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Analytic and C^k approximations of norms in separable Banach spaces

by

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Abstract. We prove that in separable Hilbert spaces, in $\ell_p(\mathbb{N})$ for p an even integer, and in $L_p[0,1]$ for p an even integer, every equivalent norm can be approximated uniformly on bounded sets by analytic norms. In $\ell_p(\mathbb{N})$ and in $L_p[0,1]$ for $p \notin \mathbb{N}$ (resp. for p an odd integer), every equivalent norm can be approximated uniformly on bounded sets by $C^{[p]}$-smooth norms (resp. by C^{p-1}-smooth norms).

Introduction. It is well known that in separable Banach spaces, or more generally in weakly countably determined Banach spaces, the existence of a C^k-Fréchet differentiable bump function implies the possibility of uniform approximation of continuous functions by C^k-smooth functions (see for instance [DGZ, Theorem VIII-3-2]). Similarly, the existence of a separating polynomial implies the possibility of analytic approximations, as shown in [Ku2].

However, the more subtle question of uniform approximation on bounded sets of an arbitrary equivalent norm on a Banach space by a C^k-smooth norm—assuming the existence of some equivalent C^k-smooth norm on the space—seems to be of a different nature, and until now there have been no examples available of infinite-dimensional spaces with this property if $k > 1$.

In [DFH], we gave a positive answer to this question in separable polyhedral Banach spaces.

We show here that separable Hilbert spaces, ℓ_p spaces for p an even integer, and $L_p[0,1]$ for p an even integer allow approximations by analytic norms. This result should be compared with [D] where it is proved that every Banach space with an equivalent C^∞-smooth norm (bump) contains an isomorphic copy of c_0 or ℓ_p, with p an even integer.

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We further show that spaces with Schauder basis that admit a C^k-smooth equivalent norm such that all its derivatives of order less than or equal to k are bounded on bounded sets, also admit approximations by C^k-smooth norms (in general without bounded derivatives). We will comment on the boundedness condition later on.

Thus, in $\ell_p(\mathbb{N})$ and in $L_p[0,1]$ for $p \notin \mathbb{N}$ (resp. for p an odd integer), every equivalent norm can be approximated uniformly on bounded sets by $C^{[p]}$-smooth norms (resp. by C^{p-1}-smooth norms). This is optimal, since by [Ku1], for p not an even integer, ℓ_p does not admit a C^k equivalent norm if $k > p$.

Since there is a natural correspondence between closed, convex and bounded sets in a normed space, containing $\vec{0}$ as an interior point, and their Minkowski functionals, the previous statements can be reformulated in the language of convex sets.

The proof of the above statements is done in two steps.

First it is shown that an arbitrary closed, convex and bounded set S_1, $\vec{0} \in \operatorname{int} S_1$, can be arbitrarily well approximated by another closed, convex and bounded set $S_2 = \{x \in X : f_i(x) \le 1,\ i \in \mathbb{N}\}$, where $\{f_i\}_{i\in\mathbb{N}}$ are C^k-smooth convex functions, satisfying some other technical conditions. Above all, for every $x \in \partial S_2$ there exists an $i \in \mathbb{N}$ such that $f_i(x) = 1$. (In case f_i are linear they form the so-called boundary of the set S_2.)

Then the general Theorem 1.3 is applied. This theorem can be viewed as a nonlinear generalization of Theorem 1 from [DFH]. This theorem shows that the body $S_2 = \{x \in X : f_i(x) \le 1,\ i \in \mathbb{N}\}$ can be arbitrarily well approximated by a body $S_3 = \{x \in X : f(x) \le 1\}$, where f is a C^k-smooth convex function.

The uniform boundedness conditions on the derivatives of $\{f_i\}_{i\in\mathbb{N}}$ in Theorem 1.3 are local. Yet some global boundedness condition on the derivatives of an equivalent norm on X seems to be necessary in the first step of the construction, in order to obtain $\{f_i\}_{i\in\mathbb{N}}$ that meet the local conditions.

Related to this is an example in [NS] of an equivalent norm on ℓ_2 not allowing approximations by C^2-smooth norms whose second derivative is uniformly continuous. More recently, Petr Habala and Petr Hájek [HH] proved that if P is a polynomial on ℓ_p endowed with its natural norm, then there exists an infinite-dimensional subspace Z of ℓ_p such that P is essentially constant on the unit sphere of Z.

Throughout the paper we use the standard notation and terminology of Banach space theory. By saying that a homogeneous function is of some class of smoothness we always mean that it is so away from the origin. A Minkowski functional is always meant to correspond to a closed, convex and bounded set containing $\vec{0}$ as an interior point. By saying that a closed, convex and bounded set S_1 in $(X, \|\cdot\|)$ with $\vec{0} \in \operatorname{int} S_1$ is arbitrarily approximable by closed, convex and bounded sets from some class $\mathcal{C}$, we mean that for every $\varepsilon > 0$ there exists $S_2 \in \mathcal{C}$ such that $(1-\varepsilon)S_2 \subset S_1 \subset (1+\varepsilon)S_2$. This is equivalent to the uniform approximation of the corresponding Minkowski functional on bounded sets.

1. Smooth approximation in spaces with countable generalized boundary. Let $(X, \|\cdot\|)$ be a Banach space, D be a closed, convex and bounded set in X with $\vec{0} \in \operatorname{int} D$. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of homogeneous, continuous and convex functions on X such that $D = \{x : f_i(x) \le 1,\ i \in \mathbb{N}\}$.

Definition 1.1. We say that $\{f_i\}_{i\in\mathbb{N}}$ as above forms a *countable generalized boundary* of D if for every $x \in \partial D$ there exists some $i \in \mathbb{N}$ such that $f_i(x) = 1$.

