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Polynomial selections and separation by polynomials

by

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Abstract. K. Nikodem and the present author proved in [3] a theorem concerning separation by affine functions. Our purpose is to generalize that result for polynomials. As a consequence we obtain two theorems on separation of an n-convex function from an n-concave function by a polynomial of degree at most n and a stability result of Hyers–Ulam type for polynomials.

1. Introduction. We denote by $\mathbb{R}$, $\mathbb{N}$ the sets of all reals and positive integers, respectively. Let $I \subset \mathbb{R}$ be an interval. In this paper we present a necessary and sufficient condition under which two functions $f, g : I \to \mathbb{R}$ can be separated by a polynomial of degree at most n, where $n \in \mathbb{N}$ is a fixed number. Our main result is a generalization of the theorem concerning separation by affine functions obtained recently by K. Nikodem and the present author in [3]. To get it we use Behrends and Nikodem's abstract selection theorem (cf. [1, Theorem 1]). It is a variation of Helly's theorem (cf. [7, Theorem 6.1]).

We denote by $\mathrm{cc}(\mathbb{R})$ the family of all non-empty compact real intervals. Recall that if $F : I \to \mathrm{cc}(\mathbb{R})$ is a set-valued function then a function $f : I \to \mathbb{R}$ is called a *selection* of F iff $f(x) \in F(x)$ for every $x \in I$.

Behrends and Nikodem's theorem states that if $\mathcal{W}$ is an n-dimensional space of functions mapping I into $\mathbb{R}$ then a set-valued function $F : I \to \mathrm{cc}(\mathbb{R})$ has a selection belonging to $\mathcal{W}$ if and only if for any $n+1$ points $x_1, \ldots, x_{n+1} \in I$ there exists $f \in \mathcal{W}$ such that $f(x_i) \in F(x_i)$ for $i = 1, \ldots, n+1$.

Let us start with the notation used in this paper. Let $n \in \mathbb{N}$. If $x_1, \ldots, x_n \in I$ are distinct then for $i = 1, \ldots, n$ we define

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$$c_i(x;x_1,\ldots,x_n)=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}.$$

Note that $c_i(x_j;x_1,\ldots,x_n)$ is 0 if $i\neq j$ and 1 if $i=j$, $i,j=1,\ldots,n$. $\mathcal{P}_n$ denotes the family of all polynomials of degree at most n. If $x_1,\ldots,x_{n+1}\in I$ are distinct then the (unique) Lagrange interpolating polynomial passing through the points (x_i,y_i), $i=1,\ldots,n+1$, is

$$w(x)=\sum_{i=1}^{n+1}c_i(x;x_1,\ldots,x_{n+1})y_i. \tag{1}$$

This polynomial is a member of $\mathcal{P}_n$. Moreover, if $x<x_1<\ldots<x_{n+1}$ then $c_i(x;x_1,\ldots,x_{n+1})$ is positive if i is odd and negative if i is even.

2. Polynomial selections of set-valued functions. Now we prove a selection theorem which will be used to obtain our main result. If $n\in\mathbb{N}$ and $A_i\subset\mathbb{R}$, $i=1,\ldots,n$, then $\sum_{i=1}^{n}A_i$ denotes the algebraic sum of the sets A_i, $i=1,\ldots,n$.

THEOREM 1. *Let $n\in\mathbb{N}$. A set-valued function $F:I\to\mathrm{cc}(\mathbb{R})$ has a selection belonging to $\mathcal{P}_n$ if and only if for all $x_0,x_1,\ldots,x_{n+1}\in I$ such that $x_0<x_1<\ldots<x_{n+1}$ the following condition holds*:

$$F(x_0)\cap\Big(\sum_{i=1}^{n+1}c_i(x_0;x_1,\ldots,x_{n+1})F(x_i)\Big)\neq\emptyset. \tag{2}$$

Proof. If F has a selection belonging to $\mathcal{P}_n$ then (2) is obvious. We prove the converse. First we note that $\mathcal{P}_n$ is an $(n+1)$-dimensional space of functions. If we prove that for any $n+2$ points $x_0,x_1,\ldots,x_{n+1}\in I$ there exists $w\in\mathcal{P}_n$ such that $w(x_i)\in F(x_i)$, $i=0,1,\ldots,n+1$, then by Behrends and Nikodem's theorem F will have a desired selection. (For another Helly-type theorem which may be used here cf. also [7, Theorem 6.9].)

