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Absolute continuity for elliptic-caloric measures

by

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Abstract. A Carleson condition on the difference function for the coefficients of two elliptic-caloric operators is shown to give absolute continuity of one measure with respect to the other on the lateral boundary. The elliptic operators can have time dependent coefficients and only one of them is assumed to have a measure which is doubling. This theorem is an extension of a result of B. Dahlberg [4] on absolute continuity for elliptic measures to the case of the heat equation. The method of proof is an adaptation of Fefferman, Kenig and Pipher's proof of Dahlberg's result [8].

The purpose of this paper is to prove the following Theorem 1 on conditions for elliptic-caloric measures to be absolutely continuous on the lateral boundary of a cylinder domain in $\mathbb{R}^{n+1}$. After determining that B. Dahlberg's condition for two elliptic measures to be absolutely continuous [4] adapted readily to elliptic-caloric measures under the conditions of Theorem 1, it seemed highly probable that the same result should be valid on the entire parabolic boundary of a cylinder domain. This extension of Theorem 1 is indeed true: A Carleson-type condition can be defined across the bottom of the cylinder, as well as on the side, for the coefficients of two operators L_0, L_1 (thus obviating the condition $a_{ij} = b_{ij}$ if $t \le \delta_0^2$) and again using the method of proof of Fefferman, Kenig and Pipher [8], the absolute continuity of the associated measures can be deduced on the whole boundary. This extension of Theorem 1 appears in a later paper, along with the technical adjustments to its proof [12].

Also the center-doubling condition assumed for the measure ω_0 in Theorem 1 needs some comment. A measure ω_L satisfies a *center-doubling condition* if

$$\omega_L(\Delta_{2r}(Q,s)) \le C\,\omega_L(\Delta_r(Q,s)) \quad \text{for all } r \le r_0,\ \Delta_r(Q,s) \subseteq \partial_{\mathrm{p}}^{+} D_T,$$

where

$$\Delta_r(Q,s) = \{(x,t) \mid |x-Q| < r,\ |t-s| < r^2\} \cap \partial_{\mathrm{p}} D_T,$$

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and C is independent of r and (Q,s). The doubling condition for an elliptic-caloric measure has been shown to be crucial [7] for proving the existence and uniqueness of a kernel function, geometric decay for the kernel and the comparison of the non-tangential maximal function of a solution with the Hardy–Littlewood maximal function (see also [9]).

Since this paper was written I have been informed that M. Safonov has shown all elliptic-parabolic measures to be center-doubling measures. If his result holds for the measures associated with operators $\partial/\partial t - L$ where L's coefficients are only assumed to be bounded and measurable, then the assumption in Theorem 1 of ω_0 being a doubling measure is, of course, unnecessary. Also since ω_1 would automatically be doubling, the conclusion of the theorem is strengthened: the two measures are A^∞ with respect to each other. The A^∞ condition is stronger than absolute continuity. (See the end of Section 1 for what follows *only* from the Carleson condition.) It may be of some interest, however, that the proof of Theorem 1 only uses the doubling condition for *one* measure.

1. Background. Let $D_T = B_1(0) \times [0,T]$ be a domain in $\mathbb{R}^{n+1}$, where $B_1(0)$ is the unit ball in $\mathbb{R}^n$. Let $\partial/\partial t - L_0$ and $\partial/\partial t - L_1$ be two operators, where

$$L_0 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x,t)\frac{\partial}{\partial x_j}\right], \qquad L_1 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[b_{ij}(x,t)\frac{\partial}{\partial x_j}\right]$$

are strictly elliptic operators in divergence form with time dependent coefficients and ellipticity constant λ. Denote by u_i solutions of

$$(\partial/\partial t - L_i)u_i = 0 \quad \text{in } D_T, \qquad u_i|_{\partial_p D_T} = f, \quad i = 0,1,$$

and by ω_0 and ω_1 the associated caloric measures. Fix $(X_0,T) \in D_T$ and take $\omega_i = \omega_i^{(X_0,T)}$. The Green's functions are $G_0(x,t;y,s)$ and $G_1(x,t;y,s)$.

Here u_0 and u_1 are weak solutions; they lie in the Banach spaces $L^2[0,T; W^{1,2}(D)] \cap L^\infty[0,T; L^2(D)]$, so that $\nabla_y u_i$ and dF/dt etc. exist as distributions. Almost all arguments involving these functions as Sobolev space functions have been omitted; expressions such as L_0F and $\operatorname{div}([\varepsilon_{ij}]\nabla u_1)$ are to be understood in the appropriate sense, using integration by parts formulas and/or as limits of smooth approximations in the local L^2 norms.

Set

$$\varepsilon_{ij}(x,t) = a_{ij}(x,t) - b_{ij}(x,t),$$
$$\varepsilon(x,t) = \sup_{ij}|a_{ij}(x,t) - b_{ij}(x,t)|, \qquad a(y,s) = \sup_{(x,t)\in P_{\delta(y,s)/2}(y,s)} |\varepsilon(x,t)|,$$

where $d(x,t;y,s) = |x-y| + |t-s|^{1/2}$ is a parabolic metric, $|x-y|$ is the Euclidean metric in $\mathbb{R}^n$ and

$$\partial_p D_T = \{(x,t) \mid x \in \partial D \text{ and } t > 0\} \cup \{(x,t) \mid x \in D \text{ and } t = 0\} = \partial_p^+ D_T \cup (D \times \{0\}),$$
$$\delta(x,t) = d(x,t;\partial_p D_T),$$
$$P_r(y,s) = \{(x,t) \mid |x_i - y_i| < r,\ |s-t| < r^2\}.$$

If $(Q,s) \in \partial_p D_T$ then we set $\Delta_r(Q,s) = \partial_p D_T \cap P_r(Q,s)$ and

$$\overline{A}_r(Q,s) = (Q+r, s+2r^2), \qquad \underline{A}_r(Q,s) = (Q+r, s-2r^2).$$

For the purpose of localization take $r_0 > 0$ fixed and define

$$\Gamma_\alpha(Q,s) = \{(y,t) \mid d(y,t;Q,s) < \alpha\delta(y,t),\ |y-Q| \le r_0,\ |s-t| \le r_0^2\},$$
$$F(x,t) = u_1(x,t) - u_0(x,t), \qquad N_\alpha(u)(Q,s) = \sup_{(x,t)\in\Gamma_\alpha(Q,s)} |u(x,t)|.$$

Then

$$\widetilde{N}_\alpha(F)(Q,s) = \sup_{(x,t)\in\Gamma_\alpha(Q,s)} \left(\frac{1}{|P_{\delta(x,t)/4}(x,t)|} \int_{P_{\delta(x,t)/4}(x,t)} |F(y,\tau)|^2\,dy\,d\tau\right)^{1/2},$$

$$M_{\omega_0}(V)(x,t) = \sup_{(x,t)\in\Delta_r} \frac{1}{\omega_0(\Delta_r)} \int_{\Delta_r} |V(y,s)|\,d\omega_0(y,s)$$

and

$$S_\alpha(u)(Q,s) = \Big(\int_{\Gamma_\alpha(Q,s)} |\nabla u(x,t)|^2 \delta(x,t)^{-n}\,dx\,dt\Big)^{1/2}$$

are the averaged non-tangential maximal function, the Hardy–Littlewood maximal function and the Lusin area integral.