Example. If K is a countable compact set and δ_x denotes the Dirac measure at $x \in K$, then the set $B = \{\pm\delta_x : x \in K\}$ is a countable linear boundary of the unit ball of $C(K)$. This fact was used in [DFH] to prove that there exists on the space $C(K)$ of real-valued continuous functions on K an equivalent analytic norm.

Now, let $1 < p < +\infty$ and $(\Omega, \mathcal{B}, \mu)$ be a measure space. Denote by $C(K, L^p)$ the space of continuous functions from K into $L^p(\Omega, \mathcal{B}, \mu)$ equipped with its natural norm $\|\varphi\| = \sup\{\|\varphi(x)\|_p : x \in K\}$. For $x \in K$, let f_x be the convex continuous function on $C(K, L^p)$ defined by $f_x(\varphi) = \|\varphi(x)\|_p^p$. The set $B = \{f_x : x \in K\}$ is a generalized countable boundary of the unit ball of $C(K, L^p)$. Observe that each f_x has the same order of smoothness as the norm of L^p. We shall see below (Corollary 1.5) how this can be used to construct an equivalent norm on $C(K, L^p)$ with the same order of smoothness as the norm of L^p.

The following facts on complex spaces and functions can be found in [Ku2] and references therein. Given $(X, \|\cdot\|)$ a real normed (Banach) space, we can pass to its complexification $(X^c, \|\cdot\|_{X^c})$ which, considered as a real normed (Banach) space, is isomorphic to $X \oplus X$ with the norm $|||(x,y)||| = \|x\| + \|y\|$. For P a k-homogeneous polynomial on X, we denote by $A_P(x_1, \ldots, x_k)$ the corresponding symmetric k-linear form. The extension of A_P to the complexification is defined by multilinearity and is still denoted by A_P. We then define the complexified polynomial P^c of P on X^c by

$$P^c((x,y)) = A_P(x+iy, x+iy, \ldots, x+iy).$$

Then

$$|||P^c||| \le 2^k \|A_P\| \le 2^k \frac{k^k}{k!} \|P\|.$$

For the last inequality see [N, p. 7].

It follows from Stirling's formula that for some $K' > 0$,

$$\frac{k^k}{k!} < K' \cdot e^k \quad \text{for every } k \in \mathbb{N}.$$

Find $K > 0$ such that $K|||\cdot||| \geq \|\cdot\|_{X^c}$. Then

$$\|P^c\|_{X^c} \leq K'(2Ke)^k \|P\|.$$

Thus whenever f is a real analytic function at $x \in X$ with the radius of convergence r, we can pass to its holomorphic complexification f^c at $(x, \vec{0})$ with the $\|\cdot\|_{X^c}$-radius of convergence at least $r/(2Ke)$.

DEFINITION 1.2. Let $k \in \mathbb{N} \cup \{+\infty\} \cup \{\omega\}$. We say that a sequence $\{f_i\}_{i\in\mathbb{N}}$ of convex and continuous functions defined on $(X, \|\cdot\|)$ satisfies the *condition* (k) if the following hold:

(i) If $k \in \mathbb{N}$, then $f_i|_O$ are C^k-Fréchet differentiable and for every $l \leq k$ and every $\vec{0} \neq x \in X$, there exists a neighbourhood O of x such that $\|D^l f_i\| |_O$ are uniformly bounded.

(ii) If $k = +\infty$, then $f_i|_O$ are C^∞-Fréchet differentiable and for every $l \in \mathbb{N}$ and $\vec{0} \neq x \in X$, there exists a neighbourhood O of x such that $\|D^l f_i\| |_O$ are uniformly bounded.

(iii) If $k = \omega$, then f_i are real analytic on $X \setminus \{\vec{0}\}$, and $\{f_i\}_{i\in\mathbb{N}}$ satisfies the following equicontinuity property: for every $\vec{0} \neq x \in X$ and $\delta > 0$ there exists an $r > 0$ such that

$$|f_j^c(z)| < |f_j(x)| + \delta \quad \text{for } \|z - (x, \vec{0})\|_{X^c} < r \text{ and } j \in \mathbb{N}.$$

THEOREM 1.3. *Let $(X, \|\cdot\|)$ be a separable Banach space, $D \subset X$ be a closed convex and bounded set, $\vec{0} \in \operatorname{int} D$. Suppose $\{f_i\}_{i\in\mathbb{N}}$ is a countable generalized boundary of D satisfying the condition (k), where $k \in \mathbb{N} \cup \{+\infty\}$ (resp. satisfying condition (ω)). Then the Minkowski functional of D can be approximated by C^k-smooth (resp. analytic) Minkowski functionals.*

A first application of Theorem 1.3 concerns c_0-sums of smooth spaces. It is proved in [FPWZ] that the real Banach space $c_0(\mathbb{N})$ admits an equivalent analytic norm. We extend this result here as follows:

COROLLARY 1.4. *Let $(X_i, \|\cdot\|_i)$ be a sequence of Banach spaces, and let X be the c_0-sum of the X_i, i.e. $X = \{(x_i) : x_i \in X_i \text{ and } \lim_i \|x_i\|_i = 0\}$, endowed with its usual norm $\|x\| = \sup_i \|x_i\|_i$. Assume that the norms $\|\cdot\|_i$ are analytic on $X_i \setminus \{\vec{0}\}$ and that they satisfy the condition that for every $\vec{0} \neq x = (x_n)$ and $\delta > 0$ there exists an $r > 0$ such that*

$$|\,\|z_i\|_i^c| < \|x_i\|_i + \delta \quad \text{for } \|z - (x, \vec{0})\|_{X^c} < r \text{ and } i \in \mathbb{N}.$$

Then there exists on X an equivalent analytic norm.

Note that in the above corollary, $\|\cdot\|_{X^c}$ denotes the norm of the complexification X^c of X, while $\|\cdot\|_i^c$ is the holomorphic complexification of $\|\cdot\|_i$.

