Hence

$$\begin{aligned}(1-|z|)[h(z)+\log|W(z)|]\\ &\le (1-|z|)m\left(\frac{1+|z|}{2}\right) + \frac{(\varrho+|z|)(1-|z|)}{\varrho-|z|}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|\frac{1}{B(\varrho e^{it})}\right|dt\\ &\le \left(1-\frac{1+|z|}{2}\right)m\left(\frac{1+|z|}{2}\right) + \frac{2}{\pi}\int_{-\pi}^{\pi}\log\left|\frac{1}{B(\varrho e^{it})}\right|dt.\end{aligned}$$

Since $\int_{-\pi}^{\pi}\log|1/B(re^{it})|\,dt \to 0$ as $r \to 1^-$, we see that $(1-|z|)[h(z) + \log|W(z)|] \to 0$ as $|z| \to 1^-$.

Now for $r \in [0,1)$ we have

$$\begin{aligned}-\log|W(rz_0)| &= -\frac{1}{2\pi}\int_{t_0-\pi}^{t_0+\pi}\frac{1-r^2}{|1-rz_0e^{-it}|^2}\,d\nu(t)\\ &\le \frac{1}{2\pi}\int_{t_0-\pi}^{t_0+\pi}\frac{1-r^2}{|1-rz_0e^{-it}|^2}\,d|\nu|(t).\end{aligned}$$

Fix $\delta \in (0,\pi)$. We obtain

$$\begin{aligned}(1-r)(-\log|W(rz_0)|) &\le \frac{1-r}{2\pi}\int_{t_0-\delta}^{t_0+\delta}\frac{1-r^2}{|1-rz_0e^{-it}|^2}\,d|\nu|(t)\\ &\quad + \frac{1-r}{2\pi}\int_{\delta<|t-t_0|\le\pi}\frac{1-r^2}{|1-rz_0e^{-it}|^2}\,d|\nu|(t)\\ &\le \frac{1}{\pi}|\nu|([t_0-\delta, t_0+\delta])\\ &\quad + \frac{(1-r)^2}{\pi}|\nu|([0,2\pi])\cdot\frac{1}{1-2r\cos\delta + r^2}.\end{aligned}$$

Hence

$$\limsup_{r\to 1^-}(1-r)(-\log|W(rz_0)|) \le \frac{1}{\pi}|\nu|([t_0-\delta, t_0+\delta])$$

for every $\delta > 0$. Since $|\nu|(\{t_0\}) = \nu(\{t_0\}) = 0$, we have $|\nu|([t_0-\pi/n, t_0+\pi/n]) \to 0$ as $n \to \infty$. Hence $(1-r)(-\log|W(rz_0)|) \to 0$ as $r \to 1^-$, and so $(1-r)h(rz_0) \to 0$ as $r \to 1^-$. But $h(rz_0) = d(1+r)/(1-r)$, and we obtain a contradiction. So μ_- is nonatomic, and $S(\varphi^+)$ is a nonatomic inner function. This concludes the proof of the lemma.

LEMMA 5.3. *Let $\varphi \in \mathrm{HD}_i^2(\Gamma)$ and let $f \in B_i$ $(i = 0$ or $1)$. Then $f.\varphi \in \mathrm{HD}_i^2(\Gamma)$.*

Proof. Consider first the case where $i = 0$. There exists an Atzmon weight ω such that $\varphi \in \mathrm{HD}_\omega^2(\Gamma)$. Let $\tau(n) = \omega(n) = 1$ $(n \ge 0)$, $\tau(n) = (1+|n|)\omega(n)$ $(n < 0)$. Then τ is an Atzmon weight, and $f \in A_\tau(\Gamma)$. Let $P : \mathrm{HD}_0(\Gamma) \to \ell^\infty$, $\psi \to (\widehat{\psi}(-n))_{n\ge 0}$. For $m \ge 0$ we have

$$\|P(\alpha^m.\varphi)\|_2 = \Big(\sum_{n\ge 0}|\widehat{\varphi}(-n-m)|^2\Big)^{1/2} \le \|P(\varphi)\|_2.$$

For $m < 0$ we have

$$\|P(\alpha^m.\varphi)\|_2 = \Big(\sum_{n\geq 0}|\widehat{\varphi}(-n-m)|^2\Big)^{1/2} \leq \|P(\varphi)\|_2 + \Big(\sum_{p=1}^{-m}|\widehat{\varphi}(p)|^2\Big)^{1/2}$$
$$\leq \|P(\varphi)\|_2 + \|\varphi\|_\omega^*\Big(\sum_{q=m}^{-1}\omega(q)^2\Big)^{1/2} \leq \|P(\varphi)\|_2 + \|\varphi\|_\omega^*\tau(m).$$

Hence

$$\sum_{m\in\mathbb{Z}}|\widehat{f}(m)|\cdot\|P(\alpha^m.\varphi)\|_2 \leq \|P(\varphi)\|_2\|f\|_1 + \|\varphi\|_\omega^*\|f\|_\tau < \infty,$$

and the series $\sum_{m\in\mathbb{Z}}P[\widehat{f}(m)\alpha^m.\varphi]$ converges in ℓ^2 to some $u \in \ell^2$. But $P : \mathrm{HD}_\omega(\Gamma) \to \ell^\infty$ is continuous, and so

$$P(f.\varphi) = \lim_{p\to\infty}\sum_{|m|\leq p}\widehat{f}(m)P(\alpha^m.\varphi),$$

with respect to the norm of ℓ^∞.

Hence $P(f.\varphi) = u$, and $\|P(f.\varphi)\|_2 \leq \|P(\varphi)\|_2\|f\|_1 + \|\varphi\|_\omega^*\|f\|_\tau$. A similar argument holds for $i = 1$.

Atzmon [2] studied operators T such that $\|T^n\| = O(n^k)$ for some $k \geq 0$ as $n \to \infty$ which are annihilated by some $f \in A^\infty(D)$. His results show in particular that if $\mathrm{Sp}\,T \subset \Gamma$ is a Carleson set the above condition is equivalent to the condition

$$\sup_{0<r<1}\int_{-\pi}^{\pi}\log^+\|(T - re^{it})^{-1}\|\,dt < \infty.$$

The following theorem is related to this property.

THEOREM 5.4. *Let $\varphi \in \mathrm{HD}_1(\Gamma)$, and assume that $\mathrm{supp}\,\varphi$ is a Carleson set. Then the following conditions are equivalent.*

(1) *There exists some nonzero $f \in A^+$ such that $f.\varphi = 0$.*
(2) *There exists some nonzero $g \in A^\infty(D)$ such that $g.\varphi = 0$.*
(3) *$\varphi^+ \in \mathcal{N}$.*

If, further, $\varphi \in \mathrm{HD}_i^2(\Gamma)$ ($i = 0$ or 1) then (1)–(3) are equivalent to each of the following conditions.

(4) *There exists $S \in \mathcal{S}_i$ such that $\mathcal{T}_i(S^*) = \mathcal{T}_i(\varphi)$.*
(5) *$\mathcal{T}_i(\varphi) = \mathcal{T}_i(S(\varphi^+)^*)$.*

Proof. Clearly, (2) implies (1), and it follows from Lemma 5.1 that (1) implies (3).

Now assume that $\varphi^+ \in \mathcal{N}$ and set $S = S(\varphi^+)$. Let g be any nonzero element of $J_\infty^+(\mathrm{supp}\,\varphi)$ and set $h = g.S$. Since μ_{φ^+} is concentrated on $\mathrm{supp}\,\varphi$ (Lemma 5.2), easy and well known verifications show that $h \in A^\infty(D)$. Then $h.\varphi^+ \in \mathcal{N}$, and $S(h.\varphi^+) = 1$. It follows then from Lemma 4.10 that $(h.\varphi)^+ \in \mathcal{N}$, and that $S((h.\varphi)^+) = 1$. Also it follows from Lemma 4.11 that $(h.\varphi)^- \in H^\infty(D)$, and so $(h.\varphi)_*^+ \in L^\infty(\Gamma)$. It follows then from (4.4) that $(h.\varphi)^+ \in H^\infty(D)$. Denote by ℓ the restriction of $h.\varphi$ to $\Gamma\setminus\mathrm{supp}\,\varphi$ so that $\ell \in L^\infty(\Gamma)$. Then $\ell = (h.\varphi)_*^+$, and so $\widehat{\ell}(n) = 0$ $(n < 0)$. Also $\overline{\ell}(\zeta) = \zeta(h.\varphi)^-(\zeta)$ $(\zeta \in \Gamma\setminus\mathrm{supp}\,\varphi)$ and so $\widehat{\ell}(n) = 0$ $(n \geq 0)$. Hence $\ell = 0$ a.e. on Γ, and $(h.\varphi)(\zeta) = 0$ $(\zeta \in \Gamma\setminus\mathrm{supp}\,\varphi)$. So $h.\varphi = 0$, and (2) holds.

Now assume that $\varphi \in \mathrm{HD}_i(\Gamma)$, $i = 0$ or 1, and that $\mathcal{T}_i(\varphi) = \mathcal{T}_i(S^*)$ for some $S \in \mathcal{S}_i$. Let μ be the singular positive measure on Γ which defines S. Then μ is concentrated on $\mathrm{supp}\,S^*$ (this is well known, and follows anyway from Lemma 5.2). We have $\mathrm{supp}\,S^* = h(I_{i,S^*}) = h(I_{i,\varphi}) = \mathrm{supp}\,\varphi$, and so μ is concentrated on $\mathrm{supp}\,\varphi$, which is a Carleson set. Let g be a nonzero element of $J_\infty^+(\mathrm{supp}\,\varphi)$. Then $g.S \in A^\infty(D)$, and $(g.S).S^* = 0$, by Proposition 3.6. Since $I_{i,S^*} = I_{i,\varphi}$, it follows that $(g.S).\varphi = 0$ and so φ satisfies the equivalent conditions (1)–(3).

We only have to check that (5) implies (4) in the case $i = 0$, and this follows from Lemma 5.2.