Let $\Gamma(Q,s) = \Gamma_1(Q,s)$, $S(u)(Q,s) = S_1(u)(Q,s)$, $\widetilde{N}(F) = \widetilde{N}_1(F)$, etc.

THEOREM 1 [8, 10]. *With L_1, L_0 and $a(x,t)$ as above, assume that $a_{ij}(x,t) = b_{ij}(x,t)$ for $t \le \delta_0^2$, $x \in D$, and there is a constant $\varepsilon_0 > 0$, where ε_0 can be taken sufficiently small, so that for $r \le r_0$,*

$$\text{(C)} \qquad \sup_{\Delta_r(Q,s)\subseteq\partial_p D_T} \left(\frac{1}{\omega_0(\Delta_r(Q,s))} \times \int_{P_r(Q,s)\cap D_T} a(x,t)^2 \frac{G_0(X_0,T;x,t)}{\delta(x,t)^2}\,dx\,dt\right)^{1/2} \le \varepsilon_0.$$

Then if ω_0 satisfies a doubling condition, then ω_1 is absolutely continuous with respect to ω_0 on $\partial_p^+ D_T$ (1).

(1) The condition that $a_{ij}(x,t) = b_{ij}(x,t)$ for $t \le \delta_0^2$ can be removed and a similar result proved on the entire parabolic boundary [12].

Proof. The theorem is proved by obtaining the inequality

$$\text{(D)}\qquad \|N(u_1)\|_{L^2(d\omega_0,\partial_p^+ D_T)} \le C\|f\|_{L^2(d\omega_0,\partial_p D_T)}$$

(see Section 2). Then the absolute continuity of ω_1 with respect to ω_0 follows from writing

$$\begin{aligned}
&\|N(u_1)\|_{L^2(d\omega_0,\partial_p^+ D_T)}\\
&\quad= \int_{\partial_p^+ D_T} [\sup_{(x,t)\in\Gamma(Q,s)} |u_1(x,t)|]^2 d\omega_0^{(X_0,T)}(Q,s)\\
&\quad= \int_{\partial_p^+ D_T} \Big[\sup_{(x,t)\in\Gamma(Q,s)} \int_{\partial_p D_T} f(\widehat{Q},\widehat{s})\, d\omega_1^{(x,t)}(\widehat{Q},\widehat{s})\Big]^2 d\omega_0^{(X_0,T)}(Q,s)\\
&\quad\ge \int_{\Delta_r(Q_0,s_0)} \Big[\sup_{(x,t)\in\Gamma(Q,s)} \int_{\partial_p D_T\cap\Delta_r(Q_0,s_0)} f(\widehat{Q},\widehat{s})\\
&\qquad\times \frac{d\omega_1^{(x,t)}}{d\omega_1^{(X_0,T)}}(\widehat{Q},\widehat{s})\, d\omega_1^{(X_0,T)}(\widehat{Q},\widehat{s})\Big]^2 d\omega_0^{(X_0,T)}(Q,s) = I,
\end{aligned}$$

for any $\Delta_r(Q_0,s_0)\subseteq\partial_p^+ D_T$, where $d\omega_1^{(x,t)}/d\omega_1^{(X_0,T)}$ is the Radon–Nikodym derivative. This exists by results of Besicovitch [2] even though the kernel function for $\partial/\partial t - L_1$ may not be uniquely defined unless ω_1 is known to be doubling.

For $(Q,s)\in\Delta_r(Q_0,s_0)$ and $\Gamma=\Gamma_\alpha$ a cone of sufficiently wide aperture one can pick $(x,t)\sim\bar{A}_r(Q_0,s_0)$ so that $(x,t)\in\bigcap_{(Q,s)\in\Delta_r(Q_0,s_0)}\Gamma_\alpha(Q,s)$. The parabolic measure ω_1 is an additive set function which satisfies the necessary conditions to obtain the existence of the Radon–Nikodym derivative $(d\omega_1^{(x,t)}/d\omega_1^{(X_0,T)})(Q,s)$ as the limit of ratios of measures of boundary disks

$$\lim_{\varepsilon\downarrow 0}\frac{\omega_1^{(x,t)}(\Delta_\varepsilon(Q,s))}{\omega_1^{(X_0,T)}(\Delta_\varepsilon(Q,s))}.$$

These conditions were first established by Besicovitch [2]. The fact that Harnack's inequality holds for $\omega_1^{(x,t)}(\Delta_r)$ means that $\omega_1^{(x,t)}(\cdot)$ and $\omega_1^{(X_0,T)}(\cdot)$ are absolutely continuous with respect to each other, so

$$\frac{d\omega_1^{(x,t)}}{d\omega_1^{(X_0,T)}}(\widehat{Q},\widehat{s}) = \lim_{\varepsilon\to 0}\frac{\omega_1^{(x,t)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}{\omega_1^{(X_0,T)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}$$

and for some $\varepsilon>0$,

$$\frac{d\omega_1^{(x,t)}}{d\omega_1^{(X_0,T)}}(\widehat{Q},\widehat{s}) \ge C\frac{\omega_1^{(x,t)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}{\omega_1^{(X_0,T)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}.$$

Hence for any points $(\widehat{Q},\widehat{s}),(Q,s)\in\Delta_r(Q_0,s_0)$,

$$\begin{aligned}
\sup_{(x,t)\in\Gamma_\alpha(Q,s)}\frac{d\omega_1^{(x,t)}}{d\omega_1^{(X_0,T)}}(\widehat{Q},\widehat{s}) &\ge C\frac{\omega_1^{\bar{A}_r(Q_0,s_0)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}{\omega_1^{(X_0,T)}(\Delta_\varepsilon(\widehat{Q},\widehat{s}))}\\
&\ge \frac{C}{\omega_1^{(X_0,T)}(\Delta_{4r}(Q_0,s_0))}.
\end{aligned}$$

The second inequality follows by applying Corollary 1.2 to Theorem 1.1 in Fabes, Garofalo and Salsa [7].

For E any measurable subset of $\Delta_r(Q_0,s_0)$ this gives

$$\begin{aligned}
I \ge &\int_{\Delta_r(Q_0,s_0)} \Big[\int_{\Delta_r(Q_0,s_0)} f(\widehat{Q},\widehat{s})\\
&\times\frac{C}{\omega_1^{X_0,T}(\Delta_{4r}(Q_0,s_0))}\, d\omega_1^{X_0,T}(\widehat{Q},\widehat{s})\Big]^2 d\omega_0^{(X_0,T)}(Q,s)\\
&\ge C\omega_0^{(X_0,T)}(\Delta_r(Q_0,s_0))\Big[\frac{\omega_1^{(X_0,T)}(E\cap\Delta_r)(Q_0,s_0)}{\omega_1^{(X_0,T)}(\Delta_{4r}(Q_0,s_0))}\Big]^2
\end{aligned}$$

when f is taken to be χ_E. Then using

$$\|N(u_1)\|^2_{L^2(d\omega_0,\partial_p^+ D_T)} \le \|f\|^2_{L^2(d\omega_0,\partial_p^+ D_T)} = C\omega_0^{(X_0,T)}(E\cap\Delta_r)$$

gives the A^∞ type condition

$$\frac{\omega_1(E)}{\omega_1(\Delta_{4r})}\le C\bigg(\frac{\omega_0(E)}{\omega_0(\Delta_r)}\bigg)^{1/2},$$

where $E\subseteq$ center quarter of $\Delta_{4r}(Q_0,s_0)$. This restriction prevents obtaining $\omega_0\in A^\infty(\omega_1)$ unless ω_1 is a center-doubling measure. So ω_0 doubling is not sufficient to give a doubling condition for ω_1.