The hypothesis on the X_i's is satisfied when the spaces X_i are all equal to some space L^p, with p an even integer.

Proof of Corollary 1.4. For $x = (x_i) \in X$, put $f_i(x) = \|x_i\|_i$. The sequence $\{f_i\}$ is a countable generalized boundary of the unit ball of X satisfying the condition (ω). Consequently, the norm of X can be approximated, uniformly on bounded sets, by a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of analytic Minkowski functionals. The function

$$N(x) = \frac{\varphi_1(x) + \varphi_1(-x)}{2}$$

is an equivalent analytic norm on X.

In [H1], R. Haydon proved that if K is a countable compact set, then there exists on the space $C(K)$ of real-valued continuous functions on K an equivalent C^∞-smooth norm. In [DFH], we proved that if K is a countable compact set, then there exists on the space $C(K)$ an equivalent analytic norm. The following result is a vector-valued extension of this result. Assertions (ii) and (iii) below also follow from the work of R. Haydon [H2].

COROLLARY 1.5. *Let K be a countable compact set, $1 < p < +\infty$ and $(\Omega, \mathcal{B}, \mu)$ be a measure space.*

(i) *If p is an even integer, there exists on $C(K, L^p)$ an equivalent analytic norm.*

(ii) *If p is an odd integer, there exists on $C(K, L^p)$ an equivalent C^{p-1}-Fréchet differentiable norm.*

(iii) *If p is not an integer, there exists on $C(K, L^p)$ an equivalent $C^{[p]}$-Fréchet differentiable norm.*

Proof. For $x \in K$, let f_x be the convex continuous function on $C(K, L^p)$ defined by $f_x(\varphi) = \|\varphi(x)\|_p^p$. We already noticed at the beginning of the section that $\{f_x\}_{x\in K}$ is a countable generalized boundary of the unit ball of $C(K, L^p)$. Let us now check condition (k).

It is well known (see for instance [DGZ]) that the pth power of the norm on L^p is C^{p-1}-Fréchet differentiable with derivatives of order $l \leq p-1$ bounded on bounded sets if p is an odd integer, and $C^{[p]}$-Fréchet differentiable with derivatives of order $l \leq [p]$ bounded on bounded sets if p is not an integer. Consequently, $\{f_x\}_{x\in K}$ is a countable generalized boundary satisfying the condition (k) where $k = p - 1$ if p is an odd integer and $k = [p]$ if p is not an integer.

When p is an even integer, the pth power of the norm on L^p is a polynomial, hence it is analytic and its holomorphic extension N to the complexified space is uniformly continuous on bounded sets. Let $\varphi \in C(K, L^p)$, and let ψ in the complexified space $C(K, L^p((\Omega, \mathcal{B}, \mu), \mathbb{C}))$ satisfy $\|\varphi - \psi\| < r$. The set $\{\varphi(x) : x \in K\}$ is bounded in L^p and, for all $x \in K$, $f_x^c(\psi) - f_x(\varphi) =$

$N(\psi(x)) - N(\varphi(x))$. Therefore, condition (ω) follows from the uniform continuity of N on bounded sets of L^p.

The result follows now by applying Theorem 1.3.

Proof of Theorem 1.3. Choose $\varepsilon_i \searrow 0$. Put $\widetilde{f}_i = (1+\varepsilon_i)f_i$. It is standard to check that $\{\widetilde{f}_i\}_{i\in\mathbb{N}}$ again satisfies the condition (k). Moreover, $\{\widetilde{f}_i\}_{i\in\mathbb{N}}$ forms a countable generalized boundary of the set

$$\widetilde{D} = \{x : \widetilde{f}_i(x) \le 1,\ i \in \mathbb{N}\}.$$

Also, for every $x \in \partial\widetilde{D}$ there exists an $i \in \mathbb{N}$ such that

$$(1) \qquad j > i \quad \text{implies} \quad \widetilde{f}_j(x) \le \frac{1+\varepsilon_{i+1}}{1+\varepsilon_i} < 1.$$

Letting $\varepsilon_1 \to 0$ gives us arbitrarily good approximation of D by $\widetilde{D}$. Therefore it is enough to prove our result for $\widetilde{D}$.

Define $\psi(x) = e^{-1}e^x$, $x \in \mathbb{R}$. Put $h_i = \psi \circ \widetilde{f}_i$. It is again standard to check that $\{h_i\}_{i\in\mathbb{N}}$ satisfies the condition (k), the h_i are nonnegative, $\widetilde{D} = \{x : h_i(x) \le 1,\ i \in \mathbb{N}\}$ and, moreover, from (1), there exists a sequence $\delta_i \searrow 0$ such that for every $x \in \widetilde{D}$ there exists $i(x) \in \mathbb{N}$ and a neighbourhood $O(x) \subset X$ of x such that

$$(2) \qquad j > i(x) \quad \text{implies} \quad h_j(y) < 1 - \delta_{i(x)} \quad \text{for } y \in O(x).$$

In case $k = \omega$, it follows from condition (ω) that for some neighbourhood $O^c \subset X^c$ of $(x, \vec{0})$ where $O^c \cap X = O$ there exist $\delta > 0$ and $i \in \mathbb{N}$ such that

$$(3) \qquad |h_j^c(z)| < 1 - \delta \quad \text{for } z \in O^c \text{ and } j > i.$$

Let $\{p_i\}_{i\in\mathbb{N}}$ be an increasing sequence of even integers. It follows from (2) that

$$G(x) = \sum_{i=1}^{\infty} (h_i(x))^{p_i}$$

is a well-defined function in a neighbourhood of $\widetilde{D}$. Let

$$A = \{x \in X : G(x) \le 1\}.$$

Clearly, $A \subset \widetilde{D}$. On the other hand, let us fix $\varepsilon > 0$. We want to show that if p_1 is large enough, then $(1-\varepsilon)\widetilde{D} \subset A$. Indeed, the functions f_i being homogeneous, one has $f_i(0) = 0$, hence $h_i(0) = 1/e$ for all $i \in \mathbb{N}$. Since the functions h_i are convex, if $x \in (1-\varepsilon)\widetilde{D}$, then $h_i(x) \le 1/e + (1-1/e)(1-\varepsilon) = \alpha < 1$. Hence $G(x) \le \sum_{i=p_1}^{\infty} \alpha^i \le 1$ if p_1 is large enough. Consequently, $(1-\varepsilon)\widetilde{D} \subset A \subset \widetilde{D}$. This proves that the Minkowski functional of A approximates (in the topology of uniform convergence on bounded sets) the Minkowski functional of D.