Clearly, if $\varphi \in \mathrm{HD}_i(\Gamma)$ and if $f \in B_i$, then $I_{i,\varphi} \subset I_{i,f.\varphi}$ and so $f.\varphi \in \mathcal{T}_i(\varphi)$. Conversely, let $f \in A^+$ be outer, and let $\varphi \in \mathrm{HD}_i^2(\Gamma)$. Then if $h \in I_{i,f.\varphi}$ we have $f.(h.\varphi) = 0$. It follows from Lemma 5.3 that $h.\varphi \in \mathrm{HD}_i^2(\Gamma)$, and so $h.\varphi = 0$, since f is outer (see Remark 4.15(3)). Hence in this case $\mathcal{T}_i(f.\varphi) = \mathcal{T}_i(\varphi)$.

Now assume that $\varphi \in \mathrm{HD}_i(\Gamma)$, that $\mathrm{supp}\,\varphi$ is a Carleson set and that $\varphi^+ \in \mathcal{N}$. Let $g \in J_\infty^+(\mathrm{supp}\,\varphi)$ and let $\psi \in \mathcal{T}_i(\varphi)$. We showed above that $(g.S).\varphi = 0$, so that $(g.S).\psi = 0$, where $S = S(\varphi^+)$. Now let $g \in J_\infty^+(\mathrm{supp}\,\varphi)$ be outer. Since $(g.S).\psi^+ = (g.S).\psi^+ - [(g.S).\psi]^+ \in H^\infty(D)$, it follows from the uniqueness of the inner-outer factorization in $H^\infty(D)$ that $S(\psi^+)$ is a divisor of $S(\varphi^+)$. Now assume that $\varphi \in \mathrm{HD}_i^2(\Gamma)$, that $\mathrm{supp}\,\varphi$ is a Carleson set and that $\varphi^+ \in \mathcal{N}$. Let $f \in J_\infty^+(\mathrm{supp}\,\varphi)$ be outer, and set $\psi = f.\varphi$. Then $\mathcal{T}_i(\varphi) = \mathcal{T}_i(\psi)$, and it follows from the discussion above that $S(\psi^+) = S$, where we set $S = S(\varphi^+)$. Also it follows from Lemma 4.11 that $\psi^- \in A^\infty(D)$, so that $(\psi^+)_* \in \mathcal{C}^\infty(\Gamma)$. Let $U = S.\psi^+$. It follows from (4.6) that $U \in H^\infty(D)$. Since $\mathrm{supp}\,S \subset \mathrm{supp}\,\varphi = \mathrm{supp}\,\psi$, by Lemma 5.2, and since $|S_*| = 1$ on $\Gamma\setminus\mathrm{supp}\,\varphi$, it follows that $U_* \in \mathcal{C}^\infty(\Gamma)$ and so $U \in A^\infty(D)$.

We have

$$U.(S^*)^+ + \psi^+ = \frac{1}{S(\infty)}U - U/S + \psi^+ = \frac{1}{S(\infty)}U \in A^\infty(D).$$

Hence $(U.S^*)^+ + \psi^+ \in A^\infty(D)$, by Lemma 4.10. Let $g \in J_\infty^+(\mathrm{supp}\,\varphi)$ be outer. Then $g.[U.S^* + \psi] = 0$, since the denominator of $U.S^* + \psi$ is 1, and so $U.S^* + \psi = 0$, since $U.S^* + \psi \in \mathrm{HD}_i^2(\Gamma)$. Hence $\psi = -U.S^* \in \mathcal{T}_i(S^*)$, and $\mathcal{T}_i(\varphi) = \mathcal{T}_i(\psi) \subset \mathcal{T}_i(S^*)$. Now let $h \in B_i$ such that $h.\psi = 0$, and set $L = \{u \in A^+ \mid u.h.S^*\} = 0$. Then $U.A^+ \subset L$, and $S.H^\infty(D) \cap A^+ \subset L$ by Proposition 3.6. Let $P : \mathrm{HD}_i^2(\Gamma) \to \ell^2$, $\theta \to (\widehat{\theta}(-n))_{n\ge0}$, be the map introduced in the proof of Lemma 5.3, and for $v \in H^2(D)$ set $\widetilde{v} = (\widehat{v}(n))_{n\ge0}$. Then $h.S^* \in \mathrm{HD}_i^2(\Gamma)$, by Lemma 5.3, $\mathrm{supp}(h.S^*) \subset \mathrm{supp}(S^*) \subset \mathrm{supp}\,\varphi \subsetneq \Gamma$ and so $L = \{u \in A^+ \mid \langle \widetilde{u}, T^{*n}(P(h.S^*))\rangle = 0 \ (n \ge 0)\}$, where we denote by T the shift operator. But the set $K = \bigcap_{n\ge1}[T^{*n}(P(h.S^*))^\perp]$ is a closed subspace of ℓ^2 invariant for the shift operator. By Beurling's theorem, $K = \varrho.\widetilde{H^2}(D)$ for some inner function ϱ. So ϱ is a divisor of S, and it is also a divisor of the inner factor of U. But $U = S.\psi^+$, where S is the denominator of ψ^+, and so it follows from the uniqueness of decomposition (4.4) that $\varrho = 1$. So $K = \widetilde{H^2(D)} = \ell^2$, $L = A^+$, $1 \in L$ and $h.S^* = 0$.

Hence $I_{i,\psi} \subset I_{i,S^*}$, $\mathcal{T}_i(S^*) \subset \mathcal{T}_i(\psi)$ and so $\mathcal{T}_i(\varphi) = \mathcal{T}_i(\psi) = \mathcal{T}_i(S^*)$, which concludes the proof of the theorem.

Remark 5.5. (i) It follows from the above arguments that if S_1 and S_2 are singular inner functions defined by positive singular measures μ_1 and μ_2 supported by Carleson sets, and if $\mu_1 \le \mu_2$ then $\mathcal{T}(S_1^*) \subset \mathcal{T}_1(S_2^*)$ (and $\mathcal{T}_0(S_1^*) \subset \mathcal{T}_0(S_2^*)$ if μ_1 and μ_2 are nonatomic).

(ii) If $J_\infty^+(\mathrm{supp}\,\varphi)$ is w^*-dense in A^+, and if $\varphi \in \mathrm{HD}_i^0(\Gamma)$ and $\varphi^+ \in \mathcal{N}$ then $\mathcal{T}_i(\varphi) = \mathcal{T}_i(S(\varphi^+)^*)$. This follows from the fact that φ can be "synthetized" by elements of $\mathrm{HD}_i^2(\Gamma) \cap \mathcal{T}_i(\varphi)$ of the form $f.\varphi$, where $f \in J_\infty^+(\mathrm{supp}\,\varphi)$ is outer, and from Theorem 5.4. We leave the details to the reader. Of course, it follows also from Remark 4.15(3) that there exists nonzero $\varphi \in \mathrm{HD}_i^0(\Gamma)$ with $\mathrm{supp}\,\varphi$ a Carleson set (or even a Kronecker set), for which $S(\varphi^+) = 1$, so that $\mathcal{T}_i(S(\varphi^+)^*) = \{0\}$.

It follows from a result of Atzmon [2, Proposition 2.6] that if $\varphi \in \mathrm{HD}_1(\Gamma)$, and if $\mathrm{supp}\,\varphi$ is a finite set, then $\varphi^+ \in \mathcal{N}$. Another result of Atzmon [2, Theorem 1.1] shows that if U is a bounded operator on a Banach space such that $\|U^n\| = O(n^k)$ for some $k \ge 0$, then $f(U) = 0$ for some nonzero $f \in A^\infty$ if and only if

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ \|(U - re^{it})^{-1}\|\,dt < \infty$$

(which implies that $\mathrm{Sp}\,U \cap \Gamma$ is a Carleson set).

Now let $E \subset \Gamma$ be an uncountable closed set. By a result of Zarrabi [27] mentioned in Section 3, there exists an Atzmon weight ω such that $J_\omega(E) \cap A^+ = \{0\}$. Let $\pi : A_\omega(\Gamma) \to A_\omega(\Gamma)/J_\omega(E)$ be the canonical map, and let U be the map $\pi(f) \to \pi(\alpha f)$. Clearly, $f(U)[\pi(1)] = \pi(f)$ for every $f \in A^+$,

and it follows from Atzmon's theorem that

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log \|(\pi(\alpha) - re^{it})^{-1}\|\,dt = \infty.$$

Now if $\varphi \in \mathrm{HD}_\omega(E)$, then there exists a continuous linear form L on $A_\omega(\Gamma)/J_\omega(E)$ such that $\langle g, \varphi\rangle = \langle \pi(g), L\rangle$ $(g \in A_\omega(\Gamma))$, so that $\varphi(z) = \langle (\alpha - z)^{-1}, \varphi\rangle = \langle (\pi(\alpha) - z)^{-1}, L\rangle$ for $z \in D$ (here we use formula (2.13)). This suggests that in this situation there exists $\varphi \in \mathrm{HD}_\omega(E) \subset \mathrm{HD}_0(E)$ such that $\varphi^+ \notin \mathcal{N}$. The following theorem shows that this is indeed true, and that the fact that $\varphi^+ \in \mathcal{N}$ for every $\varphi \in \mathrm{HD}_1(E)$ holds only for finite sets E.

THEOREM 5.6. *Let $E \subset \Gamma$ be a closed set.*

(1) *If E is infinite, there exists $\varphi \in \mathrm{HD}_1(E)$ such that $\varphi^+ \notin \mathcal{N}$ and $\varphi^- \in H^\infty(D)$.*

(2) *If E is uncountable, there exists $\varphi \in \mathrm{HD}_0(E)$ such that $\varphi^+ \notin \mathcal{N}$ and $\varphi^- \in H^\infty(D)$.*

Proof. Assume that E is infinite. Then E has a nonisolated element z_0, and there exists a sequence z_n of distinct elements of E such that $z_n \to z_0$ as $n \to \infty$.