2. Proof of (D). The proof is an adaptation to the elliptic-heat equation of the proof of Theorem 2.5 in [8], so details of the argument will frequently be omitted.

From the results in Doob one can write

$$F(x,t) = u_1(x,t) - u_0(x,t) = \int_{D_T} \nabla_y G_0(x,t;y,s)\cdot[\varepsilon_{ij}(y,s)]\nabla_y u_1(y,s)\,dy\,ds$$

by using the Riesz decomposition for the parabolic operator $\partial/\partial t - L_0$ in D_T (see [6]).

The integral form for the difference function u_1-u_0 can be used to prove the following two lemmas (these are parabolic versions of Lemmas 2.9 and 2.10 of [8]).

LEMMA 1 [8, 10]. *We have* $\widetilde{N}(F)(Q,s) \le C_1\varepsilon_0 M_{\omega_0}(S(u_1))(Q,s)$ *and*

$$\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0,\partial_p^+ D_T)} \le C_3\varepsilon_0\|S(u_1)\|_{L^2(d\omega_0,\partial_p^+ D_T)}.$$

LEMMA 2 [8, 10]. *We have*

$$\|S(F)\|_{L^2(d\omega_0,\partial_p^+ D_T)} \le C_4(\|\widetilde{N}(F)\|_{L^2(d\omega_0,\partial_p^+ D_T)} + \|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0,\partial_p^+ D_T)} + \|f\|_{L^2(d\omega_0,\partial_p D_T)}).$$

Here $C_i = C_i(\lambda, n, T, \delta_0)$, $i = 1,3,4$. Then using Lemmas 1 and 2 in addition to the inequalities

$$\|S(u_0)\|_{L^2(d\omega_0,\partial_p D_T)} \le C\|f\|_{L^2(d\omega_0,\partial_p D_T)}, \tag{1}$$

$$\|N(u_0)\|_{L^2(d\omega_0,\partial_p D_T)} \le C'\|f\|_{L^2(d\omega_0,\partial_p D_T)}, \tag{2}$$

the result

$$\|N(u_1)\|_{L^2(d\omega_0,\partial_p^+ D_T)} \le C''\|f\|_{L^2(d\omega_0,\partial_p D_T)}$$

follows by the same argument as in Fefferman, Kenig and Pipher [8, p. 78], and the fact that $N_\alpha(u_i)(Q,s) \le \widetilde{N}_\beta(u_i)(Q,s)$ since u_i are solutions, $i = 0,1$, $\beta > \alpha$, β sufficiently large.

To prove (1) use Green's theorem and a standard argument on the area integral in a bounded domain [5]. It is necessary to use the doubling condition for ω_0 here.

To prove (2) use $N(u_0)(Q,s) \le CM_{\omega_0}(f)(Q,s)$, which follows by the argument in [7] and a standard argument for the Hardy–Littlewood maximal function [11]. Again ω_0 must satisfy a doubling condition to obtain the comparison of $N(u_0)$ with $M_{\omega_0}(f)$.

Proof of Lemma 1. Fix $(Q,s) \in \partial_p^+ D_T$, and $(x,t) \in \Gamma(Q,s)$, and break $F(z,\tau)$ into two parts when $(z,\tau) \in P_{\delta(x,t)/4}(x,t)$:

$$\begin{aligned} F(z,\tau) &= F_1(z,\tau) + F_2(z,\tau) \\ &= \int\limits_{P_{\delta(x,t)/2}(x,t)} \nabla_y G_0(z,\tau;\, y,s)\cdot[\varepsilon_{ij}(y,s)]\nabla_y u_1(y,s)\,dy\,ds \\ &\quad + \int\limits_{D_T\setminus P_{\delta(x,t)/2}(x,t)} \nabla_y G_0\cdot[\varepsilon_{ij}]\nabla_y u_1. \end{aligned}$$

The second integral is further broken into integrals over the regions

$$\begin{aligned} \Omega_0 &= P_{\delta(x,t)/2}(x^*,t^*), \\ \Omega_j^0 &= (P_{2^{j-1}\delta(x,t)}(x^*,t^*)\setminus P_{2^{j-2}\delta(x,t)}(x^*,t^*)) \\ &\qquad \cap [D_T\setminus(\Gamma(Q,s)\cup P_{\delta(x,t)/2}(x,t))], \\ \Omega_j^1 &= (P_{2^{j-1}\delta(x,t)}(x^*,t^*)\setminus P_{2^{j-2}\delta(x,t)}(x^*,t^*)) \\ &\qquad \cap \Gamma(Q,s)\cap(D_T\setminus P_{\delta(x,t)/2}(x,t)), \end{aligned}$$

for $j = 1,\ldots,N$ and $N\delta(x,t)\sim 1/8$, and the region

$$\Omega^2 = \Big(D_T\setminus\bigcup_{j=0}^N \Omega_j\Big)\cap(P_{\delta(x,t)/2}(x,t))^{\mathrm{C}}.$$

Here (x^*,t^*) is the projection of (x,t) onto $\partial_p^+ D_T$ (so $t = t^*$). Set $\Omega_j^0\cup\Omega_j^1 = \Omega_j$. As can be seen, $D_T\setminus P_{\delta(x,t)/2}(x,t) = \bigcup_{j=0}^N \Omega_j\cup\Omega^2$.

To estimate $\widetilde{N}(F)(Q,s)$ the averages

$$\bigg(\frac{1}{|P_{\delta(x,t)/4}(x,t)|}\int\limits_{P_{\delta(x,t)/4}(x,t)} |F_i(y,s)|^2\,dy\,ds\bigg)^{1/2}, \quad i = 1,2,$$

are used.

Lemma 1 is proved by essentially the same argument as the proof of Lemma 2.9 in [8]. First the $F_1(z,\tau)$ term can be estimated by the following adaptation of their argument.

For $(z,\tau)\in P_{\delta(x,t)/4}(x,t)$ with $|\varepsilon(z,\tau)|\le\varepsilon_0$, let $\widetilde{G}_0(z,\tau;y,s)$ be the Green's function of the domain $P_{\delta(x,t)/2}(x,t)$ and let

$$K(z,\tau;y,s) = G_0(z,\tau;y,s) - \widetilde{G}_0(z,\tau;y,s)$$

and

$$\widetilde{F}_1(z,\tau) = \int\limits_{P_{\delta(x,t)/2}(x,t)} \nabla_y\widetilde{G}_0(z,\tau;y,s)\cdot[\varepsilon_{ij}(y,s)]\nabla_y u_1(y,s)\,dy\,ds$$

and

$$\widetilde{\widetilde{F}}_1(z,\tau) = F_1(z,\tau) - \widetilde{F}_1(z,\tau).$$

If $(y,s)\in\partial_p P_{\delta(x,t)/2}(x,t)$ then $\widetilde{F}_1(y,s) = 0$, and

$$L_0\widetilde{F}_1(z,\tau) = +\operatorname{div}([\varepsilon_{ij}(z,\tau)]\nabla_y u_1(z,\tau)\chi_{P_{\delta(x,t)/2}}(z,\tau)) + (\partial\widetilde{F}_1/\partial t)(z,\tau),$$

for $(z,\tau)\in P_{\delta(x,t)/4}(x,t)$.