We will prove that if the sequence $\{p_i\}_{i\in\mathbb{N}}$ grows fast enough, the function G has the same smoothness properties as the functions h_i in a neighbourhood of $\widetilde{D} \setminus \{\vec{0}\}$. We shall then deduce from this that the Minkowski functional of A has the same smoothness properties. This will finish the proof of Theorem 1.3.

Proof in the case $k \in \mathbb{N} \cup \{+\infty\}$. Using the Lindelöf property of $(X, \|\cdot\|)$ and condition (2), choose a sequence $\{O_j\}_{j\in\mathbb{N}}$ of open subsets of X and a sequence $\{i(j)\}_{j\in\mathbb{N}}$ of integers such that

(i) $\widetilde{D} \subset \bigcup_{j\in\mathbb{N}} O_j$.

(ii) For $l \in \mathbb{N}$, $l \le k$, $\|D^l h_i(\cdot)\|$ are uniformly bounded on O_j.

(iii) For every $n > i(j)$ and for every $y \in O_j$, $h_n(y) < 1 - \delta_{i(j)}$.

Now let $\{(k_m, l_m)\}_{m\in\mathbb{N}}$ denote an enumeration of $\mathbb{N} \times \mathbb{N}$ if $k = \infty$, and an enumeration of $\mathbb{N} \times \{0, 1, \ldots, k\}$ if $k \in \mathbb{N}$.

By induction on $m \in \mathbb{N}$, we construct a system $\{p_{m,n}\}_{n\in\mathbb{N}}$ of increasing sequences of even integers such that $\{p_{m+1,n}\}_{n\in\mathbb{N}} \subset \{p_{m,n}\}_{n\in\mathbb{N}}$ and for every $m \in \mathbb{N}$ and every subsequence $\{q_n\}_{n\in\mathbb{N}}$ of $\{p_{m,n}\}_{n\in\mathbb{N}}$ the function $G(x) = \sum_{i=1}^{\infty} (h_i(x))^{q_n}$ restricted to O_{k_m} is l_m times continuously differentiable.

Put $\{p_{0,n}\}_{n\in\mathbb{N}} = \{2n\}_{n\in\mathbb{N}}$.

Induction step from m to $m+1$. According to the generalized chain rule (see [Fe, p. 222] for the notation), we compute the βth differential of a composition of a β-differentiable real function f on X with x^p, p an even integer, at $a \in X$ as follows:

$$(4) \qquad D^\beta (f(a))^p = \sum_{\alpha \in S(\beta)} \frac{D^{\Sigma\alpha}((f(a))^p) \circ ((D^1 f(a))^{\alpha_1} \odot \ldots \odot (D^k f(a))^{\alpha_k})}{\alpha!},$$

where $S(\beta)$ is the set of all β-termed sequences α of nonnegative integers such that $\sum_{i=1}^{\beta} i\alpha_i = \beta$. Notice that (4) is a formula with a fixed number of terms on the right hand side, regardless of the value of p.

If $|f(a)| < 1$, we obtain $|D^{\Sigma\alpha}((f(a))^p)| \to 0$ as $p \to +\infty$ for every $\alpha \in S(\beta)$. Consequently, $\|D^\beta (f(a))^p\| \to 0$ as well.

The induction step is as follows: We put $p_{m+1,n} = p_{m,n}$ for $n \le i(k_{m+1})$. For $n > i(k_{m+1})$ we put $p_{m+1,n}$ to be an element from $\{p_{m,n}\}_{n\in\mathbb{N}}$ so large that $\|D^\beta (h_n(y))^{p_{m+1,n}}\| < 2^{-n}$ for all $y \in O_{k_{m+1}}$ and all $\beta \le l_{m+1}$.

Setting $\{p_n\}_{n\in\mathbb{N}}$ to be $\{p_{n,n}\}_{n\in\mathbb{N}}$, the function $G(x) = \sum_{i=1}^{\infty} (h_i(x))^{p_i}$ is C^k-smooth on the open set $\bigcup_{j\in\mathbb{N}} O_j$. The Minkowski functional of the set $A \subset \bigcup_{j\in\mathbb{N}} O_j$ is the function φ given by $G(x/\varphi(x)) = 1$. The convexity of G and the fact that $G(0) < 1$ imply that $G'(x).x \ne 0$ for all x such that $G(x) = 1$. Consequently, $G'(x).x \ne 0$ whenever $G(x) = 1$. Set $F(x, \lambda) =$

$G(x/\lambda)-1$. We have $\frac{\partial F}{\partial\lambda}(x,\lambda)=-\lambda^{-2}G'(x/\lambda).x\neq 0$ whenever $G(x)=1$. The implicit function theorem ([Dieu, p. 261]) applied to the function F shows that the function φ is C^k-smooth and

$$\varphi'(x)=\frac{-\varphi(x)G'(x/\varphi(x))}{G'(x/\varphi(x)).x}.$$

Proof in the analytic case. Let us fix k an even integer greater then or equal to p_1. As $\{h_i\}_{i\in\mathbb{N}}$ satisfy condition (3), the complex series $G^c(z)=\sum_{i=1}^{\infty}(h_i^c(z))^{2i+k}$ is uniformly convergent on some neighbourhood of every point $(x,\vec{0})\in X^c$, where $x\in\widetilde{D}$. According to the uniform convergence theorem for holomorphic functions, $G^c(z)$ is holomorphic as well. The implicit function theorem for holomorphic functions ([Dieu, p. 261]) then shows that the function φ^c defined by $G^c(x/\varphi^c(x))=1$ is holomorphic, hence analytic ([M, p. 62]), and its real part φ is real analytic and is the Minkowski functional of the set $A=\{x\in X: G(x)=\sum_{i=1}^{\infty}(h_i(z))^{2i+k}\leq 1\}$.