Let $t > 0$ and set $\omega_t(n) = 1$ $(n \ge 0)$, $\omega_t(n) = e^{t\sqrt{|n|}}$ $(n < 0)$. For $\lambda \in \Gamma$ and $\varepsilon > 0$ set

$$S_{\lambda,\varepsilon}(z) = \exp\left(\varepsilon\frac{z+\lambda}{z-\lambda}\right),$$

so that $S_{\lambda,\varepsilon}$ is inner. We have

$$\log(|S_{\lambda,\varepsilon}(z)|^{-1}) \le \frac{2\varepsilon}{1-|z|} \quad (z \in D)$$

and applying Cauchy's inequalities as in [1] we see that there exists $\varepsilon > 0$, independent of λ, such that $S_{\lambda,\varepsilon}^* \in \mathrm{HD}_{\omega_t}(\Gamma)$. Since $S_{\lambda,\varepsilon}$ is inner, $[S_{\lambda,\varepsilon}^*]^- \in H^\infty(D)$.

Set $\mathrm{HD}_{\omega_t}^\infty(\Gamma) = \{\varphi \in \mathrm{HD}_{\omega_t}(\Gamma) \mid \varphi^- \in H^\infty(D)\}$. Then $\mathrm{HD}_{\omega_t}^\infty(\Gamma)$ is a Banach space with respect to the norm $\|\varphi\| = \max(\|\varphi\|_{\omega_t}^*, \|\varphi^-\|_\infty)$. Also since $\mathrm{HD}_{\omega_t}(F)$ is a closed subspace of $(\mathrm{HD}_{\omega_t}(\Gamma), \|\cdot\|_{\omega_t}^*)$, it follows that $\mathrm{HD}_{\omega_t}^\infty(F) = \mathrm{HD}_{\omega_t}^\infty(\Gamma) \cap \mathrm{HD}_{\omega_t}(F)$ is a closed subspace of $(\mathrm{HD}_{\omega_t}^\infty(\Gamma), \|\cdot\|)$ for every closed set $F \subset \Gamma$. Now set $\varphi_n = S_{z_n,\varepsilon}^*$ $(n \ge 1)$. Clearly, $\varphi_n \in \mathrm{HD}_{\omega_t}^\infty(\{z_n\})$, and $\|\varphi_n\| = \|\varphi_1\|$ $(n \ge 1)$. Hence the series $\sum_{n=1}^\infty 2^{-n}\varphi_n$ converges in $\mathrm{HD}_{\omega_t}^\infty(\Gamma)$, and if we set $\varphi = \sum_{n=1}^\infty 2^{-n}\varphi_n$ we see that $\varphi \in \mathrm{HD}_{\omega_t}^\infty(E)$.

For $m \ge 1$ set $\psi_m = \sum_{n\ne m}^\infty 2^{-n}\varphi_n$. Then $\mathrm{supp}\,\psi_m \subset \overline{\{z_n\}}_{n\ne m}$ and so $z_m \notin \mathrm{supp}\,\psi_m$. Hence $\psi_m(rz_m)$ has a limit as $r \to 1^-$, and

$$\lim_{r\to1^-}(1-r)\log|\varphi(rz_m)| = \lim_{r\to1^-}(1-r)\log|S_{z_m,\varepsilon}^{-1}(rz_m)| = 2\varepsilon.$$

Let μ be a positive, singular measure on Γ, let R be the singular inner function defined by μ and assume that $R.\varphi^+$ is bounded on D. For $m \geq 1$ set $s_m = \mu(\{z_m\})$, and let δ_{z_m} be the Dirac measure at $\{z_m\}$. Then $\mu - \sum_{m=1}^n s_m \delta_{z_m}$ is positive for every $n \geq 1$, and $\mu(\Gamma) \geq \sum_{m=1}^n s_m$.

Now let R_m be the singular inner function defined by $\mu - s_m \delta_{z_m}$. Since $(\mu - s_m \delta_{z_m})(\{z_m\}) = 0$, the decompositions of Poisson integrals given in the proof of Lemma 5.2 show that $(1-r)\log(|R_m(rz_m)|^{-1}) \to 0$ as $r \to 1^-$. Since

$$R(z) = R_m(z) \exp\left(s_m . \frac{z + z_m}{z - z_m}\right)$$

we obtain $\lim_{r\to 1^-}(1-r)\log(|R(rz_m)|^{-1}) = 2s_m$ $(m \geq 1)$. But

$$\limsup_{r\to 1^-}[\log|\varphi(z)| + \log|R(z)|] < \infty$$

and so

$$\lim_{r\to 1^-}(1-r)\log(|R(rz_m)|^{-1}) \geq \lim_{r\to 1^-}(1-r)\log|\varphi(rz_m)|.$$

Hence $s_m \geq \varepsilon$ $(m \geq 1)$ and $\mu(\Gamma) \geq n\varepsilon$ for every $n \geq 1$. This contradiction shows that $\varphi^+ \notin \mathcal{N}$, which proves the first assertion of the theorem.

Now assume that E is uncountable. By a result of Zarrabi [27] mentioned in Section 3, there exists an Atzmon weight ω and $S \in \mathcal{S}_0$ such that $(S^n)^* \in \mathrm{HD}_\omega(E)$ for every $n \geq 1$.

Let $\mathrm{HD}_\omega^\infty(E) = \{\varphi \in \mathrm{HD}_\omega(E) \mid \varphi^- \in H^\infty(D)\}$. We then see as above that $\mathrm{HD}_\omega^\infty(E)$ is a Banach space with respect to the norm $\|\varphi\| = \max(\|\varphi\|_\omega^*, \|\varphi^-\|_\infty)$.

Denote by Ω_p the set of all $\varphi \in \mathrm{HD}_\omega^\infty(E)$ for which there exists a positive, singular measure μ on Γ, with $\mu(\Gamma) \leq p$, such that $S_\mu.\varphi^+ \in H^\infty(D)$, where we denote by S_μ the singular inner function defined by μ. Notice that in this situation we have $\|S_\mu.\varphi^+\|_\infty = \|\varphi_*^+\|_\infty = \|\varphi_*^-\|_\infty \leq \|\varphi\|$. Let $(\varphi_m)_{m\geq 1}$ be a sequence of elements of Ω_p such that $\|\varphi_m - \varphi\| \to 0$ as $m \to \infty$ for some $\varphi \in \mathrm{HD}_\omega^\infty(E)$, and for $m \geq 1$ let μ_m be a positive singular measure on Γ, with $\mu_m(\Gamma) \leq p$, such that $S_{\mu_m}.\varphi_m^+ \in H^\infty(D)$.

We can identify $\mathcal{M}(\Gamma)$, the space of all complex measures on Γ, with the dual space of $\mathcal{C}(\Gamma)$. Since $\|\mu_m\| = \mu_m(\Gamma) \leq p$ $(m \geq 1)$, and since $\mathcal{C}(\Gamma)$ is separable, we can assume without loss of generality that there exists $\mu \in \mathcal{M}(\Gamma)$ such that $\int_\Gamma f\,d\mu = \lim_{m\to\infty}\int_\Gamma f\,d\mu_m$ $(f \in \mathcal{C}(\Gamma))$. Then $\int_\Gamma f\,d\mu \geq 0$ $(f \in \mathcal{C}(\Gamma),\ f \geq 0)$ and so μ is a positive measure.

Set

$$R(z) = \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right] \quad (z \in D).$$

Then $R(z) = \lim_{m\to\infty} S_{\mu_m}(z)$. Also $\varphi_m^+(z) \to \varphi^+(z)$ as $m \to \infty$ uniformly on compact subsets of D, and we obtain

$$|R(z)\varphi^+(z)| = \lim_{m\to\infty}|S_{\mu_m}(z)\varphi_m^+(z)| \leq \limsup_{m\to\infty}\|\varphi_m\| = \|\varphi\| \quad (z \in D).$$

Let $d\mu(t) = h(e^{it})dt + d\nu(t)$ be the Radon–Nikodym decomposition of μ, where $h \in L^1(\Gamma)$, $h(e^{it}) \geq 0$ a.e. and where $\nu \geq 0$ is singular with respect to Lebesgue measure. Then

$$\nu(\Gamma) \leq \mu(\Gamma) = \int_\Gamma d\mu = \lim_{m\to\infty}\int_\Gamma d\mu_m = \lim_{m\to\infty}\mu_m(\Gamma) \leq p.$$

Let $F(z) = S_\nu(z)[R(z)]^{-1}$ $(z \in D)$ so that F is outer. Then $|S_\nu(z)\varphi^+(z)| \leq \|\varphi\| \cdot |F(z)|$ $(z \in D)$. So $S_\nu.\varphi^+ \in \mathcal{N}$, and the denominator of $S_\nu.\varphi^+$ is 1. Since $|(S_\nu.\varphi^+)_*(e^{it})| = |\varphi_*^+(e^{it})| = |\varphi_*^-(e^{it})| \leq \|\varphi\|_\infty$ a.e., it follows from (4.4) that $S_\nu.\varphi^+$ is bounded. Hence Ω_p is closed for every $p \geq 1$.

Now fix $p \geq 1$, let $\psi \in \Omega_p$, let $\varepsilon > 0$ and let ϱ be the positive, singular measure on Γ which defines the singular inner function S introduced above. Let $n \geq 1$ be an integer such that $n\varrho(\Gamma) > 2p$ and set $s = \varepsilon/(2\|(S^n)^*\|)$. The denominator of $[s(S^n)^*]^+$ being clearly equal to S^n, $s(S^n)^* \notin \Omega_{2p}$. Also it follows from the definitions that the difference of two elements of Ω_p belongs to Ω_{2p}. We thus see that $\theta = \psi + s(S^n)^* \notin \Omega_p$, and $\|\psi - \theta\| < \varepsilon$. Hence the interior of Ω_p is empty for every $p \geq 1$, and it follows from the category theorem that $\bigcup_{p\geq 1}\Omega_p \subsetneq \mathrm{HD}_\omega^\infty(E)$. This concludes the proof of the theorem.

