uniformly to the identity and T_n defined by $T_n = \theta_n^{-1} \circ T_0 \circ \theta_n$ converges uniformly to T_0 as $n \to \infty$. Now $T_n'(x)$ is given by $T_0'(\theta_n(x)) \exp(n^{-1} h(\theta_n(x)))$, which may be seen to converge uniformly in x to $T_0'(x)$ as $n \to \infty$. Then we have shown that T_n converges to T_0 in the C^1 topology. Since the invariant density of T_n is given by $\exp(-n^{-1} F(\theta(x)))\varrho(\theta(x))$, the conclusion of the theorem follows. ∎

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A non-locally convex topological algebra with all commutative subalgebras locally convex

by

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Abstract. We construct a complete multiplicatively pseudoconvex algebra with the property announced in the title. This solves Problem 25 of [6].

All vector spaces and algebras in this paper are either real or complex. *A topological algebra A* is a (Hausdorff) topological vector space provided with an associative jointly continuous multiplication. It is said to be locally convex or locally pseudoconvex if the underlying topological vector space has this property. A *locally pseudoconvex space X* is a topological vector space whose topology is given by means of a family $(\|\cdot\|_\alpha)$ of p_α-homogeneous seminorms, $0 < p_\alpha \le 1$, i.e. non-negative functions $x \to \|x\|_\alpha$ such that $\|x+y\|_\alpha \le \|x\|_\alpha + \|y\|_\alpha$ and $\|\lambda x\|_\alpha = |\lambda|^{p_\alpha}\|x\|_\alpha$ for all x, y in X, all scalars λ, and all indices α (see [3] and [4]). A locally pseudoconvex algebra A is called *multiplicatively pseudoconvex* (briefly: m-pseudoconvex) if its topology is given by means of a family of submultiplicative p_α-homogeneous seminorms, i.e. seminorms satisfying $\|xy\|_\alpha \le \|x\|_\alpha\|y\|_\alpha$ for all x, y in A and all indices α. For more information on topological algebras the reader is referred to [2], [4] or [5].

In [6] we asked whether a topological algebra with the property that all of its commutative subalgebras are locally convex must itself be a locally convex algebra (Problem 25). In this paper we give a negative answer to this question by constructing a complete m-pseudoconvex algebra which is not locally convex but all of whose commutative subalgebras have this property. In the construction we use some methods introduced in [1] and [7].

Let X be a real or complex vector space and let p satisfy $0 < p < 1$. The maximal p-convex topology $\tau^p_{\max}$ on X is the vector space topology given by means of all p-homogeneous seminorms. It is known (see [1], Theorem 1) that this topology makes every vector space into a complete (Hausdorff) topological vector space. Let $(h_\alpha)_{\alpha\in\mathfrak{a}}$ be a Hamel basis for X, so that each

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element x in X can be uniquely written as $x = \sum_{\alpha\in\mathfrak{a}} \xi_\alpha h_\alpha$, where only finitely many scalar coefficients ξ_α are different from zero. Let R be the set of all numerical $\mathfrak{a}$-tuples $\mathbf{r} = (r_\alpha)_{\alpha\in\mathfrak{a}}$ with all $r_\alpha \geq 1$. We claim that the topology $\tau^p_{\max}$ on X is given by all p-homogeneous norms of the form

$$\|x\|^{(1)}_{\mathbf{r}} = \sum_{\alpha\in\mathfrak{a}} |\xi_\alpha|^p r_\alpha, \quad \mathbf{r} \in R. \tag{1}$$

In fact, let $\|\cdot\|$ be an arbitrary p-homogeneous seminorm on X. We have to show that it is continuous with respect to some norm of the form (1). But for every x in X we have

$$\|x\| = \Big\|\sum_\alpha \xi_\alpha h_\alpha\Big\| \leq \sum_\alpha |\xi_\alpha|^p \|h_\alpha\| \leq \|x\|^{(1)}_{\mathbf{r}},$$

where $\mathbf{r}$ is given by $r_\alpha = \max\{1, \|h_\alpha\|\}$, and we are done. It is known ([1], Proposition 2) that in the case of an uncountable Hamel basis the topology $\tau^p_{\max}$ is not locally convex.