Fix $x_0,x_1,\ldots,x_{n+1}\in I$ with $x_0<x_1<\ldots<x_{n+1}$. Let $L_i=c_i(x_0;x_1,\ldots,x_{n+1})$, $i=1,\ldots,n+1$. Thus (2) has the form

$$F(x_0)\cap\Big(\sum_{i=1}^{n+1}L_iF(x_i)\Big)\neq\emptyset. \tag{3}$$

As noted in Section 1, L_i is positive if i is odd and negative if i is even.

Put

$$y_0=\inf F(x_0),\qquad z_0=\sup F(x_0)$$

and for $i=1,\ldots,n+1$,

$$y_i=\begin{cases}\inf F(x_i) & \text{if } L_i>0,\\ \sup F(x_i) & \text{if } L_i<0,\end{cases}\qquad z_i=\begin{cases}\sup F(x_i) & \text{if } L_i>0,\\ \inf F(x_i) & \text{if } L_i<0.\end{cases}$$

Then $F(x_0)=[y_0,z_0]$ and for $i=1,\ldots,n+1$,

$$F(x_i)=\begin{cases}[y_i,z_i] & \text{if } L_i>0,\\ [z_i,y_i] & \text{if } L_i<0.\end{cases}$$

Since $-[\alpha,\beta]=[-\beta,-\alpha]$ for all $\alpha,\beta\in\mathbb{R}$, we have $L_iF(x_i)=[L_iy_i,L_iz_i]$, $i=1,\ldots,n+1$. If $u=L_1y_1+\ldots+L_{n+1}y_{n+1}$ and $v=L_1z_1+\ldots+L_{n+1}z_{n+1}$ then $u\le v$. Furthermore,

$$\begin{aligned}\sum_{i=1}^{n+1}L_iF(x_i)&=[L_1y_1,L_1z_1]+\ldots+[L_{n+1}y_{n+1},L_{n+1}z_{n+1}]\\ &=[L_1y_1+\ldots+L_{n+1}y_{n+1},L_1z_1+\ldots+L_{n+1}z_{n+1}]=[u,v]\end{aligned}$$

and by (3) we get

$$[y_0,z_0]\cap[u,v]\neq\emptyset. \tag{4}$$

Three cases are possible:

(a) $u\in[y_0,z_0]$,
(b) $v\in[y_0,z_0]$,
(c) $[y_0,z_0]\subset[u,v]$.

Fix $t\in[0,1]$ and consider the polynomial $\varphi_t\in\mathcal{P}_n$ passing through the $n+1$ points

$$(x_0,tu+(1-t)v)\quad\text{and}\quad(x_i,ty_i+(1-t)z_i)\quad\text{for } i=1,\ldots,n-1,n+1.$$

We shall show later that

$$\varphi_t(x_n)=ty_n+(1-t)z_n. \tag{5}$$

Hence, in case (a) for $w=\varphi_1$ we have

$$\begin{aligned}w(x_0)&=u\in[y_0,z_0]=F(x_0),\\ w(x_i)&=y_i\in F(x_i),\qquad i=1,\ldots,n-1,n,n+1.\end{aligned}$$

and similarly in case (b) for $w=\varphi_0$. In case (c), $y_0=\lambda u+(1-\lambda)v$ for some $\lambda\in[0,1]$. For $w=\varphi_\lambda$ we obtain

$$\begin{aligned}w(x_0)&=y_0\in F(x_0),\\ w(x_i)&=\lambda y_i+(1-\lambda)z_i\in F(x_i),\qquad i=1,\ldots,n-1,n,n+1.\end{aligned}$$

So in all cases there exists a $w\in\mathcal{P}_n$ such that $w(x_i)\in F(x_i)$, $i=0,\ldots,n+1$. We will complete the proof if we show that (5) holds true.