Remark 5.7. 1) It follows from recent computations by M. Rajoelina, M. Zarrabi and the author [10] that if $\zeta \in (0, 1/2)$ and if $E_\zeta = \{\exp(2i\pi\sum_{n=1}^\infty \varepsilon_n \zeta^{n-1}(1-\zeta)) : \varepsilon_n = 0 \text{ or } 1\}$ is the perfect symmetric set of constant ratio ζ, then for every Atzmon weight ω such that

$$\liminf_{n\to\infty} n^{-(\log\zeta - \log 2)/(2\log\zeta - \log 2)}\log\omega(-n) = \infty$$

there exists an inner function S such that $(S^n)^* \in \mathrm{HD}_\omega(E_\zeta)$ $(n \geq 1)$. It follows from the proof of the theorem that if ω is such a weight then there exists $\varphi \in \mathrm{HD}_\omega^\infty(E_\zeta)$ such that $\varphi^+ \notin \mathcal{N}$. In the other direction, using the methods of [8, Section 7], it is possible to show [20] that $\mathrm{HD}_\omega^2(E_\zeta) = 0$ for every Atzmon weight ω such that $\log\omega(-n) = O(n^\alpha)$ for some $\alpha < (\log\zeta - \log 2)/(2\log\zeta - \log 2)$ (when $p \in \mathbb{N}$, $p \geq 3$ this result extends to $\mathrm{HD}_\omega^0(E_{1/p})$). It would be of interest to characterize completely the Atzmon weights ω such that $\varphi^+ \in \mathcal{N}$ for every $\varphi \in \mathrm{HD}_\omega^2(E_\zeta)$ (and also the Atzmon weights ω such that $\mathrm{HD}_\omega^2(E_\zeta) = \{0\}$).

Notice also in this direction that for every Atzmon weight ω there exists a closed, perfect, Kronecker subset F_ω of F such that $\mathrm{HD}_\omega^0(F_\omega) = \{0\}$ (see [8]).

2) The notations being as in the proof of Theorem 5.6(2), it is not difficult to see that the category argument used there produces in fact some elements $\varphi = \sum_{n=1}^\infty \varepsilon_n (S^{p_n})^*$ with $\varepsilon_n > 0$ and $p_n \in \mathbb{N}$ $(n \geq 1)$ such that $\varphi \in \mathrm{HD}_\omega(E)$,

$\varphi^- \in H^\infty(D)$ and $\varphi^+ \notin \mathcal{N}$. For such elements, it is possible to prove that $\mathcal{T}_0(\varphi)$ is the w^*-closure of the linear span of $\mathcal{S}_0^* \cap \mathcal{T}_0(\varphi)$, the notations being as at the beginning of this section (see [20]). The author has not been able so far to produce any element of $\mathrm{HD}_0^2(\Gamma)$, with support of Lebesgue measure zero, for which this "synthesis by inner functions" does not hold.

3) It follows in particular from the proof of Theorem 5.6(2) that if ω is any Atzmon weight then the set $\{\varphi \in \mathrm{HD}_\omega^\infty(E) \mid \varphi^+ \in \mathcal{N}\}$ is an F_σ-set. Now let Ω be the set of elements of $H^2(D)$ which are noncyclic for the backward shift. We observed above that if $\varphi \in \mathrm{HD}^2(\Gamma)$, and if $\operatorname{supp}\varphi$ has Lebesgue measure 0, then $\varphi^+ \in \mathcal{N}$ if and only if $\varphi^- \in \Omega$. Douglas, Shapiro and Shields showed in [6] that Ω is an F_σ-set by using interesting results of Tumarkin [26] about rational approximation. In fact, if T is any bounded operator on a Banach space X, then the set $\mathcal{U}$ of elements of X which are noncyclic for T is an F_σ-set: if $\mathcal{U} = X$, this is obvious, and if $\mathcal{U} \subsetneq X$ then

$$\mathcal{U} = \bigcup_{p\ge 1} \{x \in X \mid \inf_{P\in\mathbb{C}[X]} \|x_0 - P(T)(x)\| \ge 1/p\},$$

where x_0 is any fixed cyclic vector for T.

We conclude this paper with a complete description of closed ideals of B_1 with countable hull, which shows that elements of $\mathrm{HD}_1^0(\Gamma)$ with countable support can be "synthetized" by inner functions in the sense of Remark 4.15 (since any closed countable subset E of Γ is a set of B_0-synthesis and satisfies $\mathrm{HD}_0^0(E) = \{0\}$, no such questions arise for B_0). When $\varphi \in \mathrm{HD}_1^0(\Gamma)$ is supported by a finite set the fact that φ can be "synthetized" by inner functions follows immediately from Theorems 4.13 and 5.4 since in this case $J_\infty^+(\operatorname{supp}\varphi)$ is w^*-dense in A^+ and $\varphi^+ \in \mathcal{N}$. In the other direction, it follows from Theorem 5.6(1) that the "synthetization by inner functions" given by Corollary 5.18 holds for some $\varphi \in \mathrm{HD}_1^0(\Gamma)$ for which $\varphi^+ \notin \mathcal{N}$.

DEFINITION 5.8. For $\lambda \in \Gamma$ and $t > 0$ we denote by $S_{t,\lambda}$ the inner function

$$z \to \exp\left(t\frac{z+\lambda}{z-\lambda}\right) \quad (z \in D).$$

We set $\mathcal{S}_{\mathrm{a},\lambda} = \{S_{t,\lambda}\}_{t>0}$, $\mathcal{S}_{\mathrm{a}} = \bigcup_{\lambda\in\Gamma} \mathcal{S}_{\mathrm{a},\lambda}$ and $\mathcal{S}_{\mathrm{a}}^* = \{S^*\}_{S\in\mathcal{S}_{\mathrm{a}}}$. Also if I is a closed ideal of B_1, we set $k_{\mathrm{a}}(I) = \{S \in \mathcal{S}_{\mathrm{a}} \mid S^* \in I^\perp\}$ and $I^{\mathrm{a}} = \bigcap_{S\in k_{\mathrm{a}}(I)} \operatorname{Ker} S^*$.

Clearly, $\mathcal{S}_{\mathrm{a}}$ is the set of "atomic" inner functions, i.e. the set of inner functions defined by a scalar multiple of a Dirac measure on Γ, and $I^{\mathrm{a}} \subset I^2 \subset I^0$ for every closed ideal of B_1. Notice also that if $f \in \mathcal{S}_{\mathrm{a},\lambda}$ then the derivative of $(\alpha-\lambda)^2.S$ is bounded on D, so that $(\alpha-\lambda)^2.S \in A^+$ (see for example [16, p. 56]). For $\lambda \in \Gamma$ we now write $J_1(\lambda)$, $I_1(\lambda)$, $\mathrm{HD}_1(\lambda)$ for $J_1(\{\lambda\})$, $I_1(\{\lambda\})$, $\mathrm{HD}_1(\{\lambda\})$, etc., and we set $S_{0,\lambda} = 1$, so that $S_{0,\lambda}^* = 0$.

LEMMA 5.9. *For every $\lambda \in \Gamma$, the following properties hold*:

(i) $f = \lim_{n\to\infty} f(\alpha-\lambda)^k(\alpha-\lambda-\lambda/n)^{-k}$ $(f \in I_1(\lambda),\ k \ge 1)$.

(ii) *The map $t \to (\alpha-\lambda)^2.S_{t,\lambda}$ is continuous from $[0,\infty)$ into A^+.*

(iii) *The map $t \to S_{t,\lambda}^*$ is w^*-continuous from $[0,\infty)$ into $\mathrm{HD}_1(\lambda)$.*

(iv) $(\alpha-\lambda)^2.S_{t,\lambda}.S_{s,\lambda}^* = 0$ *for $t \ge s$, and*
$(\alpha-\lambda)^2.S_{t,\lambda}.S_{s,\lambda}^* = (\alpha-\lambda)^2.S_{s-t,\lambda}^*$ *for $0 \le t \le s$.*

(v) *The formula*

$$(u-\lambda)(\alpha-\lambda)^3(\alpha-u)^{-1} = 2\lambda\int_0^\infty e^{t(\lambda+u)/(\lambda-u)}.(\alpha-\lambda)^2.S_{t,\lambda}\,dt$$

holds in A^+ for $|u| > 1$.

Proof. As in Section 2 set $\omega_p(n) = 0$ $(n \ge 0)$ and $\omega_p(n) = e^{p\sqrt{|n|}}$ $(n < 0)$ for $p \ge 1$. Let $e_n = (\alpha-\lambda)(\alpha-\lambda-1/n)^{-1}$. A direct well known computation shows that the sequence $(e_n)_{n\ge1}$ is bounded in A^+ and that $\|\alpha-\lambda-(\alpha-\lambda)e_n\|_1 \to 0$ as $n\to\infty$.

Now let $f \in I_{\omega_p}(\lambda)$. For $m \ge 1$ set $f_m = \sum_{|k|\le m} \widehat{f}(k)(\alpha^k - \lambda^k)$. Then $f_m \in (\alpha-\lambda)A_{\omega_p}(\Gamma)$, $\|f-f_m\|_{\omega_p} \to 0$ as $m\to\infty$ and, since $(e_n)_{n\ge1}$ is bounded in A^+, $\|fe_n^k - f\|_{\omega_p} \to 0$ as $n\to\infty$ $(k\ge1)$. This proves (i).

Now set $g_t = (\alpha-\lambda)^2.S_{t,\lambda}$. Then $g_t' \in H^\infty(D)$ (and $\|g_t'\|_\infty = O(t)$ as $t\to\infty$). It follows from Lebesgue's convergence theorem that the map $t \to g_t'$ is continuous from $[0,\infty[$ into $H^2(D)$. Using again the Cauchy–Schwarz inequality as in [16, p. 56] we obtain (ii).

Now fix $a > 0$. It follows from formula (3.1) that there exists $M > 0$ such that

$$|S_{t,\lambda}^*(z)| \le Me^{2a/(1-|z|)} \quad (z \in D,\ t \in [0,a])$$

and $|S_{t,\lambda}^*(z)| \le M$ $(|z| \ge 1,\ z \ne \lambda,\ t \in [0,a])$. Hence $|\widehat{S_{t,\lambda}^*}(n)| \le M$ for $0 \le t \le a$ and $n \le 0$, and it follows from Cauchy's inequalities that there exist $p \ge 1$ and $K > 0$ such that $|\widehat{S_{t,\lambda}^*}(n)| \le Ke^{p\sqrt{n}}$ for $0 \le t \le a$ and $n \ge 0$. So the family $(S_{t,\lambda}^*)_{0\le t\le a}$ is bounded in the Banach space $\mathrm{HD}_{\omega_p}(\Gamma)$.