To start our construction choose an uncountable linearly ordered set $\mathfrak{a}$ with order relation denoted by $\succ$ (it can be the ordered set of the first uncountable ordinal number), and define A_1 as the vector space spanned by a family of vectors $(e_\alpha)_{\alpha\in\mathfrak{a}}$ so that $(e_\alpha)_{\alpha\in\mathfrak{a}}$ is its Hamel basis. Fix a p satisfying $0 < p < 1$ and provide A_1 with the topology $\tau^p_{\max}$, so that we obtain a complete Hausdorff locally pseudoconvex space and it is not locally convex since it has an uncountable Hamel basis. Put $\mathfrak{a}^* = \{(\alpha,\beta) \in \mathfrak{a}\times\mathfrak{a} : \alpha \succ \beta\}$ (note that for (α,β) in $\mathfrak{a}^*$ we always have $\alpha \neq \beta$), and denote by A_0 the vector space spanned by a family of vectors $(e_{\alpha,\beta})_{(\alpha,\beta)\in\mathfrak{a}^*}$ (which is a Hamel basis for it). We equip it with the locally convex topology given by all norms of the form

$$\|x\|^{(0)}_{\mathbf{r}} = \sum_{(\alpha,\beta)\in\mathfrak{a}^*} |\xi_{\alpha,\beta}| r_\alpha^{1/p} r_\beta^{1/p}, \quad \mathbf{r} \in R, \tag{2}$$

where $x = \sum \xi_{\alpha,\beta} e_{\alpha,\beta} \in A_0$. Finally, we define A to be the direct sum of A_0 and A_1 provided with the direct sum topology. This topology can be given by means of the family

$$\|x\|_{\mathbf{r}} = \max\{(\|u\|^{(0)}_{\mathbf{r}})^p, \|v\|^{(1)}_{\mathbf{r}}\}, \quad \mathbf{r} \in R, \tag{3}$$

of p-homogeneous seminorms, where $x = u + v$, $u \in A_0$, $v \in A_1$. Thus we obtain a locally pseudoconvex space which is not locally convex, since its subspace A_1 is not.

We make A into an algebra by setting $ux = xu = 0$ for all x in A and all u in A_0, $e^2_\alpha = 0$ for all α and $e_\alpha e_\beta = -e_\beta e_\alpha = e_{\alpha,\beta}$ for all $\alpha \succ \beta$. Multiplication defined in this way is associative since the product of any three elements is zero. Moreover, the square of any element in A is zero and $xy = -yx$ for any two elements in A. This follows from the following

formula for multiplication of elements $x = \sum \xi_\alpha e_\alpha$ and $y = \sum \eta_\beta e_\beta$ in A_1:

$$xy = \sum_{(\alpha,\beta)\in\mathfrak{a}^*} (\xi_\alpha\eta_\beta - \xi_\beta\eta_\alpha) e_{\alpha,\beta}. \tag{4}$$

It can be easily seen that A_0 is the centre of A, i.e. it is the maximal subset of elements commuting with all elements in A.

Our result reads as follows:

THEOREM. *The algebra A is a complete m-pseudoconvex algebra which is not locally convex, but all of its commutative subalgebras are locally convex.*

Proof. First we prove that A is complete. We already know that A_1 is complete, so it remains to show that so is A_0. Let $(x_\mu)_{\mu\in\mathfrak{b}}$ be a Cauchy net in A_0, $x_\mu = \sum_{(\alpha,\beta)\in\mathfrak{a}^*} \xi^{(\mu)}_{\alpha,\beta} e_{\alpha,\beta}$. Observe first that the linear functionals $x = \sum \xi_{\alpha,\beta} e_{\alpha,\beta} \to \xi_{\alpha,\beta}$ are continuous in A_0 for all $(\alpha,\beta) \in \mathfrak{a}^*$. Thus the limits

$$\xi^{(0)}_{\alpha,\beta} = \lim_\mu \xi^{(\mu)}_{\alpha,\beta}, \quad (\alpha,\beta) \in \mathfrak{a}^*, \tag{5}$$

all exist and are finite.