2. Smooth approximation in Hilbert spaces.

The goal of this section is to prove:

THEOREM 2.1. *Let $(H,\|\cdot\|)$ be a separable Hilbert space. Then every Minkowski functional on H can be approximated by analytic Minkowski functionals.*

COROLLARY 2.2. *Let $(H,\|\cdot\|)$ be a separable Hilbert space. Then every equivalent norm on H can be approximated by analytic equivalent norms.*

Proof. Let $|||\cdot|||$ be an equivalent norm on H and $\varepsilon>0$. According to Theorem 2.1, there exist an analytic Minkowski functional φ such that $(1-\varepsilon)|||x|||\leq\varphi(x)\leq(1+\varepsilon)|||x|||$. The function N defined by $N(x)=(\varphi(x)+\varphi(-x))/2$ is an equivalent analytic norm on H which also satisfies $(1-\varepsilon)|||x|||\leq N(x)\leq(1+\varepsilon)|||x|||$.

Proof of Theorem 2.1. Let W be a closed, convex and bounded subset of X with $\vec{0}\in\operatorname{int}W$. Without loss of generality, we can assume that W is contained in the unit ball B of H. Our goal is to approximate W by a convex body S which admits a countable generalized boundary $\{f_k\}_{k\in\mathbb{N}}$ satisfying the condition (ω), and then to apply Theorem 1.3.

Denote by W° the polar of W, by $\{e_n\}$ an orthonormal basis of H and by $\mathbb{Z}^{<\mathbb{N}}$ the space of finite sequences of integers. Fix $0<\varepsilon<1/2$. Denote by F the weak closure of the set $W^\circ\cap\{\sum_{(\alpha_n)\in\mathbb{Z}^{<\mathbb{N}}}(\varepsilon/2^n)\alpha_n e_n\}$ and by C its closed convex hull.

CLAIM 1. $(1-2\varepsilon)W^\circ\subset C\subset W^\circ$ *and, for each n, the set $e_n(F)=\{(e_n,x):x\in F\}$ is finite.*

Proof. $e_n(F)$ is finite because it is a bounded subset of the real line included in $(\varepsilon/2^n)\mathbb{Z}$. Clearly, $C\subset W^\circ$. Let us now pick $x\in(1-\varepsilon)W^\circ$. There exist scalars b_n such that $x=\sum_{n=1}^{\infty}b_ne_n$. For each n, choose $a_n\in\mathbb{Z}$ such that $|b_n-(\varepsilon/2^n)a_n|\leq\varepsilon/2^{n+1}$. We have

$$\Big\|x-\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}a_ne_n\Big\|\leq\sum_{n=1}^{\infty}\Big|b_n-\frac{\varepsilon}{2^n}a_n\Big|\leq\frac{\varepsilon}{2}.$$

Therefore, there exists N such that, if we set $a=\sum_{n=1}^{N}(\varepsilon/2^n)a_ne_n$, then $\|x-a\|\leq\varepsilon$. So, $a\in(1-\varepsilon)W^\circ+\varepsilon B\subset W^\circ$, and consequently, $a\in C$ and $x\in C+\varepsilon B$. This proves that $(1-\varepsilon)B\subset(1-\varepsilon)W^\circ\subset C+\varepsilon B$. In particular, $(1-2\varepsilon)B\subset C$. Finally, we obtain $(1-2\varepsilon)(1-\varepsilon)W^\circ\subset(1-2\varepsilon)(C+\varepsilon B)\subset(1-\varepsilon)C$, whence the result.

Let us now denote by $[M_n]$ the subspace of H generated by the e_i, $i>n$.

CLAIM 2. *There exist a sequence $\{h_k\}$ of points in the set F, a sequence $\{n_k\}$ of integers with $n_k\to\infty$, and a decreasing sequence $\{F_\alpha\}$ of weakly closed subsets of F such that*

(i) $\bigcup_{k=1}^{\infty}((h_k+M_{n_k})\cap F_k)=F$.
(ii) $\operatorname{diam}((h_k+M_{n_k})\cap F_k)<\varepsilon$.

Proof. For every $\varepsilon>0$, F can be covered by countably many balls of radius ε, hence, by the Baire Category Theorem, there exist a point $g\in F$ and a weak neighbourhood G of g such that $G\cap F\neq\emptyset$ and $\operatorname{diam}(G\cap F)<\varepsilon$. Because of the structure of the set F, the sets $(h+M_n)\cap F$, $h\in F$, $n\in\mathbb{N}$, form a base of the weak topology on F and each such set is both closed and open in (F,weak). On the other hand, the family $\Im=\{h+M_n: h\in F, n\in\mathbb{N}\}$ contains countably many (different) sets and obviously each weakly compact subset of F has the same structure as F.