Using the argument on p. 82 of [8] one can obtain

$$\delta(x,t)\bigg(\frac{1}{|P_{\delta(x,t)/4}(x,t)|}\int\limits_{P_{\delta(x,t)/4}(x,t)} |\nabla\widetilde{F}_1|^2\bigg)^{1/2} \le C\varepsilon_0 S(u_1)(Q,s)$$

since

$$\begin{aligned} &\bigg(\frac{1}{|P_{\delta(x,t)/4}(x,t)|}\int\limits_{P_{\delta(x,t)/4}(x,t)} |\widetilde{F}_1|^2\bigg)^{1/2} \\ &\qquad \le C\delta(x,t)\bigg(\frac{1}{|P_{\delta(x,t)/4}(x,t)|}\int\limits_{P_{\delta(x,t)/4}(x,t)} |\nabla\widetilde{F}_1|^2\bigg)^{1/2} \end{aligned}$$

by the Sobolev inequality and since

$$\int_{P_{\delta(x,t)/4}(x,t)} |\nabla\widetilde{F}_1|^2 \le \lambda \int_{P_{\delta/4}} \nabla\widetilde{F}_1\cdot[a_{ij}]\nabla\widetilde{F}_1$$
$$= \lambda\Big[\int_{P_{\delta/4}} \operatorname{div}(\widetilde{F}_1[a_{ij}]\nabla\widetilde{F}_1) - \widetilde{F}_1 L_0\widetilde{F}_1\Big]$$
$$= -\lambda \int_{P_{\delta/4}} \widetilde{F}_1 L_0 \widetilde{F}_1$$

using strict ellipticity and integration by parts on $\int_{P_{\delta/4}} \operatorname{div}(\widetilde{F}_1[a_{ij}]\nabla\widetilde{F}_1)$. So it suffices to estimate $-\lambda\int_{P_{\delta/4}}\widetilde{F}_1 L_0\widetilde{F}_1$. Using the identity for $L_0\widetilde{F}_1$ the estimate becomes

$$-\lambda\int_{P_{\delta/4}} \widetilde{F}_1 L_0\widetilde{F}_1 = -\lambda\int_{P_{\delta/4}} \widetilde{F}_1 \operatorname{div}([\varepsilon_{ij}]\nabla u_1\chi_{P_{\delta/2}}) - \frac{\lambda}{2}\int_{P_{\delta/4}} \frac{\partial\widetilde{F}_1^2}{\partial t}$$
$$= -\lambda\int_{P_{\delta/4}} \operatorname{div}(\widetilde{F}_1\cdot[\varepsilon_{ij}]\nabla u_1\chi_{P_{\delta/2}})$$
$$+\lambda\int_{P_{\delta/4}} \nabla\widetilde{F}_1\cdot[\varepsilon_{ij}]\nabla u_1\chi_{P_{\delta/2}}$$
$$-\frac{\lambda}{2}\int_{P_{\delta/4}\cap\{s=t+\delta^2/4=s_0\}} \widetilde{F}_1(y,s_0)^2\,dy$$
$$\le \lambda\int_{P_{\delta/4}} \nabla\widetilde{F}_1\cdot[\varepsilon_{ij}]\nabla u_1\chi_{P_{\delta/2}}$$
$$\le C\varepsilon_0\Big(\int_{P_{\delta/4}}|\nabla\widetilde{F}_1|^2\Big)^{1/2}\Big(\int_{P_{\delta/4}}|\nabla u_1|^2\Big)^{1/2}.$$

Dividing by $(\int_{P_{\delta/2}}|\nabla\widetilde{F}_1|^2)^{1/2}$ gives

$$\Big(\frac{1}{|P_{\delta/4}|}\int_{P_{\delta/4}}|\nabla\widetilde{F}_1|^2\Big)^{1/2} \le C\varepsilon_0\Big(\frac{1}{|P_{\delta/4}|}\int_{P_{\delta/4}}|\nabla u_1|^2\Big)^{1/2}$$

or

$$\Big(\frac{1}{|P_{\delta/4}|}\int_{P_{\delta/4}}|\widetilde{F}_1|^2\Big)^{1/2} \le C\varepsilon_0 S(u_1)(Q,s).$$

Now

$$\widetilde{\widetilde{F}}_1(z,\tau) = \int_{P_{\delta(x,t)/2}(x,t)} \nabla_y K(z,\tau;y,s)[\varepsilon_{ij}(y,s)]\nabla u_1(y,s)\,dy\,ds,$$

so

$$|\widetilde{\widetilde{F}}_1(z,\tau)| \le \varepsilon_0\int_{P_{\delta/2}} |\nabla_y K|\cdot|\nabla_y u_1|$$
$$\le \frac{C\varepsilon_0}{\delta(x,t)}\Big(\int_{P_{3\delta/4}} |K(z,\tau;y,s)|^2\,dy\,ds\Big)^{1/2}\Big(\int_{P_{\delta/2}}|\nabla u_1|^2\Big)^{1/2}$$

using Cauchy–Schwarz and the energy estimate on $(\int_{P_{\delta/2}}|\nabla_y K|^2)^{1/2}$. This is legitimate since $(\partial/\partial t + L_0)K(z,\tau;y,s) = 0$ for $(y,s)\in P_{\delta(x,t)/4}(x,t)$ and (z,τ) fixed. Now using Harnack as in [8] and Aronson's estimates on G_0 and $\widetilde{G}_0$, the above is

$$\le \frac{C\varepsilon_0}{\delta(x,t)}|P_{\delta(x,t)/2}(x,t)|^{-1/2}\int_{P_{3\delta(x,t)/4}} |K(z,\tau;y,s)|\,dy\,ds\cdot\Big(\int_{P_{\delta/2}}|\nabla u_1|^2\Big)^{1/2}$$
$$\le C\frac{\varepsilon_0}{\delta(x,t)}\delta^{-(n+2)/2}\cdot\delta^{n+2}\cdot\delta^{-n}\cdot\Big(\int_{P_{\delta/2}}|\nabla u_1|^2\Big)^{1/2} \le C\varepsilon_0 S(u_1)(Q,s).$$

Altogether

$$\widetilde{N}(F_1)(Q,s) \le C\varepsilon_0 S(u_1)(Q,s).$$

Next $F_2(z,\tau)$ is estimated pointwise by estimating the integrals over Ω_0, Ω_j^0,Ω_j^1, and Ω^2 separately. The regions Ω_0,Ω_j^0, and Ω^2 are handled by the stopping time argument of [8] adapted to parabolic functions, likewise the estimates for the regions Ω_j^1 inside $\Gamma(Q,s)$ follow from the same proof as in [8]. The adaptation to D_T and elliptic-caloric operators, their solutions and Green's functions is routine. The stopping time argument for $\int_{\Omega_0}\nabla_y G_0\varepsilon_{ij}\nabla_y u_1$ is included in an appendix to this paper for the sake of completeness. Otherwise the arguments are omitted. The tools used in the parabolic case are the Carleson condition (C) of Theorem 1, the energy estimate in place of Caccioppoli's inequality, Aronson's estimates on the Green's function G_0 [1], Hölder continuity for solutions vanishing at $\partial_p^+ D_T$, local comparison for solutions vanishing on the boundary and the estimate in Theorem 1.4 of [7]. All these results hold for time dependent operators; it is necessary, however, to use the doubling property of the measure $\omega_0^{X_0,T}$ and backwards Harnack on G_0 in several places [1, 7]. For example, to obtain

$$\varepsilon_0\int_{\Omega^2\cap\partial_p D_T} S(u_1)(\widehat{Q},\widehat{s})\,d\omega_0^{(x,t)}(\widehat{Q},\widehat{s}) \le C\varepsilon_0 M_{\omega_0^{(x_0,T)}}(S(u_1))(Q,s)$$

in estimating $F_2(z,\tau)$ one needs to have geometric decay on the kernel function for the operator $\partial/\partial t - L_0$, only known to hold for the associated measure ω_0 being a doubling measure.