We now show that only finitely many coefficients $\xi^{(0)}_{\alpha,\beta}$ can be different from zero. If not, there is a sequence (α_i,β_i) in $\mathfrak{a}^*$ with $\xi^{(0)}_{\alpha_i,\beta_i} \neq 0$. Without loss of generality we can assume that all α_i are different (otherwise we could assume that all β_i are different and perform the proof in a similar way). Define an element $\mathbf{r}$ in R by setting $r_{\alpha_i} = \max\{1, |\xi^{(0)}_{\alpha_i,\beta_i}|^{-1}\}$ and $r_\alpha = 1$ if $\alpha \neq \alpha_i$ for all i. Take the corresponding norm $\|\cdot\|^{(0)}_{\mathbf{r}}$ of the form (2). Then $(\|x_\mu\|^{(0)}_{\mathbf{r}})_{\mu\in\mathfrak{b}}$ is a numerical Cauchy net, and so the (finite) limit

$$C = \lim_\mu \|x_\mu\|^{(0)}_{\mathbf{r}}$$

exists. Take any natural $n \geq C$. There exists an index μ_0 in $\mathfrak{b}$ with

$$\|x_\mu\|^{(0)}_{\mathbf{r}} < n + 1 \quad \text{for all } \mu \succeq \mu_0. \tag{6}$$

Take the indices $\alpha_1, \ldots, \alpha_{2n+2}$. For sufficiently large μ, which can be assumed to be larger than μ_0, we have

$$\frac{|\xi^{(\mu)}_{\alpha_i,\beta_i}|}{|\xi^{(0)}_{\alpha_i,\beta_i}|} > \frac{1}{2} \quad \text{for } 1 \leq i \leq 2n+2,$$

so that for such μ we have

$$\|x_\mu\|^{(0)}_{\mathbf{r}} \geq \sum_i \frac{|\xi^{(\mu)}_{\alpha_i,\beta_i}|}{|\xi^{(0)}_{\alpha_i,\beta_i}|} > n + 1,$$

which contradicts (6) and proves that only finitely many numbers in (5) can be different from zero.

Thus $x_0 = \sum_{(\alpha,\beta)\in\mathfrak{a}^*} \xi^{(0)}_{\alpha,\beta} e_{\alpha,\beta}$ is in A_0. We show that the net (x_μ) tends to x_0, proving the completeness of A_0. To this end, upon replacing (x_μ) by $(x_\mu - x_0)$, it is sufficient to show that if a Cauchy net (x_μ) satisfies $\lim_\mu \xi^{(\mu)}_{\alpha,\beta} = 0$, then it tends to the zero element in A_0. Assume that this is not the case. Since for each continuous seminorm $\|\cdot\|$ on A_0 the numerical net $(\|x_\mu\|)$ also satisfies the Cauchy condition, there is an $\mathbf{r}$ in R such that the finite limit

$$\lim_\mu \|x_\mu\|^{(0)}_{\mathbf{r}} = M > 0 \tag{7}$$

exists. Define the *support* of an element $x = \sum \xi_{\alpha,\beta} e_{\alpha,\beta}$ in A_0 to be the set $\operatorname{supp}(x) = \{(\alpha,\beta) \in \mathfrak{a}^* : \xi_{\alpha,\beta} \neq 0\}$, so that each non-zero element has a non-void support. Clearly $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$ implies

$$\|x+y\|^{(0)}_{\mathbf{r}} = \|x\|^{(0)}_{\mathbf{r}} + \|y\|^{(0)}_{\mathbf{r}} \tag{8}$$

for all $\mathbf{r}$. Now for the (fixed) $\mathbf{r}$ of (7) we find a μ_0 in $\mathfrak{b}$ with

$$\|x_\mu - x_{\mu_0}\|^{(0)}_{\mathbf{r}} < M/2 \quad \text{for all } \mu \succeq \mu_0. \tag{9}$$

Put $S_0 = \operatorname{supp}(x_{\mu_0})$ and define a (continuous) projection on A_0 by setting

$$Px = \sum_{(\alpha,\beta)\in S_0} \xi_{\alpha,\beta} e_{\alpha,\beta} \quad \text{for } x = \sum \xi_{\alpha,\beta} e_{\alpha,\beta}.$$

Denote by I the identity operator on A_0. Clearly Px and $(I-P)x$ have disjoint supports for all x in A_0. The formula (8) now implies

$$\begin{aligned}\|x_\mu - x_{\mu_0}\|^{(0)}_{\mathbf{r}} &= \|Px_\mu - x_{\mu_0} + (I-P)x_\mu\|^{(0)}_{\mathbf{r}}\\ &= \|Px_\mu - x_{\mu_0}\|^{(0)}_{\mathbf{r}} + \|(I-P)x_\mu\|^{(0)}_{\mathbf{r}}\end{aligned}$$

and, consequently, (9) implies

$$\|(I-P)x_\mu\|^{(0)}_{\mathbf{r}} < M/2 \quad \text{for } \mu \succeq \mu_0. \tag{10}$$