For each ordinal α, we define by transfinite induction sets F_α and $(h_\alpha+M_{n(\alpha)})$ as follows: $F_0=F$, $F_{\alpha+1}=F_\alpha\backslash(h_\alpha+M_{n(\alpha)})$, where $(h_\alpha+M_{n(\alpha)})\in\Im$ such that $(h_\alpha+M_{n(\alpha)})\cap F_\alpha\neq\emptyset$ and $\operatorname{diam}((h_\alpha+M_{n(\alpha)})\cap F_\alpha)<\varepsilon$. If α is a limit ordinal, then we put $F_\alpha=\bigcap_{\beta<\alpha}F_\beta$. Since the family $\Im$ is countable and each set F_α is weakly compact there exists a countable ordinal η such that $F_\eta\neq\emptyset$ and $F_{\eta+1}=\emptyset$. It is clear that

$$\bigcup_{\alpha\leq\eta}((h_\alpha+M_{n(\alpha)})\cap F_\alpha)=F.$$

Let us rewrite the countable family $\{h_\alpha+M_{n(\alpha)}\}_{\alpha\leq\eta}$ as $\{h_k+M_{n_k}\}_{k=1}^{\infty}$. Since for each integer q there exist only finitely many members $h+M_n$ of the family $\Im$ such that $n\leq q$, it follows that $n_k\to\infty$ as $k\to\infty$. Claim 2 is proved.

We are now ready to construct the convex body S which admits a countable generalized boundary $\{f_k\}_{k\in\mathbb{N}}$ satisfying the condition (ω). Denote by P_k the orthogonal projection from H onto the subspace M_{n_k}. Define

$$f_k(x) = h_k(x) + \varepsilon\|P_k(x)\| \tag{5}$$

and

$$S = \{x \in H : f_k(x) \le 1 \text{ for all } k \in \mathbb{N}\}. \tag{6}$$

CLAIM 3. *S is a good approximation of W.*

Proof. According to Claim 1, $W \subset C^\circ \subset (1-2\varepsilon)^{-1}W$.

Let $x \in S$. This implies that for each k, $h_k(x) \le 1$, so $x \in C^\circ$. This proves that $S \subset (1-2\varepsilon)^{-1}W$. Conversely, let $x \in W$. Then $x \in C^\circ$, so for each k, $h_k(x) \le 1$. It follows from the proof of Claim 1 that $C^\circ \subset (1-2\varepsilon)^{-1}B$. So $f_k(x) \le h_k(x) + \varepsilon\|x\| \le (1-\varepsilon)(1-2\varepsilon)^{-1}$. This proves that $W \subset (1-\varepsilon)(1-2\varepsilon)^{-1}S$.

CLAIM 4. *$\{f_k\}_{k\in\mathbb{N}}$ satisfies condition (ω).*

Proof. Indeed, the f_k are homogeneous convex continuous functions. Since the square of the norm of a Hilbert space is a polynomial, the norm of H is real analytic on $H\backslash\{\vec{0}\}$. Consequently, the functions f_k are real analytic on $H\backslash\{\vec{0}\}$. A holomorphic extension of f_k is given by

$$f_k^c(x+iy) = h_k(x) + ih_k(y) + \varepsilon\sqrt{\|P_k(x)\|^2 - \|P_k(y)\|^2 + 2i(P_k(x), P_k(y))},$$

where $\sqrt{\ }$ is the determination of the square root on $\mathbb{C}\backslash i\mathbb{R}^+$. The sequence $\{h_k\}$ is bounded in H and the projections P_k are Lipschitz continuous with constant 1. Hence, the functions f_k are uniformly continuous in a neighbourhood of $(x,\vec{0})$, uniformly in k. This implies condition (ω).

CLAIM 5. *$\{f_k\}_{k\in\mathbb{N}}$ is a generalized boundary of S.*

Proof. Define $S^* := \bigcup_{k\in\mathbb{N}}\{h_k + \varepsilon u : u \in M_{n_k},\ \|u\| \le 1\}$. Observe that $S = \{x \in X : h(x) \le 1 \text{ for all } h \in S^*\}$, and, according to Claim 2, $F \subset S^*$.

We first prove that S^* is weakly closed. Indeed, let $\{g_m\}$ be a sequence in S^*, weakly converging to $g_\infty \in X^*$. If there exists k such that infinitely many of the g_m are in the weakly closed set $\{h_k + \varepsilon u : u \in M_{n_k},\ \|u\| \le 1\}$, then of course g_∞ is in this set, hence also in S^*. Otherwise, there exists $k_m \to \infty$ such that $g_m = h_{k_m} + \varepsilon u_m$, with $u_m \in M_{n_k}$, $\|u\| \le 1$. Consequently, u_m weakly converges to 0 and h_{k_m} lies in the weakly closed set F, hence (weak)-$\lim g_m =$ (weak)-$\lim h_{k_m} \in F \subset S^*$.

We now prove Claim 5. Let $x \in \partial S$. Since S^* is weakly closed, there exists $h \in S^*$ such that $h(x) = 1 = \max\{g(x) : g \in S^*\}$. There exists k such that $h = h_k + \varepsilon u$, with $u \in M_{n_k}$, $\|u\| \le 1$. Consequently, $1 = h_k(x) + \varepsilon u(x) \le h_k(x) + \varepsilon\|P_k(x)\| = f_k(x) \le 1$, so (f_k) forms a generalized boundary of S.

We now conclude the proof of Theorem 2.1. By Claim 3, W can be approximated by S. Using Claim 4, Claim 5 and Theorem 1.3, the Minkowski functional of S can be approximated by analytic Minkowski functionals. Consequently, the Minkowski functional of W can be arbitrarily well approximated by analytic Minkowski functionals.

3. Smooth approximation in smooth spaces with basis. The proof of Theorem 2.1 can be extended to a more general setting that we present now. Let us recall that $\{x_i\}_{i\in\mathbb{N}}$ is a Schauder basis of the Banach space X if for every $x \in X$, there is a unique sequence of scalars $\{a_i\}_{i\in\mathbb{N}}$ such that $x = \sum_{i\in\mathbb{N}} a_i x_i$.