It follows from Cauchy's formula and from Lebesgue's convergence theorem that the map $t \to \widehat{S_{t,\lambda}^*}(n)$ is continuous on $[0,a]$ for every $n \in \mathbb{Z}$. Since the topology of pointwise convergence of Fourier coefficients and the w^*-topology $\sigma(\mathrm{HD}_{\omega_p}(\Gamma), A_{\omega_p}(\Gamma))$ agree on bounded sets, the map $t \to S_{t,\lambda}^*$ is w^*-continuous from $[0,a]$ into $\mathrm{HD}_{\omega_p}(\Gamma)$ $(a > 0)$, and (iii) follows.

Now set $\varphi_{s,t} = (\alpha-\lambda)^2.S_{t,\lambda}.S_{s,\lambda}^*$ $(t \ge 0,\ s \ge 0)$ and $\psi_{s,t} = 0$ $(t \ge s \ge 0)$,

$\psi_{s,t} = (\alpha-\lambda)^2.S^*_{s-t,\lambda}$ $(0 \le t \le s)$. Since

$$\langle f, \varphi_{s,t}\rangle = \frac{1}{2i\pi}\int_\Gamma f(\zeta)(\zeta-\lambda)^2 S_{t,\lambda}(\zeta)\overline{S_{s,\lambda}(\zeta)}\,d\zeta \quad \text{for } f \in A^+$$

(formula (3.5)), and since

$$S_{t,\lambda}(\zeta)\overline{S_{s,\lambda}(\zeta)} = S_{t-s,\lambda}(\zeta) \quad (t \ge s \ge 0,\ \zeta \in \Gamma\setminus\{\lambda\})$$

and

$$S_{t,\lambda}(\zeta)\overline{S_{s,\lambda}(\zeta)} = \overline{S_{s-t,\lambda}}(\zeta) \quad (0 \le t \le s,\ \zeta \in \Gamma\setminus\{\lambda\}),$$

we see that $\langle f,\varphi_{s,t}\rangle = \langle f,\psi_{s,t}\rangle$ $(f \in A^+,\ t \ge 0,\ s \ge 0)$. Hence $\widehat{\varphi}_{s,t}(n) = \widehat{\psi}_{s,t}(n)$ for $n \le 0$. Since $\operatorname{supp}\varphi_{s,t} \subset \{\lambda\}$ and $\operatorname{supp}\psi_{s,t} \subset \{\lambda\}$ we obtain $\varphi_{s,t} = \psi_{s,t}$ $(t \ge 0,\ s \ge 0)$, which proves (iv).

We have $\operatorname{Re}(\lambda+u)/(\lambda-u) < 0$ for $|u| > 1$. The argument given in the proof of (ii) shows that $\|(\alpha-\lambda)^2.S_{t,\lambda}\| = O(\sqrt{t})$ as $t \to \infty$, and so the Bochner integral

$$\int_0^\infty e^{t(\lambda+u)/(\lambda-u)}(\alpha-\lambda)^2.S_{t,\lambda}\,dt$$

defines an element f of A^+. Let $g = (u-\lambda)(\alpha-\lambda)^3(\alpha-u)^{-1}$.

Since characters commute with Bochner integrals a direct computation shows that $2\lambda f(z) = g(z)$ $(z \in D)$, which concludes the proof of the lemma. Notice that formula (v) is a variant of the resolvent formula for semigroups (a similar formula was established in [7, Lemma 9-9]).

Remark 5.10. 1) We can define the product $T_{s,t} = S_{t,\lambda}.S^*_{s,\lambda}$ by using formula (2.18). We obtain

$$|\widehat{T}_{s,t}(n)|^2 \le \Big[\sum_{k=0}^\infty |\widehat{S^*_{t,\lambda}}(k)|^2\Big]\Big[\sum_{k=0}^\infty \widehat{S}_{s,\lambda}(n-k)^2\Big] \quad (n \in \mathbb{Z}).$$

It follows immediately that $T_{s,t} \in \mathrm{HD}_{\omega_p}(\Gamma)$ for some $p > 0$ depending on s and t. Clearly, $(\alpha-\lambda)^2.T_{s,t} = [(\alpha-\lambda)^2.S_{t,\lambda}].S^*_{s,\lambda}$. Hence $(\alpha-\lambda)^2.T_{s,t} = (\alpha-\lambda)^2.S^*_{s-t,\lambda}$ for $s \ge t$. It follows then from Lemma 5.9(i) that $T_{s,t} - S^*_{s-t} \in [I_1(\lambda)]^\perp$. Hence $T_{s,t} - S^*_{s-t,\lambda} = \delta\varphi_\lambda$ for some $\delta \in \mathbb{C}$, where $\widehat{\varphi}_\lambda(n) = \lambda^{-n}$ $(n \in \mathbb{Z})$. Since $T_{s,t} - S^*_{s-t,\lambda} \in \mathrm{HD}^0(\Gamma)$, we obtain $T_{s,t} = S^*_{s,t}$ $(s \ge t)$. Similarly $T_{s,t} = 0$ if $t \ge s$.

2) Using the same method as in the proof of the lemma, it is easy to check that the map $(t,\lambda) \to S_{t,\lambda}$ is in fact w^*-continuous from $[0,\infty)\times\Gamma$ into $\mathrm{HD}_1(\Gamma)$.

Proposition 5.11. *Let $\lambda \in \Gamma$ and $t > 0$. Then*

$$\mathcal{T}_1(S^*_{s,\lambda}) = \overline{\operatorname{span}\{(S^*_{t,\lambda})_{0\le t\le s}\}}^{w^*}.$$

Proof. For $t \le s$ we have $(\alpha-\lambda)^2.S_{s-t}.S^*_s = (\alpha-\lambda)^2.S^*_t$. Hence if $f.S^*_s = 0$ with $f \in B_i$, then $(\alpha-\lambda)^2 f.S^*_t = 0$. It follows from Lemma 5.9(i) that $f.S^*_t \in [I_1(\lambda)]^\perp \cap \mathrm{HD}^0(\Gamma) = \{0\}$, and so $S^*_t \in \mathcal{T}_1(S^*_{s,\lambda})$ (this also follows from the results obtained at the beginning of this section).

It also follows from Lemma 5.9 that for $f \in B_1$ and $|u| > 1$ we have

$$\begin{aligned}\langle f, (u-\lambda)(\alpha-\lambda)^3(\alpha-u)^{-1}.S^*_{s,\lambda}\rangle &= 2\lambda\int_0^\infty e^{t(\lambda+u)/(\lambda-u)}\langle f,(\alpha-\lambda)^2.S_t.S^*_{s,\lambda}\rangle\,dt\\ &= 2\lambda\int_0^s e^{t(\lambda+u)/(\lambda-u)}\langle f,(\alpha-\lambda)^2.S^*_{s-t,\lambda}\rangle\,dt.\end{aligned}$$

There exists $p \ge 1$ such that $S^*_{t,\lambda} \in \mathrm{HD}_{\omega_p}(\Gamma)$ $(0 \le t \le s)$. The Bochner integral

$$\int_0^s e^{t(\lambda+u)/(\lambda-u)}.S^*_{s-t,\lambda}\,dt$$

defines a hyperdistribution $\varphi_u \in \mathrm{HD}^0_{\omega_p}(\Gamma)$, and we have

$$2\lambda(\alpha-\lambda)^2.\varphi_u = (u-\lambda)(\alpha-\lambda)^3(\alpha-u)^{-1}.S^*_{s,\lambda}.$$

By using Lemma 5.9(i) as above, we see that

$$2\lambda.\varphi_u = (u-\lambda)(\alpha-\lambda)(\alpha-u)^{-1}.S^*_{s,\lambda}.$$

Now let $f \in B_1$ be such that $\langle f, S^*_{t,\lambda}\rangle = 0$ $(0 \le t \le s)$. We obtain

$$\langle(\alpha-u)^{-1},(\alpha-\lambda)f.S^*_{s,\lambda}\rangle = 0 \quad (|u| > 1).$$

Hence $(\alpha-\lambda)f.S^*_{s,\lambda} = 0$, since $\operatorname{supp}S^*_{s,\lambda} = \{\lambda\} \subsetneq \Gamma$. It follows again from Lemma 5.9(i) that $f.S^*_{s,\lambda} \in [I_1(\lambda)]^\perp \cap \mathrm{HD}^0(\Gamma) = \{0\}$. This concludes the proof of the proposition.

Corollary 5.12. *Let I be a closed ideal of B_1. Then $f.S^* = 0$ for every $f \in I^{\mathrm{a}}$ and every $S \in k_{\mathrm{a}}(I)$.*

Proof. Let $S = S_{t,\lambda} \in k_{\mathrm{a}}(I)$. Then $g.S^* = 0$ for every $g \in I$, since I is an ideal of B_1, and so $g \in I_{1,S^*} = [\mathcal{T}_1(S^*)]^\perp$. So $S^*_{s,\lambda} \in \mathcal{T}_1(S^*) \cap \mathcal{S}^*_{\mathrm{a}}$, and $S_{s,\lambda} \in k_{\mathrm{a}}(I)$ for every $s \le t$.

Now if $f \in I^{\mathrm{a}}$, then $\langle f, S^*_{s,\lambda}\rangle = 0$ for $s \le t$. It follows from the proposition that $f \in [\mathcal{T}_1(S^*)]^\perp$, so that $f.S^* = 0$.

Let $S = S_{t,\lambda} \in \mathcal{S}_{\mathrm{a}}$ and let $f \in A^+$. Then $f.S^* = 0$ iff $f \in S.H^\infty(D)$. Since $(\alpha-\lambda)^2.\overline{S} \in A(\Gamma)$, and since $\alpha-\lambda$ is outer, it is easy to see that $f \in S.H^\infty(D)$ iff $(\alpha-\lambda)^4.f = (\alpha-\lambda)^2.S.g$ for some $g \in A^+$. The following lemma extends this criterion to the algebra B_1.