Since $\lim_\mu \xi^{(\mu)}_{\alpha,\beta} = 0$ for all $(\alpha,\beta) \in \mathfrak{a}^*$ and the set S_0 is finite, we have $\lim_\mu \|Px_\mu\|^{(0)}_{\mathbf{r}} = 0$. The formulas (7), (8) and (10) now imply

$$\begin{aligned}M &= \lim_\mu \|x_\mu\|^{(0)}_{\mathbf{r}} = \lim_\mu \|Px_\mu\|^{(0)}_{\mathbf{r}} + \lim_\mu \|(I-P)x_\mu\|^{(0)}_{\mathbf{r}}\\ &= \lim_\mu \|(I-P)x_\mu\|^{(0)}_{\mathbf{r}} \le M/2,\end{aligned}$$

a contradiction proving the completeness of A_0 and so of A.

We now show that all norms (3) are submultiplicative, which means that A is m-pseudoconvex. Let $x, y \in A$, $x = u + w$, $y = v + z$ with $u, v \in A_1$ and $w, z \in A_0$. We have $xy = uv$ and so for all $\mathbf{r}$ in R we have

$$\|xy\|_{\mathbf{r}} = (\|uv\|^{(0)}_{\mathbf{r}})^p.$$

Writing $u = \sum \xi_\alpha e_\alpha$, $v = \sum \eta_\beta e_\beta$ and using (4), we obtain

$$\begin{aligned}\|xy\|_{\mathbf{r}} &= \Big(\Big\| \sum_{\alpha\succ\beta} (\xi_\alpha\eta_\beta - \xi_\beta\eta_\alpha) e_{\alpha,\beta}\Big\|^{(0)}_{\mathbf{r}}\Big)^p\\ &= \Big(\sum_{\alpha\succ\beta} |\xi_\alpha\eta_\beta - \xi_\beta\eta_\alpha| r_\alpha^{1/p} r_\beta^{1/p}\Big)^p \le \Big(\sum_{\alpha,\beta\in\mathfrak{a}} |\xi_\alpha|\cdot|\eta_\beta| r_\alpha^{1/p} r_\beta^{1/p}\Big)^p\\ &\le \sum_{\alpha,\beta\in\mathfrak{a}} |\xi_\alpha|^p |\eta_\beta|^p r_\alpha r_\beta = \|u\|^{(1)}_{\mathbf{r}} \|v\|^{(1)}_{\mathbf{r}} \le \|x\|_{\mathbf{r}} \|y\|_{\mathbf{r}}\end{aligned}$$

and A is m-pseudoconvex.

We already know that A is not locally convex. It remains to be shown that all commutative subalgebras of A are locally convex. It is sufficient to show that all maximal commutative subalgebras of A are locally convex. Let $\mathcal{A}$ be such a subalgebra. It must contain the centre A_0, and we claim that it contains only one element in A_1 together with its scalar multiples. So suppose that it contains two linearly independent elements x and y of A_1. Since $xy = -yx$, we must have $xy = yx = 0$. Let $x = \sum \xi_\alpha e_\alpha$ and $y = \sum \eta_\beta e_\beta$. Observe that if $\xi_\alpha \neq 0$ then $\eta_\alpha \neq 0$ because otherwise (4) implies $xy \neq 0$ (the coefficient of $e_{\alpha,\beta}$ in the Hamel expansion of xy is non-zero for some β). Choose α_1 in $\mathfrak{a}$ so that $\xi_{\alpha_1} \neq 0 \neq \eta_{\alpha_1}$. There must also be some α_2 with $\xi_{\alpha_2} \neq 0$, otherwise x and y would be proportional. Since $xy = 0$ the relation (4) implies

$$\frac{\xi_{\alpha_1}}{\eta_{\alpha_1}} = \frac{\xi_{\alpha_2}}{\eta_{\alpha_2}} = \lambda$$

for some scalar λ. If $\xi_{\alpha_3} \neq 0$, then, by the same reasoning, we can replace α_2 by α_3 in the above. But this means that $x = \lambda y$ and we are done. Now $\mathcal{A}$ is the direct sum of A_0 and the one-dimensional algebra spanned by some element x in A_1, so it is locally convex. The conclusion follows.

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