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a separable Banach space, and let $\{x_i\}_{i\in\mathbb{N}}$ be a Schauder basis of X. Let $k \in \mathbb{N}\cup\{+\infty\}$, $\|\cdot\|$ be C^k-smooth, and $D^l\|\cdot\|$ be bounded on B_X for $l \in \mathbb{N}$, $l \le k$. Then every Minkowski functional (resp. equivalent norm) on X can be approximated by C^k-smooth Minkowski functionals (resp. equivalent norms).*

COROLLARY 3.2. *On spaces $L_p[0,1]$, ℓ_p, where $1 < p < +\infty$, $p \notin \mathbb{N}$, every equivalent norm can be approximated by $C^{[p]}$-Fréchet smooth norms. On spaces $L_p[0,1]$, ℓ_p, where p is odd, every equivalent norm can be approximated by C^{p-1}-Fréchet smooth norms.*

Proof. It is well known that these spaces have a Schauder basis. Let $k = [p]$ if p is not an integer, and $k = p-1$ if p is an odd integer. The explicit calculation of the derivatives of its canonical norm, carried out e.g. in [DGZ, p. 184], shows that the norm of the spaces $L^p(\Omega)$ is C^k-smooth, and that $D^l\|\cdot\|$ are bounded on B_X for $l \in \mathbb{N}$, $l \le k$.

It should be noted that this is the best possible result because, as shown in [DGZ, p. 222], these spaces do not admit equivalent norms of higher order of Fréchet smoothness than the ones used for the approximation.

Proof of Theorem 3.1. For $k = 1$ the above result is known—see [DGZ, p. 53]. Observe that in this case, the assumption of the existence of a Schauder basis on X is not needed.

Denote by $\{x_i^*\}_{i\in\mathbb{N}}$ the biorthogonal system of $\{x_i\}_{i\in\mathbb{N}}$. For $k > 1$, the space X is superreflexive [DGZ, p. 203], so the linear span of $\{x_i^*\}_{i\in\mathbb{N}}$ is dense in X^*.

Suppose W is a closed, convex and bounded subset of X with $\vec{0} \in \operatorname{int} W$. Our goal is to approximate W by a convex body S which admits a countable generalized boundary $\{f_k\}_{k\in\mathbb{N}}$ satisfying the condition (k), and then to apply Theorem 1.3.

The following lemma is an extension of Claims 1 and 2 of Section 2. It summarizes Lemmas 3.7 and 3.8 of [DFH], where its proof is given, and is close to some results of [Zp]. Before stating it, let us recall that a biorthogonal system $\{x_i, x_i^*\}_{i\in\mathbb{N}}$ is an *M-basis* of the Banach space X if the sequence $\{x_i\}$ is total in X and x_i separates points of X, i.e. for every $x \in X\setminus\{\vec{0}\}$, there exists $i \in \mathbb{N}$ such that $x_i^*(x) \neq 0$.

LEMMA 3.3. *Let X be a Banach space with separable dual X^* and let $\{x_i\}_{i\in\mathbb{N}} \subset S_X$ be an M-basis of X such that the linear span of the biorthogonal system $\{x_i^*\}_{i\in\mathbb{N}}$ is dense in X^*. Let $W \subset X$ be a closed convex body such that $\vec{0} \in \operatorname{int} W$ and $0 < \varepsilon < 1/2$. Then there exists a w^*-compact subset $F \subset W^\circ$ such that*

(i) $(1+4\varepsilon)^{-1}W^\circ \subset w^*\text{-clco}\, F \subset (1+\varepsilon)^{-1}W^\circ$.

(ii) *For each integer i the set $x_i(F)$ is finite.*

In addition, for arbitrary $\varepsilon > 0$ there exists a sequence $\{g_k\}$ of points in the set F, a sequence $\{n_k\}$ of integers with $n_k \to \infty$, and a decreasing sequence $\{F_k\}$ of w^*-closed subsets of F such that

(iii) $\bigcup_{k\in\mathbb{N}}((g_k + M_{n_k}) \cap F_k) = F$.

(iv) $\operatorname{diam}((g_k + M_{n_k}) \cap F_k) < \varepsilon$.

Here $M_n = [x_i]_1^{n\perp}$, $n \in \mathbb{N}$.

Using the notations of Lemma 3.3, we obtain

$$\bigcup_{k\in\mathbb{N}} (g_k + \varepsilon B_{X^*} \cap M_{n_k}) \supset F.$$

Define

$$S^* := \bigcup_{k\in\mathbb{N}} (g_k + \varepsilon P_k^*(B_{X^*})),$$

where P_k are the linear projections on X defined by

$$P_k\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \sum_{i=n_k+1}^{\infty} a_i x_i.$$

From Lemma 3.3(i) and the fact that the norms of P_k and P_k^* are uniformly bounded, we deduce that w^*-clco S^* approximates W° arbitrarily well.

We claim that S^* is w^*-closed. Indeed, let $\{h_m\}$ be a sequence in S^* converging to $h_\infty \in X^*$. If there exists k such that infinitely many of the h_m are in the weak*-closed set $g_k + \varepsilon P_k^*(B_{X^*})$, then of course h_∞ is in this set, hence also in S^*. Otherwise, there exists $k_m \to \infty$ such that $h_m = g_{k_m} + u_m$, with $u_m \in \varepsilon P_{k_m}^*(B_{X^*})$. Consequently, u_m weak*-converges to 0 and g_{k_m} lies in the weak*-closed set F, hence $\lim u_n = \lim g_{k_m} \in F \subset S^*$.