By (1) we get

$$\varphi_t(x) = c_0(x; x_0, x_1, \ldots, x_{n-1}, x_{n+1})(tu + (1-t)v) + \sum_{i=1}^{n-1} c_i(x; x_0, x_1, \ldots, x_{n-1}, x_{n+1})(ty_i + (1-t)z_i) + c_{n+1}(x; x_0, x_1, \ldots, x_{n-1}, x_{n+1})(ty_{n+1} + (1-t)z_{n+1}).$$

If $M_i = c_i(x_n; x_0, x_1, \ldots, x_{n-1}, x_{n+1})$, $i = 0, 1, \ldots, n-1, n+1$, then after some computation

$$\varphi_t(x_n) = \sum_{\substack{i=1\\ i\neq n}}^{n+1} (M_0L_i + M_i)(ty_i + (1-t)z_i) + M_0L_n(ty_n + (1-t)z_n).$$

One can verify (using the product formula given in the introduction) that $M_0L_n = 1$ and $M_0L_i + M_i = 0$ for $i = 1, \ldots, n-1, n+1$. Hence (5) holds and this finishes the proof. ∎

As a consequence of Theorem 1 we obtain

COROLLARY 1 [8, Theorem 1]. *A set-valued function $F : I \to \mathrm{cc}(\mathbb{R})$ has an affine selection iff for all $x, y \in I$ and $t \in [0, 1]$,*

$$F(tx + (1-t)y) \cap (tF(x) + (1-t)F(y)) \neq \emptyset.$$

Proof. The above condition is equivalent to (2) for $n = 1$, $x < y$, $x_0 = x$, $x_2 = y$, $x_1 = tx_0 + (1-t)x_2$, where $t = (x_1 - x_2)/(x_0 - x_2)$. ∎

3. Separation by polynomials.

The main result of this paper is

THEOREM 2. *Let $n \in \mathbb{N}$ and $f, g : I \to \mathbb{R}$. The following conditions are equivalent:*

(i) *there exists $w \in \mathcal{P}_n$ such that $f(x) \le w(x) \le g(x)$, $x \in I$;*

(ii) *$f(b) \le g(b)$, where $b \in I$ is the right endpoint of I (if it exists) and for all $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 \le x_1 < \ldots < x_{n+1}$,*

$$(6)\quad \begin{aligned} f(x_0) &\le \sum_{\substack{i=1\\ i\ \mathrm{odd}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1})g(x_i) + \sum_{\substack{i=1\\ i\ \mathrm{even}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1})f(x_i),\\ g(x_0) &\ge \sum_{\substack{i=1\\ i\ \mathrm{odd}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1})f(x_i) + \sum_{\substack{i=1\\ i\ \mathrm{even}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1})g(x_i).\end{aligned}$$

Proof. To prove that (i) implies (ii) fix any $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 \le x_1 < \ldots < x_{n+1}$. Since the polynomial w passes through the

points $(x_i, w(x_i))$, $i = 1, \ldots, n+1$, we have

$$w(x_0) = \sum_{i=1}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1})w(x_i).$$

Then the inequalities (6) are obvious.