In short, it is impossible to use the proof of [8] unless one measure is assumed to satisfy a doubling condition.

The dyadic surface "intervals" in $\partial_{\mathrm{p}} D_T$ are always taken to be parabolic "cubes" of dimension $r \times r^2$ in space $\times$ time as is usual for the heat equation.

The estimate for $\widetilde{N}(\delta\nabla F)$ can be obtained using the same averaging technique in the space variable used in [8, pp. 87–88]. For the ∇_y terms, averaging in the space variable is *all* that is needed. However, an identity for FL_0F brings in a time derivative which must be estimated in time and space. Specifically, if $r = \delta(x,t)$ and $B_r(x,t) = \{(y,t) \mid |x-y| < r\}$, then

$$\frac{1}{|P_{\delta(x,t)/2}(x,t)|} \int\limits_{P_{\delta(x,t)/2}(x,t)} |\delta(y,s)\nabla_y F(y,s)|^2\,dy\,ds$$

$$\le \frac{\delta(x,t)^2}{\delta(x,t)^{n+2}} \int\limits_{t-\delta^2/4}^{t+\delta^2/4} \frac{1}{r} \int\limits_{\varrho=r/2}^{\varrho=3r/4} \int\limits_{B_\varrho(x,s)} |\nabla_y F(y,s)|^2\,dy\,d\varrho\,ds,$$

$$\le \frac{1}{\delta^{n+1}} \frac{C}{\delta^2} \int\limits_{\delta^2/4}^{9\delta^2/16} \int\limits_{\varrho=r/2}^{\varrho=3r/4} \int\limits_{t-\alpha}^{t+\alpha} \int\limits_{B_\varrho(x,s)} |\nabla_y F(y,s)|^2\,dy\,ds\,d\varrho\,d\alpha$$

$$\le \frac{C}{\delta^{n+3}} \iiiint (L_0F^2 - 2FL_0F).$$

The integral of L_0F^2 can be bounded by $C\widetilde{N}_\alpha(F)(Q,s)\widetilde{N}_\alpha(\delta\nabla F)(Q,s)$ by the same argument as in [8].

Since

$$-2FL_0F = F\operatorname{div}(\varepsilon\nabla u_1) - \partial F^2/\partial t,$$

again the integral of $F\operatorname{div}(\varepsilon\nabla u_1)$ can be shown to be $\le C\varepsilon_0(\widetilde{N}_\alpha(F)(Q,s) + \widetilde{N}_\alpha(\delta\nabla F)(Q,s))S_\alpha(u_1)(Q,s)$ as in [8]; the only new term is

$$\frac{C}{\delta^{n+3}} \int\limits_{\delta^2/4}^{9\delta^2/16} \int\limits_{\varrho=r/2}^{\varrho=3r/4} \int\limits_{t-\alpha}^{t+\alpha} \int\limits_{B_\varrho(x,s)} -\frac{\partial F^2}{\partial t}\,dy\,ds\,d\varrho\,d\alpha$$

$$= \frac{C}{\delta^{n+3}} \int\limits_{\delta^2/4}^{9\delta^2/16} \int\limits_{\varrho=r/2}^{\varrho=3r/4} \int\limits_{B_\varrho(x,t\pm\alpha)} [F(y,t-\alpha)^2 - F(y,t+\alpha)^2]\,dy\,d\varrho\,d\alpha$$

$$\le \frac{C'}{\delta^{n+2}} \int\limits_{t-9\delta^2/16}^{t+9\delta^2/16} \int\limits_{B_{3r/4}(x,s)} F(y,s)^2\,dy\,ds = C(\widetilde{N}_\alpha(F)(Q,s))^2,$$

where $r = \delta(x,t)$.

Altogether

$$\text{(F)}\qquad (\widetilde{N}(\delta\nabla F)(Q,s))^2 \le C[(\widetilde{N}_\alpha(F)(Q,s))^2 + \widetilde{N}_\alpha(F)(Q,s)\cdot\widetilde{N}_\alpha(\delta\nabla F)(Q,s) + \varepsilon_0\widetilde{N}_\alpha(F)(Q,s)\cdot S_\alpha(u_1)(Q,s) + \varepsilon_0\widetilde{N}_\alpha(\delta\nabla F)(Q,s)\cdot S_\alpha(u_1)(Q,s)].$$

Then a standard argument allows one to remove the larger cone Γ_α in taking L^2 norms. Also $\widetilde{N}_\alpha(\delta\nabla F)(Q,s) < \infty$ a.e. $d\omega_0$.

By the first part of the lemma $\widetilde{N}(F)(Q,s) \le \varepsilon_0 M_{\omega_0}(S(u_1))(Q,s)$ so (F) gives

$$\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0,\partial_{\mathrm{p}}^+ D_T)} \le \varepsilon_0 \|S(u_1)\|_{L^2(d\omega_0,\partial_{\mathrm{p}}^+ D_T)}.$$

The proof of Lemma 2 follows from an adaptation to the heat equation of the proof of Lemma 2.10 in Fefferman, Kenig and Pipher [8,10]. Once again there is an extra term involving a time derivative which is easily handled by averaging.

Proof of Lemma 2. Fix $\delta_0 > 0$ (to be chosen as indicated below) and remove the core $B_{\delta_0}(X_0) \times [\delta_0^2, T] = D_{\delta_0,T}$ from D_T:

$$\int\limits_{\partial_{\mathrm{p}}^+ D_T} (S(F)(Q,s))^2\,d\omega_0^{(X_0,T)}(Q,s)$$

$$= \int\limits_{\partial_{\mathrm{p}}^+ D_T} \int\limits_{\Gamma(Q,s)} |\nabla F(y,\tau)|^2 \delta(y,\tau)^{-n}\,dy\,d\tau\,d\omega_0(Q,s)$$

$$\le \int\limits_{D_T} |\nabla F(y,\tau)|^2 \delta(y,\tau)^{-n} \omega_0(\Delta_{\delta(y,\tau)}(y^*,\tau^*))\,dy\,d\tau$$

$$= \int\limits_{D_T\setminus D_{\delta_0,T}} + \int\limits_{D_{\delta_0,T}}$$

$$\le \int\limits_{D_T\setminus D_{\delta_0,T}} |\nabla F(y,\tau)|^2 G_0(X_0,T;\,y,\tau)\,dy\,d\tau + \int\limits_{D_{\delta_0,T}} |\nabla F(y,\tau)|^2 \delta(y,\tau)^{-n} \omega_0(\Delta_{\delta(y,\tau)}(y^*,\tau^*))\,dy\,d\tau.$$

It is easy to see that the second integral is bounded above by $C\|\widetilde{N}(\delta\nabla F)\|^2_{L^2(d\omega_0)}$.