LEMMA 5.13. *Let $S \in \mathcal{S}_a$ and let $f \in B_1$. Then the following conditions are equivalent.*

(i) $f.S^* = 0$.

(ii) $(\alpha-\lambda)^4.f = (\alpha-\lambda)^2.S.g$ *for some* $g \in B_1$.

Proof. We can assume without loss of generality that $S = S_{t,1}$ for some $t > 0$. Assume that (i) holds, and set $h = f(\alpha-1)^2$, so that $h.S^* = 0$. Set $u_n = \widehat{S}^*(n)$ $(n \in \mathbb{Z})$, and set $g = h.\overline{S}$. Then $g \in \mathcal{C}(\Gamma)$. Since $(\alpha-1)^2.\overline{S} \in A(\Gamma)$ and $f \in B_1$, it follows that $g \in A(\Gamma)$.

We have $\widehat{\overline{S}}(k) = 0$ for $k > 0$. For $k < 0$ we have

$$\widehat{\overline{S}}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\overline{S(e^{it})}\,dt = \frac{1}{2i\pi}\int_\Gamma \zeta^{-k-1}\overline{S(\zeta)}\,d\zeta = \langle \alpha^{-k-1}, S^*\rangle = \widehat{S}^*(k+1) = u_{k+1}$$

(formula (3.5)). Set $a = \widehat{\overline{S}}(0)$.

For $n < 0$, we obtain

$$\widehat{g}(n) = \sum_{k\in\mathbb{Z}} \widehat{h}(k)\widehat{\overline{S}}(n-k) = a\widehat{h}(n) + \sum_{k=n+1}^{\infty} \widehat{h}(k)\widehat{\overline{S}}(n-k) = a\widehat{h}(n) + \sum_{k=n+1}^{\infty} \widehat{h}(k)u_{n-k+1}.$$

Since $h.S^* = 0$, we have

$$\sum_{k=n+1}^{\infty} \widehat{h}(k)u_{n-k+1} + \sum_{k\le n} \widehat{h}(k)u_{n-k+1} = \widehat{h.S^*}(n+1) = 0.$$

Hence

$$\widehat{g}(n) = a\widehat{h}(n) - \sum_{k\le n} \widehat{h}(k)u_{n-k+1}.$$

For $p \ge 1$, set as before $\omega_p(n) = 0$ $(n \ge 0)$ and $\omega_p(n) = e^{p\sqrt{|n|}}$ $(n < 0)$. There exists $q \ge 1$ such that $S^* \in \mathrm{HD}_{\omega_q}(\Gamma)$.

For $p \ge q$ there exists $K_p > 0$ such that

$$|u_{n-k+1}| \le K_p\,\omega_p(k-n-1) \quad (k \in \mathbb{Z}).$$

We have $\omega_p(n)\omega_p(m) \le \omega_{2p}(n+m)$ $(n \le 0,\ m \le 0)$. Hence

$$\Big|\sum_{k\le n} \widehat{h}(k)\widehat{S}^*(n-k-1)\Big| \le K_{p+1}\sum_{k\le n}|\widehat{h}(k)|\omega_{p+1}(k-n-1) \le K_{p+1}\omega_{p+1}(-1)\sum_{k\le n}|\widehat{h}(k)|\omega_{p+1}(k-n) \le \frac{K_{p+1}\omega_{p+1}(-1)}{\omega_{p+1}(n)}\sum_{k\le n}|\widehat{h}(k)|\omega_{2p+2}(k) = \frac{M_p}{\omega_{p+1}(n)} \quad (n<0),$$

where $M_p = K_{p+1}\omega_{p+1}(-1)\|h\|_{\omega_{2p+2}} < \infty$. Hence

$$\sum_{n<0}|\widehat{g}(n)|\omega_p(n) \le |a|\cdot\|h\|_{\omega_p} + M_p\sum_{n<0}\frac{\omega_p(n)}{\omega_{p+1}(n)} < \infty \quad (p \ge q).$$

But $g \in A(\Gamma)$, and so $g \in B_1$.

We have $f(\alpha-1)^4 = (\alpha-1)^2.S.g$, and so (ii) holds.

Now assume that (ii) holds. Then $(\alpha-\lambda)^4.f.S^* = (\alpha-\lambda)^2.g.S.S^* = 0$. It follows again from Lemma 5.9(i) that $f.S^* \in [I_1(\lambda)]^\perp$, so that $f.S^* = 0$, since $f.S^* \in \mathrm{HD}_1^0(\Gamma)$.

The following lemma is a reformulation of well-known results.

LEMMA 5.14. *Let $p \ge 1$ be an integer, and let $\lambda \in \Gamma$. Then there exists $q \ge 1$ such that $(\alpha-\lambda)^2.S_{q,\lambda} \in J_{\omega_p}(\lambda)$.*

Proof. Let $\pi : A_{\omega_p}(\Gamma) \to A_{\omega_p}(\Gamma)/J_{\omega_p}(\lambda)$ be the canonical map. Since $h(J_{\omega_p}(\lambda)) = \{\lambda\}$, it follows that $\mathrm{Sp}(\pi(\alpha)) = \{\lambda\}$. Let $T : A_{\omega_p}(\Gamma)/J_{\omega_p}(\lambda) \to A_{\omega_p}(\Gamma)/J_{\omega_p}(\lambda)$ be the map $\pi(f) \to \pi(\alpha f)$. Then $\mathrm{Sp}\,T = \mathrm{Sp}\,\pi(\alpha) = \{\lambda\}$, $\|T\| = \|\pi(\alpha)\| = 1$ and $\|T^{-n}\| = \|\pi(\alpha)^{-n}\| \le e^{p\sqrt{|n|}}$ $(n < 0)$. Set $f_t = (\alpha-\lambda)^2.S_{t,\lambda}$ for $t \ge 0$, so that $f_t \in A^+$. It follows from a result of Atzmon [1] that $f_q(T) = 0$ for some $q \ge 1$. Hence $\pi((\alpha-\lambda)^2.S_{q,\lambda}) = f_q(T)\pi(1) = 0$, which proves the lemma.

A way to obtain Atzmon's result consists in observing, by using Lemma 4.1 and a suitable application of the Phragmén–Lindelöf principle, that for every $p \ge 1$ there exists an integer $q \ge 1$ such that $(\alpha-\lambda)^2.S_{q,\lambda}.\varphi^+ \in H^\infty(D)$ for every $\varphi \in \mathrm{HD}_{\omega_p}(\lambda)$, so that $\varphi^+ \in \mathcal{N}$ [2, Proposition 2.6] and $\mathcal{S}(\varphi^+) = S_{s,\lambda}$ for some $s \le q$. Since $(\alpha-\lambda)^2.\varphi \in \mathrm{HD}_{\omega_p}^\infty(\Gamma) \subset \mathrm{HD}_{\omega_p}^2(\Gamma)$ (use Lemmas 4.9 and 4.10), it follows from Theorem 5.4 that $(\alpha-\lambda)^4.S_{q,\lambda}.\varphi = 0$, hence $(\alpha-\lambda)^2.S_{q,\lambda}.\varphi = 0$ for every $\varphi \in \mathrm{HD}_{\omega_p}(\lambda)$. Another approach consists in applying the usual Paley–Wiener Theorem [24, Theorem 19.2] to the entire vector-valued map $z \to [(z+1)\pi(\alpha) - \lambda(z-1)]^{-1}$, by using the formula of Theorem 5.9(v) with $z = (\lambda+u)/(\lambda-u)$. The details can be found in [7, pp. 150–158]. Related results were obtained by Gurariĭ [14].

We now deduce from Lemma 5.14 a complete classification of closed ideals I of B_1 such that $h(I)=\{\lambda\}$.

COROLLARY 5.15. *Let I be a closed ideal of B_1. For $\lambda\in\Gamma$, set $m_I(\lambda)=\sup\{t\ge 0 \mid S^*_{t,\lambda}\in I^\perp\}$, so that $S^*_{s,\lambda}\in I^\perp$ for every $s\le m_I(\lambda)$. Then the following classification holds in the case where $h(I)=\{\lambda\}$.*

(1) *If $m_I(\lambda)=0$, then $I=I_1(\lambda)$.*
(2) *If $0<m_I(\lambda)<\infty$, then $I=\{f\in B_1 \mid f.S^*_{m_I(\lambda),\lambda}=0\}$.*
(3) *If $m_I(\lambda)=\infty$, then $I=J_1(\lambda)$.*

Proof. The fact that $S^*_{s,\lambda}\in I^\perp$ for every $s\le m_I(\lambda)$ follows from Proposition 5.11 and Lemma 5.9(iii). Now assume that $h(I)=\{\lambda\}$, and that $m_I(\lambda)=\infty$. Let $f\in I$, and let $p\ge 1$ be an integer. There exists $q\ge 1$ such that $(\alpha-\lambda)^2.S_{q,\lambda}\in J_{\omega_p}(\lambda)$. Since $f.S^*_{q,\lambda}=0$, it follows from Lemma 5.13 that $(\alpha-\lambda)^4.f=(\alpha-\lambda)^2S_{q,\lambda}.g$ for some $g\in B_1$. Hence $f\in J_{\omega_p}(\lambda)$ $(p\ge 1)$ and it follows from (2.11) that $f\in J_1(\lambda)$. So $I\subset J_1(\lambda)$, and in fact $I=J_1(\lambda)$ since $h(I)=\{\lambda\}$.

Now assume that $\delta=m_I(\lambda)<\infty$. If $f\in I$, then $f.S^*_{\delta,\lambda}=0$. Conversely, assume that $f.S^*_{\delta,\lambda}=0$ if $\delta>0$. If $\delta=0$, assume that $f\in I_1(\lambda)$ (the condition $f.S^*_{\delta,\lambda}=0$ implies that $f\in I_1(\lambda)$ if $\delta>0$ since in this case $h(I_{1,S^*_{\delta,\lambda}})=\{\lambda\}$). Let $\varphi\in I^\perp$. Since $J_1(\lambda)\subset I$ we have $\operatorname{supp}\varphi\subset\{\lambda\}$. Set $\psi=(\alpha-\lambda)^2.\varphi$. It follows from Lemma 4.11 that $\psi\in \mathrm{HD}^2_1(\lambda)$. Also it follows from Lemma 5.14 and Theorem 5.4 that $\psi^+\in\mathcal N$ (this follows also from [2, Proposition 2.6]). Set $S=S(\psi^+)$. It follows from Lemma 5.2 that $S=S_{t,\lambda}$ for some $t\ge 0$, and it follows from Theorem 5.4 that $\mathcal T_1(\psi)=\mathcal T_1(S^*)$. Hence $S^*\in I^\perp$, and $t\le\delta$.