To prove the converse implication first note that replacing x_0 by x_1 in (6) we have $f(x_1) \le g(x_1)$ in both inequalities, i.e. (ii) yields $f \le g$ on I. Let

$$F(x) = [f(x), g(x)], \quad x \in I.$$

We now show that $F : I \to \mathrm{cc}(\mathbb{R})$ satisfies (2). Fix any $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 < x_1 < \ldots < x_{n+1}$. Let u and v be the right hand sides of the upper and lower inequalities of (6), respectively. Then $v \le u$ and

$$(7)\quad [f(x_0), g(x_0)] \cap [v, u] \neq \emptyset$$

(otherwise $g(x_0) < v$ or $u < f(x_0)$, a contradiction with (6)). Let $L_i = c_i(x_0; x_1, \ldots, x_{n+1})$, $i = 1, \ldots, n+1$. Then

$$L_iF(x_i) = \begin{cases} [L_if(x_i), L_ig(x_i)] & \text{if } i \text{ is odd},\\ [L_ig(x_i), L_if(x_i)] & \text{if } i \text{ is even},\end{cases}$$

and

$$[v, u] = \sum_{i=1}^{n+1} L_iF(x_i).$$

Thus (7) implies (2). By Theorem 1, F has a selection $w \in \mathcal{P}_n$. This finishes the proof. ∎

Remark 1. Inequalities (6) in Theorem 2 do not guarantee $f(b) \le g(b)$, where $b \in I$ is the right endpoint of I (if it exists). The functions

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \le x < 1,\\ 1 & \text{for } x = 1,\end{cases} \quad \text{and} \quad g(x) = \begin{cases} x & \text{for } 0 \le x < 1,\\ \frac{1}{2} & \text{for } x = 1,\end{cases}$$

satisfy (6) for $n = 1$ but $f(1) > g(1)$. Of course, f and g cannot be separated by a straight line.

As a consequence of Theorem 2 we obtain

COROLLARY 2 [3, Theorem 1]. *Let $f, g : I \to \mathbb{R}$. The following conditions are equivalent:*

(i) *there exists an affine function $h : I \to \mathbb{R}$ such that $f(x) \le h(x) \le g(x)$, $x \in I$;*

(ii) *for all $x, y \in I$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \le tg(x) + (1-t)g(y)$$

and

$$g(tx+(1-t)y) \geq tf(x)+(1-t)f(y).$$

Proof. The above inequalities are equivalent to (ii) of Theorem 2 (cf. the proof of Corollary 1). ∎

4. Applications. One can verify that $f: I \to \mathbb{R}$ is convex iff for all $x_0, x_1, x_2 \in I$ such that $x_0 < x_1 < x_2$,

$$f(x_0) \geq c_1(x_0; x_1, x_2) f(x_1) + c_2(x_0; x_1, x_2) f(x_2).$$

We adopt the following definition (cf. [6, §83], [2], [4], [5]).

DEFINITION. Let $n \in \mathbb{N}$. A function $f: I \to \mathbb{R}$ is *n-convex* iff for all $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 < x_1 < \ldots < x_{n+1}$,

$$(-1)^n f(x_0) \leq (-1)^n \sum_{i=1}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) f(x_i).$$

f is *n-concave* iff $-f$ is n-convex. ∎

If f is both n-convex and n-concave then f is a polynomial belonging to $\mathcal{P}_n$ (passing through the points $(x_i, f(x_i))$, $i = 0, 1, \ldots, n+1$).

COROLLARY 3. *Let $n \in \mathbb{N}$. If $f: I \to \mathbb{R}$ is n-convex, $g: I \to \mathbb{R}$ is n-concave and $f(x) \leq g(x)$, $x \in I$, then there exists $w \in \mathcal{P}_n$ such that $f(x) \leq w(x) \leq g(x)$, $x \in I$.*

Proof. Fix any $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 \leq x_1 < \ldots < x_{n+1}$. If n is even then by n-convexity of f,

$$\begin{aligned} f(x_0) &\leq \sum_{i=1}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) f(x_i) \\ &\leq \sum_{\substack{i=1 \\ i \text{ odd}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) g(x_i) + \sum_{\substack{i=1 \\ i \text{ even}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) f(x_i). \end{aligned}$$