The first integral is

$$(*)\quad \int\limits_{D_T\setminus D_{\delta_0,T}} |\nabla F(y,\tau)|^2 G_0(X_0,T;\, y,\tau)\,dy\,d\tau$$
$$\le C(\lambda) \int\limits_{D_T\setminus D_{\delta_0,T}} (G_0\nabla F\cdot[A_0]\nabla F)\,dy\,d\tau$$
$$= C(\lambda) \int\limits_{D_T\setminus D_{\delta_0,T}} \left(\frac{1}{2}G_0L_0(F^2) - G_0FL_0F\right),$$

which equals, using the identity $L_0F = -\operatorname{div}([\varepsilon_{ij}]\nabla u_1) + \partial F/\partial t$ and integration by parts,

$$\frac{1}{\eta T}\int\limits_T \Big[C(\lambda)\int\limits_{D_T\setminus D_{\delta_0,T}} \frac{1}{2}G_0L_0(F^2) + \int\limits_{\partial D_{\delta_0,T}} G_0F\cdot[\varepsilon_{ij}]\nabla u_1\cdot\vec{n}$$
$$- \int\limits_{D_T\setminus D_{\delta_0,T}} \nabla(G_0F)\cdot[\varepsilon_{ij}]\nabla u_1 - \int\limits_{D_T\setminus D_{\delta_0,T}} G_0F\frac{\partial F}{\partial t}\Big].$$

The middle two integrals are bounded above by

$$C(\lambda)\int\limits_{D_T\setminus D_{\delta_0,T}} (|\nabla G_0|\cdot|F|\cdot|\varepsilon|\cdot|\nabla u_1| + |G_0|\cdot|\nabla F|\cdot|\varepsilon|\cdot|\nabla u_1|)$$
$$+ \int\limits_{\partial_p^+ D_{\delta_0,T}} |G_0|\cdot|F|\cdot|\varepsilon|\cdot|\nabla u_1\cdot\vec{n}|.$$

These integrals can be estimated as follows (see [10]):

$$\int\limits_{\partial_p^+ D_{\delta_0,T}} |G_0|\cdot|F|\cdot|\varepsilon|\cdot|\nabla u_1\cdot\vec{n}| \le C\varepsilon_0 \int\limits_{\partial_p^+ D_T} \widetilde{N}(F)(Q,s)S(u_1)(Q,s)\,d\omega_0(Q,s)$$

by averaging and an argument similar to the one estimating the boundary integral in the proof of Lemma 2.9 in [8], and

$$\int\limits_{D_T\setminus D_{\delta_0,T}} |\nabla G_0 F\varepsilon\nabla u_1|$$
$$\le C\varepsilon_0 \int\limits_{\partial_p^+ D_T} \Big|\int\limits_{\Gamma(Q,s)} F(y,\tau)^2|\nabla u_1(y,\tau)|^2\delta(y,\tau)^{-n}\,dy\,d\tau\Big|^{1/2} d\omega_0(Q,s)$$

by using the stopping time argument on the first integral with $F\nabla u_1$ in place of ∇u_1.

Then using the fact that

$$N_\alpha(F)(Q,s) = \sup_{(x,t)\in\Gamma_\alpha(Q,s)} |F(x,t)|$$

and for $\beta > \alpha$ fixed,

$$|F(x,t)| \le |u_1(x,t)| + |u_0(x,t)|$$
$$\le C\bigg(\frac{1}{|P_{\delta(x,t)/2}(x,t)|}\int\limits_{P_{\delta(x,t)/2}(x,t)} |u_1(y,s)|^2\,dy\,ds\bigg)^{1/2}$$
$$+ CN_\alpha(u_0)(Q,s)$$
$$\le C(\widetilde{N}_\beta(F)(Q,s) + \widetilde{N}_\beta(u_0)(Q,s))$$

plus the inequality (2) we get

$$C\varepsilon_0\int\limits_{\partial_p^+ D_T} \Big|\int\limits_{\Gamma(Q,s)} F(y,\tau)^2|\nabla u_1(y,\tau)|^2\delta(y,\tau)^{-n}\,dy\,d\tau\Big|^{1/2} d\omega_0(Q,s)$$
$$\le C\varepsilon_0\int\limits_{\partial_p^+ D_T} [\widetilde{N}_\beta(F)(Q,s) + N_\beta(u_0)(Q,s)]S(u_1)(Q,s)\,d\omega_0(Q,s)$$
$$\le C\varepsilon_0[\|\widetilde{N}(F)\|_{L^2(d\omega_0,\partial_p^+D_T)} + \|f\|_{L^2(d\omega_0,\partial_pD_T)}]\cdot\|S(u_1)\|_{L^2(d\omega_0,\partial_p^+D_T)}.$$

For $\int_{D_T\setminus D_{\delta_0,T}} G_0|\nabla F|\cdot|\varepsilon|\cdot|\nabla u_1|$ a slight variation of the stopping time argument (Cauchy–Schwarz is used on $\int_{T_j^k}|\nabla F|\cdot|\nabla u_1|\delta^{-n}$ instead of being used at the beginning on $\int_{T_j^k} G_0|\nabla F|\cdot|\varepsilon|\cdot|\nabla u_1|$; see Appendix) gives an upper bound of

$$\varepsilon_0\int\limits_{\partial_p^+ D_T} \widetilde{N}(\delta\nabla F)(Q,s)S(u_1)(Q,s)\,d\omega_0(Q,s)$$
$$\le C\varepsilon_0\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0,\partial_p^+D_T)}\|S(u_1)\|_{L^2(d\omega_0,\partial_pD_T)}.$$

Finally, the first and last integrals in $(*)$ can be combined to give

$$C(\lambda)\int\limits_{D_T\setminus D_{\delta_0,T}} \frac{1}{2}G_0\Big[L_0F^2 - \frac{\partial F^2}{\partial t}\Big]$$
$$= C(\lambda)\Big(\int\limits_{\partial_p^+ D_{\delta_0,T}} G_0[A_0]\nabla F^2\cdot\vec{n} - \int\limits_{\partial_p^+ D_{\delta_0,T}} [A_0]\nabla G_0\cdot nF^2\Big)$$
$$+ \int\limits_{D_T\setminus D_{\delta_0,T}} \Big(\Big(\frac{\partial}{\partial t} + L_0\Big)G_0\Big)F^2$$
$$+ \Big(\int\limits_{D_{\delta_0,T}\cap\{t=\delta_0^2\}} G_0(X_0,T;y,\delta_0^2)F(y,\delta_0^2)^2\,dy\Big)$$
$$- \Big(\int\limits_{(D\setminus D_{\delta_0,T})\cap\{t=T\}} G_0(X_0,T;y,T)F(y,T)^2\,dy\Big).$$

The first three boundary integrals can be estimated as before to obtain upper bounds of

$$\|\widetilde{N}(F)\|_{L^2(d\omega_0)}\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0)}+\|\widetilde{N}(F)\|^2_{L^2(d\omega_0)}$$

using averaging in the space variable and the time variable. The remaining two integrals are ≤ 0 since $(\partial/\partial t+L_0)G_0(X_0,T;y,s)=0$ if $(y,s)\in D_T\setminus D_{\delta_0,T}$.