There exists $g\in B_1$ such that $(\alpha-\lambda)^4.f=(\alpha-\lambda)^2.S_{\delta,\lambda}.g$, and so $(\alpha-\lambda)^4.f.S^*=g(\alpha-\lambda)^2.S_{\delta,\lambda}.S^*_{t,\lambda}=0$. Hence $(\alpha-\lambda)^6.f.\varphi=(\alpha-\lambda)^4.f.\psi=0$, since $\mathcal T_i(\psi)=\mathcal T_i(S^*)$. But $f\in I_1(\lambda)$, and it follows then from Lemma 5.9(i) that $f.\varphi=0$. So $f\in I$, which concludes the proof of the corollary.

The following lemma plays a crucial role in the description of closed ideals I of B_1 such that $h(I)$ is countable. It shows that $J_1(\lambda)$ enjoys a property analogous to the classical Ditkin condition for regular Banach algebras [19].

LEMMA 5.16. *For every $f\in J_1(\lambda)$, there exists a sequence $(e_p)_{p\ge1}$ of elements of $J_1(\lambda)$ such that $f=\lim_{p\to\infty} f\,e_p$.*

Proof. Fix $p\ge 1$. There exists $q\ge 1$ such that $(\alpha-\lambda)^2.S_{q,\lambda}\in J_{\omega_p}(\lambda)$. Since $J_1(\lambda)$ is dense in $(J_{\omega_p}(\lambda),\|\cdot\|_{\omega_p})$ (see Section 2), there exists a sequence $(u_k)_{k\ge1}$ of elements of $J_1(\lambda)$ such that $\|(\alpha-\lambda)^2.S_{q,\lambda}-u_k\|_{\omega_p}\to 0$ as $k\to\infty$, so that

$$\|(\alpha-\lambda)^8.S_{q,\lambda}-(\alpha-\lambda)^6.u_k\|_{\omega_p}\xrightarrow[k\to\infty]{}0.$$

Since $u_k\in J_1(\lambda)$, we have $u_k.S^*_{2q,\lambda}=0$, and there exists $v_k\in B_1$ such that $(\alpha-\lambda)^2.S_{2q,\lambda}.v_k=(\alpha-\lambda)^4.u_k$.

We obtain

$$(\alpha-\lambda)^6.u_k=[(\alpha-\lambda)^2S_{q,\lambda}.v_k].(\alpha-\lambda)^2.S_{q,\lambda}.$$

We have $(\alpha-\lambda)^2.S_{q,\lambda}.v_k\in J_{\omega_p}(\lambda)$, and so there exists $w_k\in J_1(\lambda)$ such that $\|(\alpha-\lambda)^2.S_{q,\lambda}.v_k-w_k\|_{\omega_p}<1/k$. Hence

$$\|(\alpha-\lambda)^6.u_k-(\alpha-\lambda)^2.S_{q,\lambda}.w_k\|_{\omega_p}\le\frac{\|(\alpha-\lambda)^2.S_{q,\lambda}\|_1}{k}.$$

So $\|(\alpha-\lambda)^8.S_{q,\lambda}-(\alpha-\lambda)^2.S_{q,\lambda}.w_k\|_{\omega_p}\to 0$ as $k\to\infty$.

Let $f\in J_1(\lambda)\subset I_1(\lambda)$. It follows from Lemma 5.9 that

$$\|f-f(\alpha-\lambda)^{10}.(\alpha-\lambda-\lambda/n)^{-10}\|_{\omega_p}\xrightarrow[n\to\infty]{}0.$$

Choose $n\ge 1$ such that

$$\|f-f(\alpha-\lambda)^{10}.(\alpha-\lambda-\lambda/n)^{-10}\|_{\omega_p}<\frac{1}{2p}.$$

Since $f\in J_1(\lambda)$, we have $f.S^*_{q,\lambda}=0$ and it follows from Lemma 5.13 that

$$(\alpha-\lambda)^4.f=(\alpha-\lambda)^2.S_{q,\lambda}.g\quad\text{ for some } g\in B_1.$$

Hence

$$(\alpha-\lambda)^{10}.(\alpha-\lambda-\lambda/n)^{-10}f=(\alpha-\lambda)^8.S_{q,\lambda}.g(\alpha-\lambda-\lambda/n)^{-10},$$

and

$$\begin{aligned}&\|(\alpha-\lambda)^{10}.(\alpha-\lambda-\lambda/n)^{-10}.f-(\alpha-\lambda)^4(\alpha-\lambda-\lambda/n)^{-10}.f.w_k\|_{\omega_p}\\&\quad\le\|g.(\alpha-\lambda-\lambda/n)^{-10}\|_{\omega_p}\|(\alpha-\lambda)^8.S_{q,\lambda}-(\alpha-\lambda)^2.S_{q,\lambda}.w_k\|_{\omega_p}\xrightarrow[k\to\infty]{}0.\end{aligned}$$

If we set $e_p=(\alpha-\lambda)^4(\alpha-\lambda-\lambda/n)^{-10}.w_k$ with k sufficiently large, we obtain $\|f-fe_p\|_{\omega_p}<1/p$. Hence $f=\lim_{p\to\infty}fe_p$ with respect to the topology of B_1. Since $e_p\in J_1(\lambda)$ $(p\ge 1)$, this concludes the proof of the lemma.

THEOREM 5.17. *Let I be a closed ideal of B_1. If $h(I)$ is countable, then $I^0=I^{\mathrm a}$, and $I=I_1(h(I))\cap I^{\mathrm a}$.*

Proof. Let $f\in I^{\mathrm a}$ and let $J=\{g\in B_1 \mid f.g\in I^0\}$. Assume, if possible, that $h(J)$ has an isolated point λ. Since B_1 is a regular algebra (see Section 2), there exists $u\in B_1$ such that $\lambda\notin\operatorname{supp}(1-u)$ and such that $\operatorname{supp}u\cap[h(J)\setminus\{\lambda\}]=\emptyset$. Then $u(1-u)\in J_1(h(I))$, and so $u(1-u)\in J$.

Let $H=\{g\in B_1 \mid gu\in J\}=\{g\in B_1 \mid guf\in I^0\}$. Then $h(H)\subset h(J)$ and $1-u\in J$, so that $h(H)\subset\{\lambda\}$.

If $m_I(\lambda)=\infty$, then $S_{t,\lambda}\in k_{\mathrm a}(I)$ and so $f.S^*_{t,\lambda}=0$ for every $t>0$ (Corollary 5.12). It follows then from Corollary 5.15 that $f\in J_1(\lambda)$, and it follows from Lemma 5.16 that there exists a sequence $(e_p)_{p\ge1}$ of elements of $J_1(\lambda)$ such that $f = \lim_{p\to\infty} fe_p$. Since $h(H) \subset \{\lambda\}$, we have $J_1(\lambda) \subset H$ and so $e_p uf \in I^0$ $(p \ge 1)$. So $uf \in I^0$.

If $m_I(\lambda) < \infty$, set $\delta = m_I(\lambda)$ and let $\varepsilon > 0$. There exists $g \in I$ such that $g.S^*_{\delta+\varepsilon,\lambda} \neq 0$. We have $g.S^*_{\delta,\lambda} = 0$, and $f.S^*_{\delta,\lambda} = 0$ since $f \in I^{\mathrm{a}}$. So there exist $h, l \in B_1$ such that $(\alpha-\lambda)^4 f = (\alpha-\lambda)^2 S_{\delta,\lambda}.h$ and $(\alpha-\lambda)^4 g = (\alpha-\lambda)^2.S_{\delta,\lambda}.l$. We have $(\alpha-\lambda)^4.l.f = (\alpha-\lambda)^2.S_{\delta,\lambda}.h.l = (\alpha-\lambda)^4 hg$. Hence $(\alpha-\lambda)^4.l \in J \supset H$.

We have $\lambda \in h(J) \subset h(I)$, and it follows from Lemma 5.9(i) that

$$g = \lim_{n\to\infty} (\alpha-\lambda)^6(\alpha-\lambda-\lambda/n)^{-6}.$$

So

$$l.(\alpha-\lambda)^4.S^*_{\varepsilon,\lambda} = l.(\alpha-\lambda)^4.S_{\delta,\lambda}.S^*_{\delta+\varepsilon,\lambda} = (\alpha-\lambda)^6.g.S^*_{\delta+\varepsilon} \neq 0.$$

Hence $m_H(\lambda) = 0$, and it follows from Corollary 5.15 that $I_1(\lambda) \subset H$.

Now let $\varphi \in (I^\perp)_0 = I^\perp \cap \mathrm{HD}_1^0(\Gamma)$ and set $K = \{v \in A^+ \mid v.u.f.\varphi = 0\}$. Since $u.f.\varphi \in \mathrm{HD}_1^0(\Gamma)$, K is w^*-closed in A^+. But $I^+(\lambda) \subset K$ and the sequence $((\alpha-\lambda)(\alpha-\lambda-\lambda/n)^{-1})_{n\ge1}$ is w^*-convergent to 1 in A^+. Hence $1 \in K$, $uf.\varphi = 0$ and $uf \in I^0$. Since $u(\lambda) = 1$, this contradicts the fact that $\lambda \in h(J)$. We see that $h(J)$ is a countable closed set without isolated points. So $h(J) = \emptyset$, $J = B_1$, and $f = 1 \cdot f \in I^0$. This shows that $I^0 = I^{\mathrm{a}}$. It then follows from Corollary 2.27 that $I = I^A \cap I^{\mathrm{a}}$. But countable sets are sets of synthesis, and so $I^A = I_1(h(I))$ and $I = I_1(h(I)) \cap I^{\mathrm{a}}$, which concludes the proof of the theorem.