If n is odd then by n-concavity of g,

$$\begin{aligned} f(x_0) \leq g(x_0) &\leq \sum_{i=1}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) g(x_i) \\ &\leq \sum_{\substack{i=1 \\ i \text{ odd}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) g(x_i) + \sum_{\substack{i=1 \\ i \text{ even}}}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) f(x_i). \end{aligned}$$

The proof of the second inequality in (6) is analogous. Theorem 2 completes the proof. ∎

In the same way we get

COROLLARY 4. *Let $n \in \mathbb{N}$. If $f: I \to \mathbb{R}$ is n-concave, $g: I \to \mathbb{R}$ is n-convex and $f(x) \leq g(x)$, $x \in I$, then there exists $w \in \mathcal{P}_n$ such that $f(x) \leq w(x) \leq g(x)$, $x \in I$.* ∎

For $n = 1$ the above two results are well known.

Finally, we prove a stability result for polynomials (cf. the Hyers–Ulam type stability theorem for affine functions in [3]). First observe that if $n \in \mathbb{N}$ and $w(x) = 1$, $x \in I$, then $w \in \mathcal{P}_n$ and for any distinct $x_1, \ldots, x_{n+1} \in I$, (1) has the form

$$\sum_{i=1}^{n+1} c_i(x; x_1, \ldots, x_{n+1}) = 1, \quad x \in I.$$

COROLLARY 5. *Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $f: I \to \mathbb{R}$. If for all $x_0, x_1, \ldots, x_{n+1} \in I$ such that $x_0 \leq x_1 < \ldots < x_{n+1}$,*

$$\left| f(x_0) - \sum_{i=1}^{n+1} c_i(x_0; x_1, \ldots, x_{n+1}) f(x_i) \right| \leq \varepsilon \tag{8}$$

then there exists a polynomial $w \in \mathcal{P}_n$ such that

$$|f(x) - w(x)| \leq \varepsilon/2, \quad x \in I. \tag{9}$$

Proof. If f satisfies (8) then (ii) of Theorem 2 holds for $g(x) = f(x) + \varepsilon$, $x \in I$. So there exists $\varphi \in \mathcal{P}_n$ such that $f(x) \leq \varphi(x) \leq f(x) + \varepsilon$, $x \in I$. For

$$w(x) = \varphi(x) - \varepsilon/2, \quad x \in I,$$

we obtain (9). ∎

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Invariant densities for C^1 maps

by

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Abstract. We consider the set of C^1 expanding maps of the circle which have a unique absolutely continuous invariant probability measure whose density is unbounded, and show that this set is dense in the space of C^1 expanding maps with the C^1 topology. This is in contrast with results for C^2 or $C^{1+\varepsilon}$ maps, where the invariant densities can be shown to be continuous.

For expanding maps of the circle which are C^2 or $C^{1+\varepsilon}$ (that is, differentiable with Hölder continuous derivative), there is always a unique absolutely continuous invariant probability measure whose density is continuous and strictly positive. These functions will be called *invariant densities*. These maps with their unique absolutely continuous invariant measures form exact systems (see [4]). Our paper deals with the case of C^1 expanding maps.

Throughout this paper, let $E^1(M)$ denote the space of expanding C^1 mappings of a compact manifold M to itself with the C^1 topology. In [3], Krzyżewski showed that the subset $A \subset E^1(M)$ of those mappings which have no absolutely continuous invariant probability measure with strictly positive continuous density is residual or of second category in $E^1(M)$. This means that topologically "most" mappings fail to have absolutely continuous invariant probability measures which have continuous densities bounded away from 0. Clearly there are a number of ways in which this failure can take place: One way is for there to be no absolutely continuous invariant probability measure. In the case where M is the unit circle, S^1, Góra and Schmitt showed that this can occur (see [1]). A second possibility is that there may be examples which have absolutely continuous invariant densities which fail to be continuous or fail to be bounded away from 0, although no examples of this type are in the literature. In particular, the question might be asked as to whether there are examples of C^1 expanding maps which have

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