Note. δ_0 must be chosen so that

$$\int\limits_{\partial_p^+D_{\delta_0,T}} G_0[A_0]\nabla F^2\cdot\vec{n}\le C\frac{1}{2\eta}\int\limits_{\delta_0-\eta}^{\delta_0+\eta}\Big(\int\limits_{\partial_p^+D_{\beta,T}} G_0[A_0]\nabla F^2\cdot\vec{n}\Big)\,d\beta$$

and likewise for $\int_{\partial_p^+D_{\delta_0,T}}\nabla G_0[A_0]F^2$ and $\int_{\partial_p^+D_{\delta_0,T}}|G_0|\cdot|F|\cdot|\varepsilon|\cdot|\nabla u_1|$ to allow averaging in the space variable—or averaging can be done when the time average is introduced as in $(*)$, in which case no restrictions on δ_0 are necessary.

Altogether

$$\begin{aligned}&\|S(F)\|^2_{L^2(d\omega_0)}\\&\quad\le C\varepsilon_0[\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0)}+\|\widetilde{N}(F)\|_{L^2(d\omega_0)}+\|f\|_{L^2(d\omega_0)}]\cdot\|Su_1\|_{L^2(d\omega_0)}\\&\qquad+\|\widetilde{N}(\delta\nabla F)\|_{L^2(d\omega_0)}\|\widetilde{N}(F)\|_{L^2(d\omega_0)}+\|\widetilde{N}(F)\|^2_{L^2(d\omega_0)}.\end{aligned}$$

Writing

$$\begin{aligned}\|S(u_1)\|_{L^2(d\omega_0)}&\le\|S(F)\|_{L^2(d\omega_0)}+\|S(u_0)\|_{L^2(d\omega_0)}\\&\le\|S(F)\|_{L^2(d\omega_0)}+C\|f\|_{L^2(d\omega_0)}\end{aligned}$$

by (1), for ε_0 sufficiently small the conclusion of Lemma 2 follows.

Appendix. The basic stopping time argument for parabolic functions [8]. Let $(Q_0,s_0)\in\partial_p^+D_T$, $\Delta_r(Q_0,s_0)\subseteq\partial_p^+D_T$ and $(x,t)\in\Gamma_\alpha(Q_0,s_0)$, let $(x^*,t^*)=$ the projection of (x,t) onto $\partial_p^+D_T$ (see Section 1 for definitions). Define

$$\begin{aligned}\Omega_0&=\{(y,s)\mid d(y,s;x^*,t^*)<\delta(x,t)/2\}\cap D_T,\\\Delta_0&=\Omega_0\cap\partial_p^+D_T=\{(Q,s)\in\partial_p^+D_T\mid d(Q,s;x^*,t^*)<\delta(x,t)/2\}.\end{aligned}$$

Then Ω_0 can be covered by parabolic boxes whose dimension compares with their distance from $\partial_p^+D_T$ and so that the projection of such a box onto $\partial_p^+D_T$ is a dyadic surface "cube" contained in $2\Delta_0$. Let $\bigcup_{j,k}I_j^k=$ union of all dyadic cubes in $2\Delta_0$ and $\bigcup_{j,k}T_j^k=$ union of all corresponding boxes in D_T, that is,

$$\begin{aligned}I_j^k&=\{(Q,s)\in\partial_p^+D_T\mid |Q_i-Q_i^{j*}|<2^{-k}r\text{ and }|s-s^{j*}|<2^{-2k}r^2\},\\T_j^k&=\{(y,\tau)\in D_T\mid 2^{-k}r\le|y_i-y_i^{j*}|\le2^{-(k-1)}r\\&\qquad\text{and }|\tau-\tau^{j*}|\le(2^{-k}r)^2\},\qquad i=1,\dots,n,\end{aligned}$$

where $Q\in\partial D$, $Q=(Q_1,\dots,Q_{n-1})$, $y\in D$, $y=(y_1,\dots,y_n)$, (Q^{j*},s^{j*}) is the center of I_j^k, (y^{j*},τ^{j*}) is the center of T_j^k and (Q^{j*},s^{j*}) is the projection of (y^{j*},τ^{j*}) onto $\partial_p^+D_T$.

So $\Delta_0\subseteq\bigcup_{j=1}^{N(k)}I_j^k$ for each k and $\Omega_0\subseteq\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{N(k)}T_j^k$. Now write

$$\begin{aligned}&\Big|\int\limits_{\Omega_0}\nabla G_0(x,t;y,s)\cdot[\varepsilon_{ij}(y,s)]\nabla u_1(y,s)\,dy\,ds\Big|\\&\quad=\Big|\sum_{j,k}\int\limits_{T_j^k}\nabla G_0\cdot[\varepsilon_{ij}]\nabla u_1\Big|\le C\sum_{j,k}\int\limits_{T_j^k}|\nabla G_0|\cdot|\varepsilon|\cdot|\nabla u_1|\\&\quad\le C\sum_{j,k}\sup_{(y,s)\in T_j^k}|\varepsilon(y,s)|\Big(\frac{1}{2^{-2k}r^2}\int\limits_{(1+\eta)T_j^k}|G_0(x,t;y,s)|^2\,dy\,ds\Big)^{1/2}\\&\qquad\times\Big(\int\limits_{T_j^k}|\nabla u_1(y,s)|^2\,dy\,ds\Big)^{1/2}\end{aligned}$$

using Cauchy–Schwarz and the energy estimate on $(\int_{T_j^k}|\nabla G_0|^2)^{1/2}$. The inequality of Theorem 1.4 of [7] applied to the Green's function, doubling for ω_0 and a local comparison theorem (see Theorem 2.5 of [7]) gives

$$|G_0(x,t;y,s)|\le\frac{CG(X_0,T;y,s)}{\omega_0(\Delta_0)}$$

so the last sum is bounded above by

$$C\sum_{j,k}\frac{C'}{\omega_0(\Delta_0)}\Big(\int\limits_{(1+\eta)T_j^k}\frac{|G_0(X_0,T;y,s)|^2}{(\delta(y,s))^2}a(y,s)^2\,dy\,ds\Big)^{1/2}\times\Big(\int\limits_{T_j^k}|\nabla u_1(y,s)|^2\,dy\,ds\Big)^{1/2}.$$

Let

$$\begin{aligned}\mathcal{O}_l&=\{(Q,s)\in2\Delta_0:S(u_1)(Q,s)>2^l\},\\\widetilde{\mathcal{O}}_l&=\{(Q,s)\in2\Delta_0:M_{\omega_0}(\chi_{\mathcal{O}_l})(Q,s)>1/2\},\end{aligned}$$

where
$$M_{\omega_0}(\chi_{\mathcal{O}_l})(Q,s)=\sup_{\substack{I\\ I\ni(Q,s)}}\frac{1}{\omega_0(I)}\int_I \chi_{\mathcal{O}_l}(Q,s)\,d\omega_0(Q,s).$$
Also let $\mathcal{T}_l=\{I_j^k\}$, where $I_j^k\in\mathcal{T}_l$ if $\omega_0(I_j^k\cap\widetilde{\mathcal{O}}_l)>\frac12\omega_0(I_j^k)$ but $\omega_0(I_j^k\cap\mathcal{O}_{l+1})\le\frac12\omega_0(I_j^k)$. Then $I_j^k\in\mathcal{T}_l\Rightarrow I_j^k\subseteq\widetilde{\mathcal{O}}_l$ (see [8, p. 84]).