Put

$$S = \{x \in X : h(x) \leq 1 \text{ for all } h \in S^*\} \tag{7}$$

and

$$f_k(x) = \sup\{y(x) : y \in g_k + \varepsilon P_k^*(B_{X^*})\} = g_k(x) + \varepsilon\|P_k(x)\|. \tag{8}$$

We claim that $\{f_k\}_{k\in\mathbb{N}}$ forms a generalized boundary of S. Indeed, let $x \in \partial S$. Since S^* is weak*-closed, there exists $h \in S^*$ such that $h(x) = 1 = \max\{g(x) : g \in S^*\}$. There exists k such that $h = g_k + \varepsilon u$ with $u \in P_k^*(B_{X^*})$. Consequently,

$$1 = g_k(x) + \varepsilon u(x) \leq g_k(x) + \varepsilon\|P_k(x)\| = f_k(x) \leq 1,$$

so $\{f_k\}$ forms a generalized boundary of S. It follows from (8) and the chain rule that $\{f_k\}_{k\in\mathbb{N}}$ satisfies the condition (k). By Theorem 1.3 we have thus finished the proof of Theorem 3.1.

THEOREM 3.4. *Let $(X, \|\cdot\|)$ be a separable Banach space with Schauder basis $\{x_i\}_{i\in\mathbb{N}}$. Assume that there exist an even $p \in \mathbb{N}$ and a convex homogeneous p-polynomial $P(\cdot)$ on X such that $\|\cdot\| = P(\cdot)^{1/p}$. Then every Minkowski functional (resp. equivalent norm) on X can be approximated by analytic Minkowski functionals (resp. equivalent norms).*

Proof. The construction of $\{f_k\}_{k\in\mathbb{N}}$ is exactly the same as in Theorem 3.1. In order to verify that $\{f_k\}_{k\in\mathbb{N}}$ satisfy the condition (ω), it is enough to realize, as in the proof of Claim 4 in Section 2, that

$$f_k^c = g_k^c + \varepsilon(P^c(P_k^c))^{1/p},$$

where g_k^c, P_k^c are uniformly continuous on a neighbourhood of $(x, \vec{0})$ and P^c is uniformly continuous on every bounded set. As, for $x \in X$, we have $\|P_k^c((x,\vec{0}))\|^c \to 0$ for $k \to +\infty$ and g_k lie in the polar of $\widetilde{D}$, we are done.

COROLLARY 3.5. *On spaces $L_p[0,1]$, ℓ_p, where p is an even integer, every Minkowski functional (resp. equivalent norm) can be approximated by analytic Minkowski functionals (resp. analytic norms).*

References

[D] R. Deville, *Geometrical implications of the existence of very smooth bump functions in Banach spaces*, Israel J. Math. 6 (1989), 1–22.

[DFH] R. Deville, V. Fonf and P. Hájek, *Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces*, to appear.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs Surveys Pure Appl. Math. 64, Longman, 1993.

[Dieu] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1960.

[FPWZ] M. Fabian, D. Preiss, J. H. M. Whitfield and V. Zizler, *Separating polynomials on Banach spaces*, Quart. J. Math. Oxford Ser. (2) 40 (1989), 409–422.

[Fe] H. Federer, *Geometric Measure Theory*, Springer, 1969.

[HH] P. Habala and P. Hájek, *Stabilization of polynomials*, C. R. Acad. Sci. Paris Sér. I 320 (1995), 821–825.
[H1] R. Haydon, *Normes infiniment différentiables sur certains espaces de Banach*, ibid. 315 (1992), 1175–1178.
[H2] —, *Smooth functions and partitions of unity on certain Banach spaces*, to appear.
[Ku1] J. Kurzweil, *On approximation in real Banach spaces*, Studia Math. 14 (1954), 213–231.
[Ku2] —, *On approximation in real Banach spaces by analytic operations*, ibid. 16 (1957), 124–129.
[M] P. Mazet, *Analytic Sets in Locally Convex Spaces*, North-Holland Math. Stud. 89, North-Holland, 1984.
[N] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*, Springer, 1969.
[NS] A. M. Nemirovskiĭ and S. M. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. Sb. 92 (1973), 257–281 (in Russian).
[Zp] M. Zippin, *The separable extension problem*, Israel J. Math. 26 (1977), 372–387.

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Polynomial selections and separation by polynomials

by

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Abstract. K. Nikodem and the present author proved in [3] a theorem concerning separation by affine functions. Our purpose is to generalize that result for polynomials. As a consequence we obtain two theorems on separation of an n-convex function from an n-concave function by a polynomial of degree at most n and a stability result of Hyers–Ulam type for polynomials.

1. Introduction. We denote by $\mathbb{R}$, $\mathbb{N}$ the sets of all reals and positive integers, respectively. Let $I \subset \mathbb{R}$ be an interval. In this paper we present a necessary and sufficient condition under which two functions $f, g : I \to \mathbb{R}$ can be separated by a polynomial of degree at most n, where $n \in \mathbb{N}$ is a fixed number. Our main result is a generalization of the theorem concerning separation by affine functions obtained recently by K. Nikodem and the present author in [3]. To get it we use Behrends and Nikodem's abstract selection theorem (cf. [1, Theorem 1]). It is a variation of Helly's theorem (cf. [7, Theorem 6.1]).

We denote by $\mathrm{cc}(\mathbb{R})$ the family of all non-empty compact real intervals. Recall that if $F : I \to \mathrm{cc}(\mathbb{R})$ is a set-valued function then a function $f : I \to \mathbb{R}$ is called a *selection* of F iff $f(x) \in F(x)$ for every $x \in I$.

Behrends and Nikodem's theorem states that if $\mathcal{W}$ is an n-dimensional space of functions mapping I into $\mathbb{R}$ then a set-valued function $F : I \to \mathrm{cc}(\mathbb{R})$ has a selection belonging to $\mathcal{W}$ if and only if for any $n+1$ points $x_1, \ldots, x_{n+1} \in I$ there exists $f \in \mathcal{W}$ such that $f(x_i) \in F(x_i)$ for $i = 1, \ldots, n+1$.

Let us start with the notation used in this paper. Let $n \in \mathbb{N}$. If $x_1, \ldots, x_n \in I$ are distinct then for $i = 1, \ldots, n$ we define

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