Corollary 5.18. *Let $\varphi \in \mathrm{HD}_1^0(\Gamma)$. If $\operatorname{supp}\varphi$ is countable, then*

$$\mathcal{T}_1(\varphi) = \overline{\operatorname{span}(\mathcal{T}_1(\varphi) \cap \mathcal{S}^*_{\mathrm{a}})}^{w^*}.$$

Proof. Denote by U the w^*-closure of $\operatorname{span}(\mathcal{T}_1(\varphi) \cap \mathcal{S}^*_{\mathrm{a}})$. We have $U \subset \mathcal{T}_1(\varphi)$. Conversely, let $f \in U^\perp$, and set $I = I_{1,\varphi} = [\mathcal{T}_1(\varphi)]^\perp$. Then $I = \{g \in B_1 \mid g.\varphi = 0\}$ (see Remark 4.15) and so $I = I^0$. But $\mathcal{T}_1(\varphi) \cap \mathcal{S}^*_{\mathrm{a}} = k_{\mathrm{a}}(I)^*$, the notations being as in Definition 5.8, and so $f \in I^{\mathrm{a}} = I^0 = I = [\mathcal{T}_1(\varphi)]^\perp$. This shows that $\mathcal{T}_1(\varphi) = U$.

We conclude the paper by a few remarks.

Remark 5.19. 1) If $\varphi \in \mathrm{HD}_1(\Gamma)$, and if $\operatorname{supp}\varphi$ is countable, it follows from the theorem that $\mathcal{T}_1(\varphi)$ is the w^*-closure of $\operatorname{span}(\mathcal{T}_1(\varphi) \cap (\mathcal{S}^*_{\mathrm{a}} \cup h(I)^*))$, where we denote by $h(I)^*$ the set $\{\varphi_\lambda\}_{\lambda\in h(I)}$ with $\widehat{\varphi}_\lambda(n) = \lambda^{-n}$ $(n \in \mathbb{Z})$ (so that $\langle f, \varphi_\lambda\rangle = f(\lambda)$ for $f \in B_1$). In other words, elements of $\mathrm{HD}_1(\Gamma)$ with countable support can be synthetized by "inner functions and characters".

2) Let I be a closed ideal of B_1, and define $m_I(\lambda)$ as in Corollary 5.15 for $\lambda \in \Gamma$. For $f \in B_1$ set $L_f = \{(t,\lambda) \in [0,\infty) \times \Gamma \mid f.S^*_{t,\lambda} = 0\}$. It follows from Remark 5.10(2) that L_f is closed. Also it follows from the first assertion of Corollary 5.15 that $\{\lambda \in \Gamma \mid m_I(\lambda) \ge t_0\} = \bigcap_{f\in I}\{\lambda \in \Gamma \mid (t_0,\lambda) \in L_f\}$ for $t_0 \ge 0$. So the set $\{\lambda \in \Gamma \mid m_I(\lambda) \ge t_0\}$ is closed, which shows that the map $\lambda \to m_I(\lambda)$ is upper semicontinuous on Γ [24, p. 37].

3) Let $\varphi \in \mathrm{HD}_1(\Gamma)$, and set $m_\varphi(\lambda) = m_I(\lambda)$ $(\lambda \in \Gamma)$, where $I = I_{1,\varphi} = \{f \in B_1 \mid f.\varphi = 0\}$. Let $\lambda \in \Gamma \setminus \operatorname{supp}\varphi$. Since B_1 is regular, there exists $u \in B_1$ such that $u(\lambda) = 1$ and $\operatorname{supp} u \cap \operatorname{supp}\varphi = \emptyset$. Now if $t > 0$ and if $S = S_{t,\lambda}$ then $I_{1,S} \subsetneq B_1$, and so $h(I_{1,S}) = \{\lambda\}$. Hence $u.S \neq 0$. Since $u.\varphi = 0$, we see that $m_\varphi(\lambda) = 0$. Hence m_φ is concentrated on $\operatorname{supp}\varphi$. Now if L is an open arc contained in $\operatorname{supp}\varphi$, let $f \in B_1$ such that $f.\varphi = 0$. Set $E = \{\lambda \in \Gamma \mid f(\lambda) = 0\}$. Since B_1 is regular, $J_1(E) \subset \overline{f.B_1}$ by (2.12), and so $\varphi \in J_1(E)^\perp = \mathrm{HD}_1(E)$, so that $L \subset E$. In particular, $f \in J_1(\lambda)$, so that $f.S^*_{t,\lambda} = 0$ for every $t > 0$ and for every $\lambda \in L$. We thus see that $m_\varphi(\lambda) = 0$ for $\lambda \in \Gamma \setminus \operatorname{supp}\varphi$, and that $m_\varphi(\lambda) = \infty$ for every interior point of $\operatorname{supp}\varphi$.

In particular, we can consider $\varphi = \sum_{n=1}^\infty 2^{-n}\delta_{z_n}$, where z_n is a dense sequence in Γ and where δ_{z_n} is the Dirac measure at z_n. It is easily checked that $\operatorname{supp}\varphi = \Gamma$ and so $m_I(\lambda) = \infty$ for every $\lambda \in \Gamma$, despite the fact that $|\varphi^+(z)| = O(1/(1-|z|))$ as $|z| \to 1^-$, so that $(1-|z|)\log|\varphi^+(z)| \to 0$ as $|z| \to 1^-$.

In the other direction, assume that $\operatorname{supp}\varphi = \{\lambda\}$. Then $\varphi^+ \in \mathcal{N}$, and it follows from Lemma 5.2 that $S(\varphi^+) = S_{t,\lambda}$ for some $t \ge 0$.

The computations on Poisson kernels given in the proof of Lemma 5.2 show in fact that

$$t = \tfrac{1}{2}\limsup_{r\to1^-}(1-r)\log|\varphi(r\lambda)|.$$

Set $\psi = (\alpha-\lambda)^2.\varphi$. If $f.\varphi = 0$, then $f.\psi = 0$. Conversely, if $f.\psi = 0$, and if $\psi \neq 0$, then $f(\lambda) = 0$ so that $f = \lim_{n\to\infty}(\alpha-\lambda)^2(\alpha-\lambda-\lambda/n)^{-2} f$ and $f.\varphi = 0$. Also it follows from the results of Section 4 that $S(\varphi^+) = S(\psi^+)$, since $\psi^+ - (\alpha-\lambda)^2.\varphi^+ \in H^\infty(D)$ and since $(\alpha-\lambda)^2$ is outer. Since $\mathcal{T}_1(\psi) = \mathcal{T}_1(S(\psi^+)^*)$ (Theorem 5.4) there are two possibilities:

(a) $t > 0$. In this case $\psi \neq 0$, $\mathcal{T}_1(\varphi) = \mathcal{T}_1(S^*_{t,\lambda})$ and $I_{1,\varphi} = I_{1,\psi} = \{f \in B_1 \mid (\alpha-\lambda)^4.f \in (\alpha-\lambda)^2.S_{t,\lambda}.B_1\}$. Then $\mathcal{T}_1(\varphi) \cap \mathcal{S}^*_{\mathrm{a}} = \{S^*_{s,\lambda}\}_{0\le s\le t}$ and $m_\varphi(\lambda) = t$.

(b) $t = 0$. In this case $\psi = 0$, and $\varphi \perp I_1(\lambda)$ (use Lemma 5.9(i)). Then $\varphi = c\varphi_\lambda$ for some $c \neq 0$, and $I_{1,\varphi} = I_1(\lambda)$. In this case, we obtain again $m_\varphi(\lambda) = 0 = t$.

So if $\operatorname{supp}\varphi = \{\lambda\}$, then

$$m_\varphi(\lambda) = \tfrac{1}{2}\limsup_{r\to1^-}(1-r)\log|\varphi(r\lambda)|.$$

Now assume that λ is an isolated point of $\operatorname{supp}\varphi$ and let $u \in B_1$ be such that $[\operatorname{supp}\varphi \setminus \{\lambda\}] \cap \operatorname{supp} u = \emptyset$ and $\lambda \notin \operatorname{supp}(1-u)$. Set $\varphi_1 = u.\varphi$ and $\varphi_2 = (1-u).\varphi$ so that $\varphi = \varphi_1 + \varphi_2$. We have $\mathcal{T}_1(\varphi_1) \subset \mathcal{T}_1(\varphi)$. Now if

$S^*_{t,\lambda} \in \mathcal{T}_1(\varphi)$, let $f \in I_{1,\varphi_1}$. We have $(1-u)S^*_{t,\lambda} = 0$ so that $f.S^*_{t,\lambda} = fu.S^*_{t,\lambda}$. But $fu \in I_{1,\varphi}$ and so $f.S^*_{t,\lambda} = 0$. This shows that $m_\varphi(\lambda) = m_{\varphi_1}(\lambda)$. Also $u\varphi = \varphi - (1-u)\varphi$, and $\lambda \notin \operatorname{supp}(1-u)\varphi$. Hence

$$\limsup_{r\to 1^-}(1-r)\log|\varphi(r\lambda)| = \limsup_{r\to 1^-}(1-r)\log|\varphi_1(r\lambda)| = m_\varphi(\lambda).$$

So we have the following situation:

(a) If $\lambda \notin \operatorname{supp}\varphi$, then $m_\varphi(\lambda) = 0$.
(b) If λ is an interior point of $\operatorname{supp}\varphi$, then $m_\varphi(\lambda) = \infty$.
(c) If λ is an isolated point of $\operatorname{supp}\varphi$ then

$$m_\varphi(\lambda) = \tfrac{1}{2}\limsup_{r\to 1^-}(1-r)\log|\varphi(r\lambda)|.$$

One would like to get more information, in particular in the case where $\operatorname{supp}\varphi$ is countable. Also it would be probably interesting to get more "concrete" versions of Theorem 5.17 and Corollary 5.18, in particular when $h(I)$, or $\operatorname{supp}\varphi$, is given by a convergent sequence.

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