Two facts:

(i) $I_j^k\in\mathcal{T}_l\Rightarrow\omega_0(I_j^k)\le 2\omega_0(I_j^k\cap(\widetilde{\mathcal{O}}_l\setminus\mathcal{O}_{l+1}))$ and

(ii) $\omega_0(\widetilde{\mathcal{O}}_l)\le 2\omega_0(\mathcal{O}_l)$,

which will be used below, follow easily from the definitions.

The estimate in Theorem 1.4 of [7] in addition to backwards Harnack on G_0 gives
$$\Big(\int_{(1+\eta)T_j^k}\frac{|G_0(X_0,T;y,s)|^2}{\delta(y,s)^2}a(y,s)^2\,dy\,ds\Big)^{1/2}$$
$$\le C\Big(\frac{\omega_0^{(X_0,T)}(I_j^k)}{l(I_j^k)^n}\Big)^{1/2}\Big(\int_{(1+\eta)T_j^k}\frac{G_0(X_0,T;y,s)}{\delta(y,s)^2}a(y,s)^2dy\,ds\Big)^{1/2},$$
where $l(I_j^k)=$ side length of I_j^k.

Using this inequality and rewriting the sum over i,j as $\sum_l\sum_{\mathcal{T}_l}$ gives
$$\Big|\int_{\Omega_0}\nabla G_0\cdot[\varepsilon_{ij}]\nabla u_1\Big|$$
$$\le\frac{C}{\omega_0(\Delta_0)}\sum_l\sum_{\mathcal{T}_l}\Big(\int_{(1+\eta)T_j^k}\frac{|G_0|a^2}{\delta^2}\Big)^{1/2}\Big(\frac{\omega_0(I_j^k)}{l(I_j^k)^n}\Big)^{1/2}\Big(\int_{T_j^k}|\nabla u_1|^2\Big)^{1/2}$$
$$\le\frac{C}{\omega_0(\Delta_0)}\sum_l\Big(\sum_{I_j^k\in\mathcal{T}_l}\int_{(1+\eta)T_j^k}\frac{|G_0|a^2}{\delta^2}\Big)^{1/2}\Big(\sum_{I_j^k\in\mathcal{T}_l}\frac{\omega_0(I_j^k)}{l(I_j^k)^n}\int_{T_j^k}|\nabla u_1|^2\Big)^{1/2}$$
$$\overset{(C)}{\le}\frac{C}{\omega_0(\Delta_0)}\sum_l\Big(\sum_{\mathcal{T}_l}C\varepsilon_0^2\omega_0(I_j^k)\Big)^{1/2}$$
$$\times\Big(\sum_{\widetilde{\mathcal{O}}_l}\omega_0(I_j^k)\int_{T_j^k}|\nabla u_1(y,s)|^2\delta(y,s)^{-n}\,dy\,ds\Big)^{1/2}$$
$$\overset{(i)}{\le}\frac{C\varepsilon_0}{\omega_0(\Delta_0)}\sum_l\omega_0(\widetilde{\mathcal{O}}_l)^{1/2}\Big(\sum_{\mathcal{T}_l}\omega_0(I_j^k\cap(\widetilde{\mathcal{O}}_l\setminus\mathcal{O}_{l+1}))\cdot\int_{T_j^k}|\nabla u_1|^2\delta^{-n}\Big)^{1/2}$$

$$\overset{(ii)}{\le}\frac{C\varepsilon_0}{\omega_0(\Delta_0)}\sum_l\omega_0(\mathcal{O}_l)^{1/2}$$
$$\times\Big(\int_{\widetilde{\mathcal{O}}_l\setminus\mathcal{O}_{l+1}}\Big(\int_{\Gamma(Q,s)}|\nabla u_1(y,\tau)|^2\delta(y,\tau)^{-n}\,dy\,d\tau\Big)\,d\omega_0(Q,s)\Big)^{1/2}$$
$$\le\frac{C\varepsilon_0}{\omega_0(\Delta_0)}\sum_l\omega_0(\mathcal{O}_l)^{1/2}2^{l+1}\omega_0(\widetilde{\mathcal{O}}_l\setminus\mathcal{O}_{l+1})^{1/2}$$
$$\overset{(ii)}{\le}\frac{C\varepsilon_0}{\omega_0(\Delta_0)}\sum_l 2^{l+1}\omega_0(\mathcal{O}_l)$$
$$\le C\varepsilon_0\frac{1}{\omega_0(\Delta_0)}\int_{2\Delta_0}S(u_1)(Q,s)\,d\omega_0(Q,s)$$
$$\le C\varepsilon_0 M_{\omega_0}(S(u_1))(Q_0,s_0).$$
Notice that the center doubling property of ω_0 has been used several times. The constant C varies from line to line, but depends only on λ,n,T_0,r_0, but not on $(s,t),u_i$ or $r,(Q_0,s_0)$.

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Closed ideals in certain Beurling algebras, and synthesis of hyperdistributions

by

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Abstract. We consider the ideal structure of two topological Beurling algebras which arise naturally in the study of closed ideals of A^+. Even in the case of closed ideals I such that $h(I) = E_{1/p}$, the perfect symmetric set of constant ratio $1/p$, some questions remain open, despite the fact that closed ideals J of A^+ such that $h(J) = E_{1/p}$ can be completely described in terms of inner functions. The ideal theory of the topological Beurling algebras considered in this paper is related to questions of synthesis for hyperdistributions such that $\limsup_{n\to-\infty} |\widehat{\varphi}(n)| < \infty$ and such that $\limsup_{n\to\infty}(\log^+ |\widehat{\varphi}(n)|)/\sqrt{n} < \infty$.

1. Introduction. Let $\mathcal{C}(\Gamma)$ be the algebra of all continuous, complex-valued functions on the unit circle Γ, and let

$$A(\Gamma) = \Big\{ f \in \mathcal{C}(\Gamma) \,\Big|\, \|f\|_1 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty \Big\}$$

be the usual Wiener algebra. By identifying continuous functions on the closed unit disc $\overline{D}$ which are analytic on D with their restrictions to Γ, we can interpret A^+, the algebra of absolutely convergent Taylor series, to be the algebra

$$\{f \in A(\Gamma) \mid \widehat{f}(n) = 0 \ (n < 0)\},$$

a closed subalgebra of $A(\Gamma)$.

There was some recent progress [8], [11], [12] in the theory of closed ideals of A^+. If I is a closed ideal of A^+, set $h(I) = \{z \in \overline{D} \mid f(z) = 0 \ (f \in I)\}$ and denote by $I^{A(\Gamma)}$ the set of elements of A^+ which belong to the closed ideal generated by I in $A(\Gamma)$.

Also, when $I \neq \{0\}$, denote by S_I the inner factor of I (i.e. the G.C.D. of the inner factors of all nonzero elements of I, see [15, p. 85]) and set $S_{\{0\}} = 1$. Bennett and Gilbert had conjectured in [3] (see also [17]) that all closed ideals I of A^+ satisfy

$$I = I^{A(\Gamma)} \cap S_I \cdot H^\infty(D), \tag{1}$$

where $H^\infty(D)$ is the algebra of bounded analytic functions on D.

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